On curvature-type invariants in hyperbolic Kähler spaces

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Abstract. The holomorphically semi-symmetric connections on anti-Kähler spaces and product spaces have been already intensively studied, and their invariants have been determined ([4, 8, 9]). In the case of (F, g)-connections and F-connections, the curvature-type invariant was shown to be equal to a conformal invariant of such spaces. Since a hyperbolic Kähler space is geometrically rather different (it is an integrable unit of a space with a positive definite and a space with negative definite metrics, and hence it has isotropic subspaces), the connections of such kind are more complicated. As well, so are their torsion tensors, since the covariant structure tensor is skew-symmetric. It is shown that both the (F, g)-connections and the F-connections have curvature-type invariants which are not mutually equal, like on anti-Kähler spaces and on product spaces.

M.S.C. 2010: 53A55, 53B15, 53B30, 53B35.
Key words: Hyperbolic Kähler space; curvature tensor; structure tensor; Ricci tensor; scalar curvature.

1 Introduction

The author has intensively been studying hyperbolic Kähler spaces (spaces with Norden metrics) and related structures ([2, 3, 4, 5, 6, 7, 8, 9]). The present work investigates such kind of spaces following an idea from [1].

A hyperbolic Kähler (or Kähler) space is an even-dimensional space with involutive and skew-symmetric structure. On such a space, just like on the standard (or elliptic) Kählerian space, it is impossible to introduce a conformal transformation in a natural way.

The metric tensor g_{ij} and the structure tensor F^i_j on a hyperbolic Kählerian space satisfy the following conditions:

$$F^t_j F^s_i g_{ts} = -g_{ji}; \tag{1.1}$$

$$F^t_j F^i_t = \delta^i_j; \tag{1.2}$$

Differential Geometry - Dynamical Systems, Vol.17, 2015, pp. 110-125.

$$\overset{\circ}{\nabla}_k F^i_j = 0. \tag{1.3}$$

Here the symbol $\overset{\circ}{\nabla}$ denotes the covariant derivative with respect to the Levi-Civita connection. The scalar square of any tangent vector on such a space is opposite to the scalar square of its image by the structure. So, such a kind of space can be considered as a product of a space with positive definite and a space with negative definite metrics. Moreover, as the structure has n (even dimension of the space) eigenvectors, they must be null-vectors (isotropic).

As there is no way to define a conformal transformation of such a space by an exponential change of metrics, we shall introduce the following transformation

$$\overline{g}_{ji} = e^{2p} g_{ji}; \quad \overline{F}^i_j = F^i_j; \quad \overline{F}_{ji} = e^{2p} F_{ji}. \tag{1.4}$$

Then it is easy to obtain a metric connection (for $\overline{g}$) with torsion tensor $\overline{S}^a_{ji} = 2F_{ji}q^a$, where $(\star) q^a = p_b F^{ab}$ and p_k is $\frac{\partial p}{\partial x^k}$. The coefficients of such a connections are

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + p_k\delta^i_j + p_j\delta^i_k - g_{jk}p^i + F^i_j q_k + F^i_k q_j + F_{jk}q^i. \tag{1.5}$$

Since $(\star)$ holds, such a connection is also an $F-$connection. We call such a connection a conformal connection on a hyperbolic Kähler space. Properties of such a connection and its curvature tensor were investigated in [6].

In [4], [8] and [9] we investigated properties of metric and $F-$connections, which are holomorphically semi-symmetric on anti-Kähler and product spaces. Some of these connections (namely, (F, g) and F) have invariants of curvature type, which are equal to conformal invariants on such spaces. But, these connections are holomorphically semi-symmetric and the covariant structure tensors of these spaces are symmetric. So, the analogous problem on hyperbolic Kähler space will be much more complicated.

2 Holomorphically-structurally semi-symmetric connections on a hyperbolic Kähler space

We are going to construct connections whose coefficients are consisting of the following adders: Christoffel symbol, $p_j\delta^i_k$ and $p^i g_{jk}$, where p_j is a gradient vector field and $q_j F^i_k$, $q^i F_{jk}$, where $q_j = F^a_j p_a$. The general look of the coefficients of such a connection will be

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + Ap_j\delta^i_k + Bp^i g_{jk} + Cq_j F^i_k + Dq^i F_{jk}, \tag{2.1}$$

where the coefficients A, B, C, D are equal to ± 1. We would like to construct the coefficients of three connections: (i) a connection towards to which metric and structural tensors are parallel (an (F, g) connection); (ii) a connection towards to which only structural tensor is parallel (an $F-$connection); (iii) a connection towards to which only metric tensor is parallel (a $g-$connection). Generally, all connections with coefficients of the form (2.1) on a hyperbolic Kähler space will be called holomorphically-structurally semi-symmetric connections.

Now we are going to find coefficients of the three connections which have been listed.

It is easy to be shown that the connection with coefficients given by (2.1) is metric if $\nabla_k g_{ij} = -(A+B)p_i g_{jk} - (A+B)p_j g_{ik} + (C-D)q_i F_{jk} + (C-D)q_j F_{ik} = 0$.

For the scalar values $A = 1; B, C, D = \pm 1$, we shall have the possibilities:

$$(1)\quad A = 1, B = -1, C = 1, D = 1,$$

$$(2)\quad A = 1, B = -1, C = -1, D = -1.$$

If we want a connection with coefficients (2.1) to be an $F-$connection, then the following should be satisfied: $\nabla_k F^i_j = (A-C)q_j\delta^i_k + (B+D)p^i F_{jk} + (C-A)p_j F^i_k + (B+D)q^i g_{jk} = 0$. Here, A, B, C, D we have also the same two possibilities (1) and (2) from above. So, we consider three connections with the coefficients:

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + p_j\delta^i_k - p^i g_{jk} + q_j F^i_k + q^i F_{jk}, \tag{2.2}$$

which is a metric and F, or (F, g) holomorphically-structurally semi-symmetric connection,

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + p_j\delta^i_k + p^i g_{jk} + q_j F^i_k - q^i F_{jk}, \tag{2.3}$$

which is an $F-$holomorphically-structurally semi-symmetric connection and

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + p_j\delta^i_k - p^i g_{jk} - q_j F^i_k - q^i F_{jk}, \tag{2.4}$$

which is a $g-$holomorphically-structurally semi-symmetric connection.

In this paper we shall find the curvature-type invariants of the connections with coefficients given by (2.2) and (2.3), which are not mutually equal in the case of such a space. The case of curvature-type invariant of the connection (2.4) will be studied in a subsequent paper of the author.

3 The curvature tensor of (F, g)-holomorphically-structurally semi-symmetric connections

We shall further consider an $(F, g)-$ holomorphically-structrally semi-symmetric connection on a hyperbolic Kähler space. Its coefficients are given by (2.2). The components of its curvature tensor are

$$\begin{aligned} R_{ijkl} \;=\; & K_{ijkl} + g_{ki}p_{lj} - g_{li}p_{kj} + g_{lj}p_{ki} - g_{kj}p_{li} \\ & + F_{ki}q_{lj} - F_{li}q_{kj} + F_{lj}q_{ki} - F_{kj}q_{li} + \beta_{ji}F_{kl}, \end{aligned} \tag{3.1}$$

where we use the following abbreviations

$$p_{kj} = \overset{\circ}{\nabla}_k p_j - p_k p_j - q_k q_j + \frac{1}{2} p_s p^s g_{kj} \tag{3.2}$$

and

$$q_{kj} = \overset{\circ}{\nabla}_k q_j - p_k q_j - q_k p_j - \frac{1}{2} p_s p^s F_{kj}. \tag{3.3}$$

The tensor K_{ijkl} is the curvature tensor of the Levi-Civita connection. It is obvious that $q_{kj} = p_{ka}F_j^a$ holds. Also, since we assumed that (p_j) is a gradient, it is obvious that p_{kj} is a symmetric tensor. Moreover, we have

$$\beta_{ji} = 2(p_j q_i - p_i q_j).$$

It is evident that the components of curvature tensor of this connection are skew-symmetric in the last two indices. Also, they are skew-symmetric in their first two indices:

$$\begin{aligned} R_{jikl} &= K_{jikl} + g_{kj}p_{li} - g_{lj}p_{ki} + g_{li}p_{kj} - g_{ki}p_{lj} \\ &\quad + F_{kj}q_{li} - F_{lj}q_{ki} + F_{li}q_{kj} - F_{ki}q_{lj} + \beta_{ij}F_{kl}, \end{aligned}$$

since the tensor β is skew-symmetric.

Now we want the components of the curvature tensor of this connection to be invariant under changing places of the first and of the second pair of indices; this leads to

$$\begin{aligned} R_{klij} &= K_{klij} + g_{ik}p_{jl} - g_{jk}p_{il} + g_{jl}p_{ki} - g_{il}p_{jk} \\ &\quad + F_{ik}q_{jl} - F_{jk}q_{il} + F_{jl}q_{ik} - F_{il}q_{jk} + \beta_{lk}F_{ij}. \end{aligned} \tag{3.4}$$

The first lines of (3.1) and (3.4) are equal. So, when we consider their equality ($R_{ijkl} = R_{klij}$), the first lines cancel each other. We obtain hence the equality in the following way: we put on the left hand side of such an equality every adder which contains the components of the structure tensor and of the tensor q. If we take into account the skew-symmetry of the structural tensor, we obtain

$$\begin{aligned} &F_{ki}(q_{lj} + q_{jl}) - F_{li}(q_{kj} + q_{jk}) + F_{lj}(q_{ki} + q_{ik}) - F_{kj}(q_{li} + q_{il}) \\ =\ &\beta_{lk}F_{ij} - \beta_{ji}F_{kl}. \end{aligned}$$

By transvecting the last equality with F^{ik}, we obtain

$$(n-2)(q_{lj} + q_{jl}) = -\beta_{lj} - \beta_{jl} = -(\beta_{jl} + \beta_{lj}) = 0,$$

since the tensor β is skew-symmetric. Then the tensor q is also skew-symmetric, i.e.,

$$q_{lj} = -q_{jl}. \tag{3.5}$$

If we want the first Bianchi identity to be satisfied, then we get

$$\begin{aligned} 0 &= K_{ijkl} + g_{ki}p_{lj} - g_{li}p_{kj} + g_{lj}p_{ki} - g_{kj}p_{li} \\ &\quad + F_{ki}q_{lj} - F_{li}q_{kj} + F_{lj}q_{ki} - F_{kj}q_{li} + \beta_{ji}F_{kl} \\ &\quad + K_{iklj} + g_{li}p_{jk} - g_{ji}p_{lk} + g_{jk}p_{li} - g_{lk}p_{ji} \\ &\quad + F_{li}q_{jk} - F_{ji}q_{lk} + F_{jk}q_{li} - F_{lk}q_{ji} + \beta_{ki}F_{lj} \\ &\quad + K_{iljk} + g_{ji}p_{kl} - g_{ki}p_{jl} + g_{kl}p_{ji} - g_{jl}p_{ki} \\ &\quad + F_{ji}q_{kl} - F_{ki}q_{jl} + F_{kl}q_{ji} - F_{jl}q_{ki} + \beta_{li}F_{jk}. \end{aligned}$$

All the terms in the odd lines of the above formula cancel each other; moreover, the curvature tensor of the Levi-Civita connection satisfies the first Bianchi identity. Then, from the even lines, and taking into account the skew-symmetry of the structural tensor and of the tensor q, we obtain

$$\begin{aligned}(3.6)\qquad 0 &= 2(F_{ki}q_{lj} - F_{li}q_{kj} + F_{lj}q_{ki} - F_{kj}q_{li} + F_{ji}q_{kl} - F_{lk}q_{ji}) \\ &\quad +\beta_{ji}F_{kl} + \beta_{ki}F_{lj} + \beta_{li}F_{jk}.\end{aligned}$$

If we transvect (3.6) by F^{ik} and take into account the skew-symmetry of β_{ij} and the fact that $\beta_{ki}F^{ik} = -4p_s p^s$, we obtain

$$(n-4)q_{lj} - p_s^s F_{lj} - 2p_s p^s F_{lj} - \beta_{lj} = 0.$$

From the last equation, we infer

$$(3.7)\qquad q_{lj} = \frac{1}{n-4}\left[(p_s^s + 2p_s p^s)F_{lj} + \beta_{lj}\right].$$

Since there holds

$$p_s^s = \overset{\circ}{\nabla}_s p^s - p_s p^s - q_s q^s + \frac{n}{2}p_s p^s = \overset{\circ}{\nabla}_s p^s + \frac{n}{2}p_s p^s$$

and also

$$p_s^s + 2p_s p^s = \overset{\circ}{\nabla}_s p^s + \frac{n+4}{2}p_s p^s,$$

then, substituting the last equation into (3.7), we obtain

$$(3.8)\qquad q_{lj} = \frac{1}{n-4}\left[(\overset{\circ}{\nabla}_s p^s + \frac{n+4}{2}p_s p^s)F_{lj} + 2(p_l q_j - q_l p_j)\right].$$

Since $p_{lj} = q_{la}F_j^a$ holds, we get

$$(3.9)\qquad p_{lj} = \frac{1}{n-4}\left[-(\overset{\circ}{\nabla}_s p^s + \frac{n+4}{2}p_s p^s)g_{lj} + 2(p_l p_j - q_l q_j)\right].$$

If we transvect the last equality by g^{lj}, we infer

$$p_s^s = -\frac{n}{n-4}(\overset{\circ}{\nabla}_s p^s + \frac{n+4}{2}p_s p^s) + \frac{4}{n-4}p_s p^s.$$

Taking into account one of the upper equations, we get

$$\overset{\circ}{\nabla}_s p^s + \frac{n}{2}p_s p^s = -\frac{n}{n-4}\overset{\circ}{\nabla}_s p^s - \frac{n(n+4)}{2(n-4)}p_s p^s + \frac{4}{n-4}p_s p^s.$$

From the upper equation, we obtain

$$\begin{aligned}\frac{2(n-2)}{n-4}\overset{\circ}{\nabla}_s p^s &= \frac{8 - n(n+4) - n(n-4)}{2(n-4}p_s p^s = \\ &= -\frac{(n-2)(n+2)}{n-4}p_s p^s\end{aligned}$$

and consequently,

$$\overset{\circ}{\nabla}_s\, p^s = -\frac{n+2}{2} p_s p^s. \tag{3.10}$$

Then there holds

$$\overset{\circ}{\nabla}_s\, p^s + \frac{n+4}{2} p_s p^s = p_s p^s$$

and

$$\begin{aligned} p_{lj} &= -\frac{p_s p^s}{n-4} g_{lj} + \frac{2}{n-4}(p_l p_j - q_l q_j) \\ q_{lj} &= \frac{p_s p^s}{n-4} F_{lj} + \frac{2}{n-4}(p_l q_j - q_l p_j). \end{aligned} \tag{3.11}$$

Now we can substitute the formula (3.11) into (3.1). Then, we infer

$$\begin{aligned} & R_{ijkl} \\ = \; & K_{ijkl} - \frac{2}{n-4} p_s p^s (g_{ik} g_{lj} - g_{il} g_{kj} + F_{li} F_{kj} - F_{ki} F_{lj}) \\ & + \frac{2}{n-4}[g_{ki}(p_l p_j - q_l q_j) - g_{li}(p_k p_j - q_k q_j) + g_{lj}(p_k p_i - q_k q_i) \\ & - g_{kj}(p_l p_i - q_l q_i) + F_{ki}(p_l q_j - q_l p_j) - F_{li}(p_k q_j - q_k p_j) \\ & + F_{lj}(p_k q_i - q_k p_i) - F_{kj}(p_l q_i - q_l p_i)] + 2(p_j q_i - q_j p_i) F_{kl}. \end{aligned} \tag{3.12}$$

Now we shall the upper equality by g^{il}, and get the following relation between the Ricci tensors

$$\begin{aligned} R_{jk} &= K_{jk} + \frac{2n}{n-4} p_s p^s g_{kj} - \frac{4}{n-4} p_s p^s g_{kj} \\ & \quad - \frac{2n}{n-4}(p_k p_j - q_k q_j) + 2(p_k p_j - q_k q_j) \end{aligned}$$

and consequently,

$$R_{jk} = K_{jk} + \frac{2(n-2)}{n-4} p_s p^s g_{jk} - \frac{8}{n-4}(p_k p_j - q_k q_j). \tag{3.13}$$

Now, if we transvect (3.12) by g^{jk}, we obtain

$$R = K + 2\frac{n^2 - 2n - 8}{n-4} p_s p^s,$$

which means

$$R = K + \frac{2(n-4)(n+2)}{(n-4)} p_s p^s,$$

and consequently

$$p_s p^s = \frac{R - K}{2(n+2)}. \tag{3.14}$$

If we substitute (3.10) into (3.1), we get the following equality

$$\frac{8}{n-4}(p_k p_j - q_k q_j) = K_{jk} - R_{jk} + \frac{n-2}{n-4}\frac{R-K}{n+2}g_{jk}.$$

After multiplying this equation by $\frac{n-4}{8}$, we infer

$$p_k p_j - q_k q_j = \frac{n-4}{8}(K_{jk} - R_{jk}) + \frac{n-2}{8(n+2)}(R-K)g_{kj}. \tag{3.15}$$

After transvecting $p_k p_s - q_k q_s$ by F^s_j, we obtain that there holds

$$p_k q_j - q_k p_j = \frac{n-4}{8}(K_{ka} - R_{ka})F^a_j - \frac{n-2}{8(n+2)}(R-K)F_{kj}. \tag{3.16}$$

Now we can substitute (3.14), (3.15) and (3.16) into (3.10). We put all the tensors and scalar functions depending on the $(F,g)-$holomorphically-structurally semi-symmetric connection on the left-hand side of the equality, and all the tensors and quantities depending on the Levi-Civita connection on the right-hand side of the equality. Then the following holds

$$\begin{aligned} &R_{ijkl} - \frac{R}{2(n+2)}(g_{ik}g_{lj} - g_{il}g_{kj} + F_{li}F_{kj} - F_{ki}F_{lj}) \\ &+\frac{1}{4}[R_{lj}g_{ki} - R_{kj}g_{li} + R_{ki}g_{lj} - R_{li}g_{kj} + R_{la}F^a_j F_{ki} \\ &-R_{ka}F^a_j F_{li} + R_{ka}F^a_i F_{lj} - R_{la}F^a_i F_{kj} \\ &+(n-4)R_{ja}F^a_i F_{kl} + \frac{n-2}{n+2}RF_{ji}F_{kl}] \\ = \; &K_{ijkl} - \frac{K}{2(n+2)}(g_{ik}g_{lj} - g_{il}g_{kj} + F_{li}F_{kj} - F_{ki}F_{lj}) \\ &+\frac{1}{4}[K_{lj}g_{ki} - K_{kj}g_{li} + K_{ki}g_{lj} - K_{li}g_{kj} + K_{la}F^a_j F_{ki} \\ &-K_{ka}F^a_j F_{li} + K_{ka}F^a_i F_{lj} - K_{la}F^a_i F_{kj} \\ &+(n-4)K_{ja}F^a_i F_{kl} + \frac{n-2}{n+2}KF_{ji}F_{kl}]. \end{aligned} \tag{3.17}$$

So, we proved that there holds

Theorem 3.1. *If the generator of the holomorphically-structurally semi-symmetric (F,g)-connection is a gradient and if its curvature tensor is invariant under changing places of first and second pair of indices (i.e., $R_{ijkl} = R_{klij}$) and if it satisfies the first Bianchi identity, then the tensor from the left-hand side of (3.17) does not depend on the generator.*

4 Properties of the curvature tensor

Now we shall consider a connection with its coefficients given by (2.3). We shall calculate the coefficients of its curvature tensor. After lowering the upper index, we obtain

$$R_{ijkl} = K_{ijkl} + g_{ki}p_{lj} - g_{li}p_{kj} + g_{kj}\overline{p}_{li} - g_{lj}\overline{p}_{ki} + F_{ki}q_{lj} - F_{li}q_{kj} + F_{kj}\overline{q}_{li} - F_{lj}\overline{q}_{ki} + 2(p_iq_j - p_jq_i)F_{kl}, \tag{4.1}$$

where

$$p_{kj} = \overset{\circ}{\nabla}_k p_j - p_kp_j - q_kq_j - \frac{1}{2}p_sp^sg_{kj}; \tag{4.2}$$

$$\overline{p}_{li} = \overset{\circ}{\nabla}_l p_i + p_lp_i + q_lq_i + \frac{1}{2}p_sp^sg_{li};$$

$$q_{kj} = \overset{\circ}{\nabla}_k q_j - q_kp_j - p_kq_j + \frac{1}{2}p_sp^sF_{kj};$$

$$\overline{q}_{li} = \overset{\circ}{\nabla}_l q_i + p_lq_i + q_lp_i - \frac{1}{2}p_sp^sF_{li}.$$

It is easy to see that the following holds

$$p_{ka}F_j^a = q_{kj}; \overline{p}_{la}F_i^a = \overline{q}_{li}.$$

Also, both tensors p_{kj} and $\overline{p}_{li}$ are symmetric.

It is obvious that the tensor given by (4.1) is skew-symmetric in its last two indices. We want it to be also skew-symmetric in its first two indices. Then we infer that

$$R_{jikl} = K_{jikl} + g_{kj}p_{li} - g_{lj}p_{ki} + g_{ki}\overline{p}_{lj} - g_{li}\overline{p}_{kj} + F_{kj}q_{li} - F_{lj}q_{ki} + F_{ki}\overline{q}_{lj} - F_{li}\overline{q}_{kj} + 2(p_jq_i - p_iq_j)F_{kl}.$$

If we add the upper equation to (4.1), we obtain that there holds

$$\begin{aligned} R_{ijkl} + R_{jikl} &= g_{ki}(p_{lj} + \overline{p}_{lj}) - g_{li}(p_{kj} + \overline{p}_{kj}) + g_{kj}(p_{li} + \overline{p}_{li}) \\ &\quad - g_{lj}(p_{ki} + \overline{p}_{ki}) + F_{ki}(q_{lj} + \overline{q}_{lj}) - F_{li}(q_{kj} + \overline{q}_{kj}) \\ &\quad + F_{kj}(q_{li} + \overline{q}_{li}) - F_{lj}(q_{ki} + \overline{q}_{ki}) \\ &= 0. \end{aligned}$$

It is easy to see that $p_{lj} + \overline{p}_{lj} = 2 \overset{\circ}{\nabla}_l p_j, q_{lj} + \overline{q}_{lj} = 2 \overset{\circ}{\nabla}_l q_j$. If we divide the upper equation by 2, we obtain

$$\begin{aligned} 0 &= g_{ki} \overset{\circ}{\nabla}_l p_j - g_{li} \overset{\circ}{\nabla}_k p_j + g_{kj} \overset{\circ}{\nabla}_l p_i \\ &\quad - g_{lj} \overset{\circ}{\nabla}_k p_i + F_{ki} \overset{\circ}{\nabla}_l q_j - F_{li} \overset{\circ}{\nabla}_l q_j \\ &\quad + F_{kj} \overset{\circ}{\nabla}_l q_i - F_{lj} \overset{\circ}{\nabla}_k q_i. \end{aligned} \tag{4.3}$$

If we transvect the upper equality by g^{ik}, we get

$$\begin{aligned} 0 &= n \overset{\circ}{\nabla}_l p_j - \overset{\circ}{\nabla}_l p_j + \overset{\circ}{\nabla}_l p_j - g_{lj} \overset{\circ}{\nabla}_s p^s - F_l^k \overset{\circ}{\nabla}_k q_j \\ &\quad - F_j^i \overset{\circ}{\nabla}_l q_i - F_{lj} \overset{\circ}{\nabla}_s q^s. \end{aligned}$$

We supposed that the generator (p_j) of this connection is a gradient, hence

$$\overset{\circ}{\nabla}_s q^s = \overset{\circ}{\nabla}_s p_k F^{sk} = 0;$$

$$F_l^k \overset{\circ}{\nabla}_k q_j = F_l^k F_j^a \overset{\circ}{\nabla}_k p_a = F_l^k F_j^a \overset{\circ}{\nabla}_a p_k;$$

$$F_j^i \overset{\circ}{\nabla}_l q_i = \overset{\circ}{\nabla}_l p_a F_i^a F_j^i = \overset{\circ}{\nabla}_l p_j.$$

Then, it is readily follows

$$(n-1) \overset{\circ}{\nabla}_l p_j - g_{lj} \overset{\circ}{\nabla}_s p^s - F_l^k F_j^a \overset{\circ}{\nabla}_a p_k = 0,$$

and, consequently,

$$\overset{\circ}{\nabla}_l p_j = \frac{1}{n-1}(\overset{\circ}{\nabla}_s p^s g_{lj} + F_l^k F_j^a \overset{\circ}{\nabla}_k p_a). \tag{4.4}$$

Now we shall substitute in (4.4) the expression for $\overset{\circ}{\nabla}_k p_a$, by the same pattern; we get

$$\begin{aligned}
\overset{\circ}{\nabla}_l p_j &= \frac{1}{n-1}(\overset{\circ}{\nabla}_s p^s g_{lj} + \frac{1}{n-1} F_l^k F_j^a (\overset{\circ}{\nabla}_s p^s g_{ka} + F_k^t F_a^r \overset{\circ}{\nabla}_t p_r)) \\
&= \frac{1}{n-1}(\overset{\circ}{\nabla}_s p^s g_{lj} + \frac{1}{n-1} \overset{\circ}{\nabla}_s p^s F_{la} F_j^a + \frac{1}{n-1} F_l^k F_j^a F_k^t F_a^r \overset{\circ}{\nabla}_t p_r) \\
&= \frac{1}{n-1} \overset{\circ}{\nabla}_s p^s g_{lj} - \frac{1}{(n-1)^2} \overset{\circ}{\nabla}_s p^s g_{lj} + \frac{1}{(n-1)^2} \overset{\circ}{\nabla}_l p_j.
\end{aligned}$$

From this equation, we obtain

$$\frac{n(n-2)}{(n-1)^2} \overset{\circ}{\nabla}_l p_j = \frac{n-2}{(n-1)^2} \overset{\circ}{\nabla}_s p^s g_{lj}$$

and finally,

$$\overset{\circ}{\nabla}_l p_j = \frac{1}{n} \overset{\circ}{\nabla}_s p^s g_{lj}. \tag{4.5}$$

From (4.5), we infer

$$\overset{\circ}{\nabla}_l q_j = -\frac{1}{n} \overset{\circ}{\nabla}_s p^s F_{lj}. \tag{4.6}$$

Then the tensor $\overset{\circ}{\nabla}_l q_j$ is skew-symmetric.

If we contract the equality (4.3) by F^{ik}, we get

$$(n-1) \overset{\circ}{\nabla}_l q_j - \overset{\circ}{\nabla}_j q_l = -F_{lj} \overset{\circ}{\nabla}_s p^s,$$

which gives us the same result. Then, there holds

$$\overset{\circ}{\nabla}_l q_j + \overset{\circ}{\nabla}_j q_l = 0. \tag{4.7}$$

We further use the following notation

$$p_{kj} = \overset{\circ}{\nabla}_k p_j - S_{kj}; \quad \overline{p}_{kj} = \overset{\circ}{\nabla}_k p_j + S_{kj} \tag{4.8}$$

and, consequently,

$$\begin{aligned} q_{kj} &= \overset{\circ}{\nabla}_k q_j - S_{ka}F^a_j = p_{ka}F^a_j, \\ \overline{q}_{kj} &= \overset{\circ}{\nabla}_k q_j + S_{ka}F^a_j = \overline{p}_{ka}F^a_j. \end{aligned} \tag{4.9}$$

We also want the curvature tensor (4.1) to be invariant under changing places of the first and the second pair of indices (i.e., $R_{ijkl} = R_{klij}$). Then,

$$\begin{aligned} R_{klij} \;=\; & K_{klij} + g_{ik}p_{jl} - g_{jk}p_{il} + g_{il}\overline{p}_{jk} - \\ & g_{jl}\overline{p}_{ik} + F_{ik}q_{jl} - F_{jk}q_{il} + F_{il}\overline{q}_{jk} \\ & -F_{jl}\overline{q}_{ik} + 2(p_k q_l - p_l q_k). \end{aligned} \tag{4.10}$$

By substracting (4.10) from (4.1), we obtain that there holds

$$\begin{aligned} 0 \;=\; & g_{ik}(p_{lj} - p_{jl}) - g_{li}(p_{kj} + \overline{p}_{jk}) + g_{kj}(\overline{p}_{li} + p_{il}) \\ & -g_{lj}(\overline{p}_{ki} - \overline{p}_{ik}) + F_{ki}(q_{lj} + q_{jl}) + F_{jk}(q_{il} - \overline{q}_{li}) \\ & -F_{li}(q_{kj} - \overline{q}_{jk}) - F_{lj}(\overline{q}_{ki} + \overline{q}_{ik}) + 2(p_i q_j - p_j q_i)F_{kl} \\ & -2(p_k q_l - p_l q_k)F_{ij}. \end{aligned}$$

Taking into account symmetry of tensors p_{lj} and $\overline{p}_{li}$, we obtain from the upper equation

$$\begin{aligned} 0 \;=\; & -g_{li}(p_{kj} + \overline{p}_{jk}) + g_{kj}(\overline{p}_{li} + p_{il}) + F_{ki}(q_{lj} + q_{jl}) \\ & -F_{li}(q_{kj} - \overline{q}_{jk}) + F_{kj}(\overline{q}_{li} - q_{il}) - F_{lj}(\overline{q}_{ki} + \overline{q}_{ik}) \\ & +2(p_i q_j - p_j q_i)F_{kl} - 2(p_k q_l - p_l q_k)F_{ij}. \end{aligned}$$

Using the relations (4.5), (4.6), (4.8) and (4.9), we get

$$\begin{aligned} 0 \;=\; & 2g_{kj} \overset{\circ}{\nabla}_l p_i - 2g_{li} \overset{\circ}{\nabla}_k p_j \\ & +F_{ki}(\overset{\circ}{\nabla}_l q_j - S_{la}F^a_j + \overset{\circ}{\nabla}_j q_l - S_{ja}F^a_l) \\ & -F_{li}(\overset{\circ}{\nabla}_k q_j - S_{ka}F^a_j - \overset{\circ}{\nabla}_j q_k - S_{ja}F^a_k) \\ & +F_{kj}(\overset{\circ}{\nabla}_l q_i + S_{la}F^a_i - \overset{\circ}{\nabla}_i p_l + S_{ia}F^a_l) \\ & -F_{lj}(\overset{\circ}{\nabla}_k q_i + S_{ka}F^a_i + \overset{\circ}{\nabla}_i q_k + S_{ia}F^a_k) \\ & +2(p_i q_j - p_j q_i)F_{kl} - 2(p_k q_l - p_l q_k)F_{ij} \end{aligned}$$

and, consequently,

$$\begin{gathered} -F_{ki}(S_{la}F^a_j + S_{ja}F^a_l) - F_{li}(-\tfrac{2}{n} \overset{\circ}{\nabla}_s p^s F_{kj} - S_{ka}F^a_j - S_{ja}F^a_k) \\ +F_{kj}(-\tfrac{2}{n} \overset{\circ}{\nabla}_s p^s F_{li} + S_{la}F^a_i + S_{ia}F^a_l) \\ -F_{lj}(S_{ka}F^a_i + S_{ia}F^a_k) + 2(p_i q_j - p_j q_i)F_{kl} - \\ -2(p_k q_l - p_l q_k)F_{ij} = 0. \end{gathered}$$

Since the terms containing $\overset{\circ}{\nabla}_s\, p^s$ cancel, and since

$$S_{la}F^a_j = p_l q_j + q_l p_j - \frac{1}{2}p_s p^s F_{lj},$$

$$S_{ja}F^a_l = p_j q_l + q_j p_l - \frac{1}{2}p_s p^s F_{jl},$$

we get, after dividing by 2:

$$(4.11)\qquad \begin{aligned} 0 \;=\; & F_{li}(p_k q_j + q_k p_j) - F_{ki}(p_l q_j + q_l p_j) \\ & + F_{kj}(p_l q_i + q_l p_i) - F_{lj}(p_k q_i + p_i q_k) \\ & + F_{kl}(p_i q_j - p_j q_i) + F_{ij}(p_k q_l - p_l q_k). \end{aligned}$$

If we transvect the above equality by F^{il}, we obtain

$$(n-2)(p_k q_j + q_k p_j) - (p_k q_j - q_k p_j) + (p_k q_j - q_k p_j) = 0$$

and, consequently,

$$(4.12)\qquad p_k q_j + q_k p_j = 0.$$

If we transvect (4.11) by F^{kj}, we obtain that there holds

$$n(p_j q_i - p_i q_j) = 2p_s p^s F_{ij}.$$

From the above equality and (4.12), we get

$$(4.13)\qquad p_j q_i = \frac{p_s p^s}{n}F_{ij}; \quad p_i p_j = \frac{p_s p^s}{n}g_{ij}; \quad q_i q_j = -\frac{p_s p^s}{n}g_{ij}.$$

Then, there holds

$$(4.14)\qquad S_{kj} = \frac{1}{2}p_s p^s g_{kj}; \quad S_{ka}F^a_j = -\frac{1}{2}p_s p^s F_{kj},$$

Now we shall find the scalar square of the generator in another way:

$$p_j p^j = p_j q_i F^{ji} = \frac{p_s p^s}{n}F_{ji}F^{ji} = -p_s p^s.$$

This means that the generator is an isotropic vector, and, if we use (4.13), it is obvious that then all the components of the generator and of its image by the structure vanish. So, we proved the following

Theorem 4.1. *The curvature tensor of $F-$holomorphically$-$structurally semi-symmetric connection on a hyperbolic Kähler space cannot be invariant under changing places of first and second pair of indices (i.e., $R_{ijkl} \neq R_{klij}$).*

We also have to check the possibility for the curvature tensor of this connection to satisfy the first Bianchi identity. Then, using the formula (4.1), we obtain that, if the first Bianchi identity is satisfied, there must hold

$$\begin{aligned} 0 \;=\; & F_{li}S_{ka}F^a_j - F_{ki}S_{la}F^a_j + F_{kj}S_{la}F^a_i - F_{lj}S_{ka}F^a_i + F_{ji}S_{la}F^a_k \\ & - F_{kl}S_{ja}F^a_i + (p_i q_j - p_j q_i)F_{kl} + (p_i q_k - p_k q_i)F_{lj} \\ & + (p_i q_l - p_l q_i)F_{jk}. \end{aligned}$$

We transvect this equation by F^{il} and, taking into account that $S^s_s = \frac{n}{2}p_s p^s$ (since the vectors p and q are mutually orthogonal), we obtain that there holds

$$\begin{aligned} 0 &= (n-4)(p_k q_j + q_k p_j - \frac{1}{2}p_s p^s F_{kj}) - \frac{n}{2}F_{kj}p_s p^s \\ &+2(p_j q_k - p_k q_j) - 2p_s p^s F_{kj}. \end{aligned}$$

The final form of such equation is

$$(n-4)(p_k q_j + q_k p_j) = np_s p^s F_{kj} + 2(p_k q_j - q_k p_j). \tag{4.15}$$

The left side of (4.15) is symmetric and the right side is skew-symmetric. Then, each of them will vanish. The case $n = 4$ will be considered a bit later. Then, from vanishing of both sides of (4.15), we shall obtain that there holds

$$p_k q_j = \frac{n}{4}p_s p^s F_{kj}. \tag{4.16}$$

Then we have

$$p_k q_j F^{jk} = -p_k q_j F^{kj} = \frac{n}{4}p_s p^s F_{kj}F^{jk} = \frac{n^2}{4}p_s p^s = -p_k p^k.$$

From this fact, we have $p_s p^s = 0$. Then the relation (4. 15) takes the form

$$(n-4)(p_k q_j + q_k p_j) = 2(p_k q_j - q_k p_j),$$

or,

$$(n-6)p_k q_j = -(n-2)q_k p_j.$$

If we apply this relation once again, we obtain

$$p_k q_j = (\frac{n-2}{n-6})^2 p_k q_j. \tag{4.17}$$

From this fact, we have either $n-2 = n-6$, which is impossible, or $n-2 = 6-n$. The second relation implies $n = 4$. Hence, in the case of low dimension, the relation (4.15) implies that

$$p_s p^s F_{kj} = -\frac{1}{2}(p_k q_j - p_j q_k).$$

If we transvect the last relation by F^{jk}, we obtain

$$4p_s p^s = -\frac{1}{2}(q_j q^j - p_j p^j) = p_s p^s.$$

From the last relation, we also have that $p_s p^s = 0$. Then, for $n = 4$, the following holds

$$p_k q_j = p_j q_k,$$

which means that in such a case the generator is an eigenvector for the structure. From (4.16) then we obtain that the all components of the generator vanish. In such a case, obviously, there does not exist such a connection. So, we have proved the following

Theorem 4.2. *The curvature tensor of a $F-$ holomorhically-structurally semi-symmetric connection on a hyperbolic Kähler space does not satisfy the first Bianchi identity.*

5 Constructing a curvature-like invariant of the considered semi-symmetric connection

If we substitute (4.8), (4.9), (4.5) and (4.6) into (4.1), we obtain

$$
\begin{aligned}
R_{ijkl} &= K_{ijkl} + g_{li}S_{kj} - g_{ki}S_{lj} + g_{kj}S_{li} \\
&\quad -g_{lj}S_{ki} + F_{li}S_{ka}F_j^a - F_{ki}S_{la}F_j^a \\
&\quad +F_{kj}S_{la}F_i^a - F_{lj}S_{ka}F_i^a + 2(p_iq_j - p_jq_i)F_{kl},
\end{aligned} \tag{5.1}
$$

because all the adders containing $\overset{\circ}{\nabla}_s\ p^s$ and the metric tensor, or $\overset{\circ}{\nabla}_s\ p^s$ and the structural tensor cancel each other. If we transvect (5.1) by g^{il}, we get

$$
\begin{aligned}
R_{jk} &= K_{jk} + (n-1)S_{kj} + \frac{n}{2}p_sp^sg_{kj} - S_{kj} + \\
&\quad +p_sp^sg_{kj} + 2(q_kq_j - p_kp_j).
\end{aligned}
$$

Further, by using the relations

$$
(i)S_{la}F^{la} = 0; \quad (ii)S_s^s = \frac{n}{2}p_sp^s; \quad (iii)S_{la}F_k^lF_j^a = S_{kj} - p_sp^sg_{kj},
$$

we get a better expression

$$
R_{jk} = K_{jk} + (n-2)S_{kj} + \frac{n+2}{2}p_sp^sg_{kj} + 2(q_kq_j - p_kp_j). \tag{5.2}
$$

If we transvect once again (5.2) by g^{jk}, we obtain

$$
R = K + (n-2)S_s^s + \frac{n(n+2)}{2}p_sp^s - 4p_sp^s,
$$

and, consequently

$$
p_sp^s = \frac{R-K}{(n-2)(n+2)}. \tag{5.3}
$$

We proved that there holds the following

Lemma 5.1. *The vector p is isotropic if and only if the scalar curvature of $F-$holomorphically-structurally semi-symmetric connection and the scalar curvature depending on the Levi-Civita connection are equal.*

Then, substituting (4.2) and (5.3) into (5.2), we obtain

$$
\begin{aligned}
R_{jk} &= K_{jk} + (n-2)p_kp_j + (n-2)q_kq_j + \frac{R-K}{2(n+2)}g_{kj} \\
&\quad +\frac{R-K}{2(n-2)}g_{kj} + 2q_kq_j - 2p_kp_j
\end{aligned}
$$

and, consequently,

$$
R_{jk} = K_{jk} + nq_kq_j + (n-4)p_kp_j + \frac{n}{(n-2)(n+2)}(R-K)g_{kj}.
$$

From here, we can easily get that there holds

$$q_k q_j = \frac{1}{n}[R_{jk} - K_{jk} - (n-4)p_k p_j] - \frac{R-K}{(n-2)(n+2)} g_{kj}.$$

From this equality, it is easy to get

$$q_k p_j = \frac{1}{n}[R_{ak}F_j^a - K_{ak}F_j^a - (n-4)p_k q_j] + \frac{R-K}{(n-2)(n+2)} F_{kj}$$

and

$$q_j p_k = \frac{1}{n}[R_{aj}F_k^a - K_{aj}F_k^a - (n-4)p_j q_k] - \frac{R-K}{(n-2)(n+2)} F_{kj}.$$

Now it is easy to calculate

$$\begin{aligned} q_k p_j - q_j p_k &= \frac{1}{n}[R_{ak}F_j^a - R_{aj}F_k^a - K_{ak}F_j^a + K_{aj}F_k^a] + \\ &+ \frac{n-4}{n}(q_k p_j - q_j p_k) + \frac{2(R-K)}{(n-2)(n+2)} F_{kj}. \end{aligned}$$

From here, we obtain

$$\begin{aligned} (1 - \frac{n-4}{n})(q_k p_j - q_j p_k) &= \frac{1}{n}[R_{ak}F_j^a - R_{aj}F_k^a - K_{ak}F_j^a + K_{aj}F_k^a] \\ &+ \frac{2(R-K)}{(n-2)(n+2)} F_{kj} \end{aligned}$$

and, finally

$$\begin{aligned} q_k p_j - q_j p_k &= \frac{1}{4}[R_{ak}F_j^a - R_{aj}F_k^a - K_{ak}F_j^a + K_{aj}F_k^a] \\ &+ \frac{n(R-K)}{2(n-2)(n+2)} F_{kj}. \end{aligned} \tag{5.4}$$

From (5.4), we obtain that there holds

$$\begin{aligned} q_k q_j - p_k p_j &= \frac{1}{4}[R_{jk} - K_{jk} - R_{ab}F_k^a F_j^b + K_{ab}F_k^a F_j^b] \\ &- \frac{n(R-K)}{2(n-2)(n+2} g_{kj}. \end{aligned} \tag{5.5}$$

If we substitute this into the expression (5.2), we obtain

$$S_{kj} = \frac{1}{n-2}[R_{jk} - K_{jk} - \frac{n+2}{2} p_s p^s g_{kj} - 2(q_k q_j - p_k p_j)].$$

If we use (5.3) and (5.5), there will hold

$$\begin{aligned} S_{kj} &= \frac{1}{n-2}(R_{jk} - K_{jk}) - \frac{n+2}{2(n-2)} \frac{R-K}{n-2} g_{kj} - \\ &- \frac{1}{2(n-2)}(R_{jk} - K_{jk} - R_{ab}F_k^a F_j^b + K_{ab}F_k^a F_j^b) \\ &+ \frac{n}{(n-2)^2(n+2)}(R-K) g_{kj}. \end{aligned}$$

Finally, we obtain the following two equations

$$S_{kj} = \frac{1}{2(n-2)}[R_{jk} - K_{jk} - R_{ab}F_k^a F_j^b + K_{ab}F_k^a F_j^b + \frac{R-K}{n+2}g_{kj}] = \frac{1}{2(n-2)}[R_{jk} - K_{jk} - R_{ab}F_k^a F_j^b + K_{ab}F_k^a F_j^b] + \frac{R-K}{2(n-2)(n+2)}g_{kj}; \tag{5.6}$$

$$S_{ka}F_j^a = \frac{1}{2(n-2)}[R_{ak}F_j^a - K_{ak}F_j^a - R_{aj}F_k^a + K_{aj}F_k^a] - \frac{R-K}{2(n-2)(n+2)}F_{kj}. \tag{5.7}$$

It is obvious that, although the curvature tensor of this connection is not invariant under changing places of the first and the second pair of indices, its Ricci tensor is still symmetric.

Now, we substitute the expressions (5.6), (5.7) and (5.5) into (5.1), and obtain

$$\begin{aligned} & R_{ijkl} + \frac{R}{(n-2)(n+2)}(g_{ki}g_{lj} - g_{kj}g_{li} + F_{kj}F_{li} - F_{ki}F_{lj} - nF_{ji}F_{kl}) \\ & + \frac{1}{2(n-2)}[(R_{ab}F_k^aF_j^b - R_{jk})g_{li} - (R_{ab}F_l^aF_j^b - R_{jl})g_{ki} \\ & + (R_{ab}F_l^aF_i^b - R_{il})g_{kj} - (R_{ab}F_k^aF_i^b - R_{ik})g_{lj} \\ & + (R_{aj}F_k^a - R_{ak}F_j^a)F_{li} - (R_{aj}F_l^a - R_{al}F_j^a)F_{ki} \\ & + (R_{ai}F_l^a - R_{al}F_i^a)F_{kj} - (R_{ai}F_k^a - R_{ak}F_i^a)F_{lj} \\ & + (n-2)(R_{ai}F_j^a - R_{aj}F_i^a)F_{kl}] \\ = \; & K_{ijkl} + \frac{K}{(n-2)(n+2)}(g_{ki}g_{lj} - g_{kj}g_{li} + F_{kj}F_{li} - F_{ki}F_{lj} - nF_{ji}F_{kl}) \\ & + \frac{1}{2(n-2)}[(K_{ab}F_k^aF_j^b - K_{jk})g_{li} - (K_{ab}F_l^aF_j^b - K_{jl})g_{ki} \\ & + (K_{ab}F_l^aF_i^b - K_{il})g_{kj} - (K_{ab}F_k^aF_i^b - K_{ik})g_{lj} \\ & + (K_{aj}F_k^a - K_{ak}F_j^a)F_{li} - (K_{aj}F_l^a - K_{al}F_j^a)F_{ki} \\ & + (K_{ai}F_l^a - K_{al}F_i^a)F_{kj} - (K_{ai}F_k^a - K_{ak}F_i^a)F_{lj} \\ & + (n-2)(K_{ai}F_j^a - K_{aj}F_i^a)F_{kl}]. \end{aligned} \tag{5.8}$$

So, we proved that there holds the following

Theorem 5.2. *If the generator (p_j) of a holomorphically-structurally semi-symmetric connection on a hyperbolic Kähler space is a gradient, and if its curvature tensor is* *skew-symmetric in the first two indices (i.e., $R_{ijkl} = -R_{jikl}$), then the tensor from the left-hand side of (5.8) does not depend on the generator, but only on the metric and on the structure tensors.*

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