Characteristic Jacobi operator on contact Riemannian 3-manifolds

Jong Taek Cho and Jun-ichi Inoguchi

Abstract. The Ricci tensor, φ-Ricci tensor and the characteristic Jacobi operator on contact Riemannian 3-manifolds are investigated.

Key words: contact manifolds; characteristic Jacobi operator; φ-Ricci tensor.

1 Introduction

In contact Riemannian geometry, the Jacobi operator $\ell$ along the Reeb vector field $\xi$ plays an important role. The class of contact Riemannian manifolds with $\ell = 0$ is particularly large. For instance, Bang [1] showed that the normal bundle of a Legendre submanifold in a Sasakian manifold admits a contact Riemannian structure with $\ell = 0$ (See [4, Theorem 9.16]). Contact Riemannian 3-manifolds with vanishing $\ell$ were studied by Gouli-Andreou [16] and Perrone [31] (and called $M_\ell$-manifolds).

Koufogiorgos and Tsichlias [25] showed that complete, simply connected, contact Riemannian 3-manifolds with vanishing $\ell$ and positive constant $|Q\xi|$ are Lie groups. Here $Q$ is the Ricci operator.

In this paper, we study model spaces for the class of 3-dimensional $M_\ell$-manifolds with constant $|Q\xi|$.

Ghosh and Sharma [15] introduced a new class of contact Riemannian manifolds. According to [15], a contact Riemannian manifold is said to be a Jacobi $(\kappa, \mu)$-contact space if it satisfies

$$\ell = -\kappa \varphi^2 + \mu h$$

for some constants $\kappa$ and $\mu$. This class includes both the class of contact $(\kappa, \mu)$-spaces and that of $K$-contact manifolds.

The only examples of Jacobi $(\kappa, \mu)$-contact spaces which are neither contact $(\kappa, \mu)$ nor $K$-contact given in [15] are $M_\ell$-manifolds, i.e., Jacobi $(0, 0)$-contact spaces.

In the final section of this paper, we provide new examples of Jacobi $(\kappa, \mu)$-contact 3-spaces which are neither contact $(\kappa, \mu)$-spaces nor $M_\ell$-manifolds.
2 Preliminaries

2.1
Let $M$ be a manifold and $\eta$ a 1-form on $M$. Then the exterior derivative $d\eta$ is defined by

$$2d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]), \quad X, Y \in \mathfrak{X}(M).$$

Here $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on $M$.

Now let $(M, g)$ be a Riemannian manifold with its Levi-Civita connection $\nabla$. Then the Riemannian curvature $R$ of $M$ is defined by

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}.$$ 

On a Riemannian manifold $(M, g)$, we define a curvaturelike tensor field $(X, Y, Z) \mapsto (X \wedge Y)Z$ on $M$ by

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y.$$ 

Note that a Riemannian manifold $(M, g)$ is of constant curvature $c$ if and only if its Riemannian curvature $R$ satisfies $R(X, Y) = c(X \wedge Y)$ for all $X, Y \in \mathfrak{X}(M)$.

2.2
Let $(M, g)$ be a Riemannian manifold. For a nonzero tangent vector $v \in T_p M$, the tidal force operator $F_v$ associated to $v$ is a linear endomorphism on $(\mathbb{R}v)^\perp$ defined by

$$F_v(w) := -R(w, v)v$$ for $w \perp v$ ([28, p. 219]). One can see that $F_v$ is self-adjoint on $(\mathbb{R}v)^\perp$ and has the trace $\text{tr} F_v = -\rho(v, v)$. Here $\rho$ is the Ricci tensor field of $(M, g)$.

For a geodesic $\gamma$ in $(M, g)$, a vector field $X$ along $\gamma$ is said to be a Jacobi field along $\gamma$ if it satisfies the Jacobi equation:

$$\nabla_{\gamma'} \nabla_X X = -F_{\gamma'}(X).$$

2.3
On a Riemannian 3-manifold $(M, g)$, the Riemannian curvature $R$ is described by the Ricci tensor field $\rho$ and corresponding Ricci operator $Q$ by

$$(2.1) \quad R(X, Y)Z = \rho(Y, Z)X - \rho(Z, X)Y + g(Y, Z)QX - g(Z, X)QY - \frac{s}{2}(X \wedge Y)Z$$

for all vector fields $X, Y$ and $Z$ on $M$. Here $s$ is the scalar curvature.

2.4
Let $G$ be a Lie group equipped with an invariant Riemannian metric $\langle \cdot, \cdot \rangle$. Then the Levi-Civita connection $\nabla$ of $(G, \langle \cdot, \cdot \rangle)$ is described by the Koszul formula:

$$2\langle \nabla_X Y, Z \rangle = -\langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle, \quad X, Y, Z \in \mathfrak{g}.$$ 

Here $\mathfrak{g}$ is the Lie algebra of $G$. Let us define a symmetric bilinear map $U : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$2\langle U(X, Y), Z \rangle = \langle X, [Z, Y] \rangle + \langle Y, [Z, X] \rangle.$$
and call it the natural-reducibility obstruction of \((G, \langle \cdot, \cdot \rangle)\). One can see that the metric \(g\) is right-invariant if and only if \(U = 0\).

A Lie group \(G\) is said to be unimodular if its left invariant Haar measure is right invariant. J. Milnor gave an infinitesimal reformulation of unimodularity for 3-dimensional Lie groups. We recall it briefly here.

Let \(g\) be a 3-dimensional oriented Lie algebra with an inner product \(\langle \cdot, \cdot \rangle\). Denote by \(\times\) the vector product operation of the oriented inner product space \((g, \langle \cdot, \cdot \rangle)\). The vector product operation is a skew-symmetric bilinear map \(\times : g \times g \rightarrow g\) which is uniquely determined by the following conditions:

(i) \(\langle X, X \times Y \rangle = \langle Y, X \times Y \rangle = 0\),

(ii) \(|X \times Y|^2 = \langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2\),

(iii) if \(X\) and \(Y\) are linearly independent, then \(\det(X, Y, X \times Y) > 0\),

for all \(X, Y \in g\). On the other hand, the Lie-bracket \([\cdot, \cdot] : g \times g \rightarrow g\) is a skew-symmetric bilinear map. Comparing these two operations, we get a linear endomorphism \(L_g\) which is uniquely determined by the formula

\[ [X, Y] = L_g(X \times Y), \quad X, Y \in g. \]

Now let \(G\) be an oriented 3-dimensional Lie group equipped with a left invariant Riemannian metric. Then the metric induces an inner product on the Lie algebra \(g\). With respect to the orientation on \(g\) induced from \(G\), the endomorphism field \(L_g\) is uniquely determined. The unimodularity of \(G\) is characterized as follows.

**Proposition 2.1. ([27])** Let \(G\) be an oriented 3-dimensional Lie group with a left invariant Riemannian metric. Then \(G\) is unimodular if and only if the endomorphism \(L_g\) is self-adjoint with respect to the metric.

### 3 Contact 3-manifolds

#### 3.1

Let \(M\) be a \((2n+1)\)-dimensional manifold. A contact form is a 1-form \(\eta\) which satisfies \((d\eta)^n \wedge \eta \neq 0\) on \(M\).

A plane field \(\mathcal{D} \subset TM\) with rank \(2n\) is said to be a contact structure on \(M\) if for any point \(p \in M\), there exists a contact form \(\eta\) defined on a neighbourhood \(U_p\) of \(p\) such that \(\text{Ker} \, \eta = \mathcal{D}\) on \(U_p\).

A \((2n+1)\)-manifold \(M\) together with a contact structure is called a contact manifold.

In this note we assume that there exists a globally defined contact form \(\eta\) which annihilates \(\mathcal{D}\), i.e., \(\text{Ker} \, \eta = \mathcal{D}\). Moreover we fix a contact form \(\eta\) on \(M\).

On a contact manifold \((M, \eta)\) with a fixed contact form \(\eta\), there exists a unique vector field \(\xi\) such that

\[ \eta(\xi) = 1, \quad d\eta(\xi, \cdot) = 0. \]
The vector field $\xi$ is called the \textit{Reeb vector field} of $(M, \eta)$. Note that $\xi$ is traditionally called the \textit{characteristic vector field} of $M$ in analytical mechanics. Moreover, $(M, \eta)$ admits a Riemannian metric $g$ and an endomorphism field $\varphi$ such that

\begin{align}
\varphi^2 &= -I + \eta \otimes \xi, \quad \varphi \xi = 0, \\
g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y),
\end{align}

and

\[ d\eta = \Phi, \]

where $\Phi$ is a 2-form

\[ \Phi(X, Y) = g(X, \varphi Y). \]

The structure $(\varphi, \xi, \eta, g)$ is called a \textit{contact Riemannian structure} of $M$ associated to the contact form $\eta$. A contact manifold $(M, \eta)$ together with its associated contact Riemannian structure is called a \textit{contact Riemannian manifold} and denoted by $(M, \varphi, \xi, \eta, g)$. Note that on a contact manifold $M$, the structure group $\text{GL}_{2n+1}\mathbb{R}$ of the linear frame bundle $L(M)$ is reducible to $\text{U}_n \times \{1\}$.

3.2

More generally, an \textit{almost contact Riemannian structure} of a $(2n+1)$-manifold $M$ is a quartet $(\varphi, \xi, \eta, g)$ of structure tensor fields which satisfies (3.1)--(3.2). A $(2n+1)$-manifold $M = (M, \varphi, \xi, \eta, g)$ equipped with an almost contact Riemannian structure is called an \textit{almost contact Riemannian manifold}.

\textbf{Definition 3.1.} Let $(M, \varphi, \xi, \eta, g)$ be an almost contact Riemannian manifold. A tangent plane $\Pi_x$ at $x \in M$ is said to be \textit{holomorphic} if it is invariant under $\varphi_x$.

It is easy to see that a tangent plane $\Pi_x$ is holomorphic if and only if $\xi_x$ is orthogonal to $\Pi_x$. The sectional curvature $H_x := K(\Pi_x)$ of a holomorphic plane $\Pi_x$ is called the \textit{holomorphic sectional curvature} of $M$ at $x$.

3.3

Now we recall an endomorphism field $h$ which is useful for the study of contact Riemannian manifolds.

\[ hX = \frac{1}{2}(\mathcal{L}_\xi \varphi)X = \frac{1}{2}[[\xi, \varphi X] - \varphi[X, X]]. \]

Let $M = (M, \varphi, \xi, \eta, g)$ be a contact Riemannian 3-manifold, then $M$ satisfies ([38]):

\[ (\nabla_X \varphi)Y = (\xi \wedge (1 + h)X)Y, \quad X, Y \in \mathfrak{X}(M). \]

The \textit{Webster curvature} $W$ of a contact Riemannian 3-manifold $M$ is defined by

\[ W = \frac{1}{8}(s - \rho(\xi, \xi) + 4). \]

Here $\rho$ is the Ricci tensor and $s$ is the scalar curvature of $M$, respectively. The \textit{torsion invariant} of $M$ introduced by Chern and Hamilton [7] is the square norm $|\tau|^2$ of $\tau = \mathcal{L}_\xi g$. The torsion invariant is computed as

\[ |\tau|^2 = -2\rho(\xi, \xi) + 4. \]
3.4

Contact Riemannian 3-manifolds with $h = 0$ has been paid much attention from differential geometers.

**Definition 3.2.** A contact Riemannian 3-manifold is said to be a *Sasakian 3-manifold* if $h = 0$.

**Proposition 3.1.** A contact Riemannian 3-manifold is Sasakian if and only if $\tau = 0$.

**Definition 3.3.** A complete Sasakian 3-manifold $M$ is said to be a *Sasakian space form* if it is of constant holomorphic sectional curvature.

3.5

On a $(2n + 1)$-dimensional contact Riemannian manifold $(M, \varphi, \xi, \eta, g)$, we define a self-adjoint endomorphism field $\ell$ by

$$\ell(X) = R(X, \xi)\xi, \quad X \in \mathfrak{X}(M).$$

One can see that $\ell = -F_\xi$ on the contact distribution. Moreover $\ell$ and $h$ satisfy the following relations;

$$h\xi = \ell(\xi) = 0, \quad \eta \circ h = 0, \quad \text{tr } h = \text{tr } (h\varphi) = 0, \quad h\varphi + \varphi h = 0,$$

$$\nabla \xi h = \varphi(I - \ell - h^2), \quad \text{tr } \ell = 2n - \text{tr } (h^2),$$

The self adjoint operator $\ell$ is called the *characteristic Jacobi operator* of $M$. In case $\dim M = 3$, we have the following equations:

$$\tau(X, Y) = 2g(\varphi X, hY), \quad (\nabla \xi \tau)(X, Y) = 2g(\varphi X, (\nabla \xi h)Y).$$

**Lemma 3.2.** (cf. [6], [19]) Let $M$ be a contact Riemannian 3-manifold. Then there exists a local orthonormal frame field $E = \{e_1, e_2, e_3\}$ such that:

$$he_1 = \lambda e_1, \quad e_2 = \varphi e_1, \quad e_3 = \xi, \quad \lambda \geq 0.$$ 

With respect to $E$, the Levi-Civita connection $\nabla$ is given by

$$\nabla_{e_1} e_1 = be_2, \quad \nabla_{e_1} e_2 = -be_1 + (1 + \lambda)e_3, \quad \nabla_{e_1} e_3 = -(1 + \lambda)e_2,$$

$$\nabla_{e_2} e_1 = -\alpha e_2 + (\lambda - 1)e_3, \quad \nabla_{e_2} e_2 = \alpha e_1, \quad \nabla_{e_2} e_3 = (1 - \lambda)e_1,$$

$$\nabla_{e_3} e_1 = \alpha e_2, \quad \nabla_{e_3} e_2 = -\alpha e_1, \quad \nabla_{e_3} e_3 = 0.$$ 

The Ricci operator $Q$ is given by

$$Qe_1 = \rho_{11} e_1 + \xi(\lambda)e_2 + (2\lambda - e_2(\lambda))\xi,$$

$$Qe_2 = \xi(\lambda)e_1 + \rho_{22} e_2 + (2\lambda - e_1(\lambda))\xi,$$

$$Q\xi = (2\lambda - e_2(\lambda))e_1 + (2\lambda - e_1(\lambda))e_2 + 2(1 - \lambda^2)\xi,$$

where

$$\rho_{11} = \frac{s}{2} + \lambda^2 - 2\alpha \lambda - 1, \quad \rho_{22} = \frac{s}{2} + \lambda^2 + 2\alpha \lambda - 1.$$
The covariant derivative $\nabla \phi$ of $\phi$ by $\xi$ is given by

$$\nabla \phi = -2\phi \nabla \phi + \nu h.$$

The function $\nu$ is given by $\nu = 0$ when $M$ is Sasakian and $\nu = (\lambda)(\lambda)$ when $M$ is non-Sasakian.

**Remark 3.4.** Perrone obtained the following result.

**Proposition 3.3.** ([31, Proposition 2.1]) On a contact Riemannian metric manifold $M$, the following four conditions are mutually equivalent.

- $\nabla \phi = 0$,
- $\nabla \tau = 0$,
- $\nabla \ell = 0$,
- $\ell \phi = \phi \ell$.

3.6

Let $M$ be an almost contact Riemannian manifold. We define a tensor field $\rho^*$ on $M$ by

$$\rho^*(X, Y) := \frac{1}{2} \text{trace} \ R(X, \phi Y) \phi.$$

One can see that $\rho^*(X, \xi) = 0$ for all $X \in \mathfrak{X}(M)$. Next we denote by $\rho^\phi$ the symmetric part of $\rho^*$, that is,

$$\rho^\phi(X, Y) = \frac{1}{2} \{\rho^*(X, Y) + \rho^*(Y, X)\}.$$

We call $\rho^\phi$ the $\phi$-Ricci tensor field of $M$ [11].

**Definition 3.5.** An almost contact Riemannian manifold $M$ is said to be a weakly $\phi$-Einstein manifold if

$$\rho^\phi(X, Y) = \lambda g^\phi(X, Y), \quad X, Y \in (M)$$

for some function $\lambda$. Here the symmetric tensor field $g^\phi$ is defined by

$$g^\phi(X, Y) = g(\phi X, \phi Y), \quad X, Y \in \mathfrak{X}(M).$$

When $\lambda$ is a constant, then $M$ is said to be a $\phi$-Einstein manifold. The function $s^\phi = \text{trace} \rho^\phi$ is called the $\phi$-scalar curvature of $M$.

**Remark 3.6.** An almost contact Riemannian manifold $M$ is said to be weakly $*$-Einstein if

$$\rho^*(X, Y) = \lambda g(X, Y), \quad X, Y \in \mathcal{D}$$

for some function $\lambda$. The function $s^\phi = \text{trace} \rho^\phi$ is called the $*$-scalar curvature of $M$. A weakly $*$-Einstein manifold of constant $*$-scalar curvature is called a $*$-Einstein manifold. Clearly $s^\phi = s^\phi$. 

4  $H$-contact manifolds

4.1

In this section we study Ricci tensor and $\varphi$-Ricci tensor of $H$-contact 3-manifolds.

First we recall the notion of $H$-contact manifold. Let $(M, g)$ be a Riemannian manifold with unit tangent sphere bundle $T_1 M$. We equip the Sasaki-lift metric on $T_1 M$. Denote by $\mathfrak{X}_1(M)$ the space of all smooth unit vector fields on $M$. A unit vector field $V \in \mathfrak{X}_1(M)$ is said to be harmonic if it is a critical point of the energy functional restricted to $\mathfrak{X}_1(M)$.

In particular, a contact Riemannian manifold $M$ is said to be an $H$-contact manifold if its Reeb vector field is harmonic in above sense.

The $H$-contact property is characterized in terms of Ricci operator as follows.

**Theorem 4.1** ([35]). A contact Riemannian 3-manifold $M$ is $H$-contact if and only if $\xi$ is an eigenvector field of $Q$, that is, $Q\xi = \sigma \xi$ for some function $\sigma$.

To characterize the class of $H$-contact manifolds, we here recall the following definition.

**Definition 4.1.** A contact Riemannian manifold $M$ is said to be a contact generalized $(\kappa, \mu, \nu)$-space if its Riemannian curvature $R$ satisfies

$$R(X, Y)\xi = (\kappa I + \mu h + \nu \varphi h)\{\eta(Y)X - \eta(X)Y\}$$

for all $X, Y \in \mathfrak{X}(M)$. Here $\kappa, \mu$ and $\nu$ are smooth functions. In particular, A contact generalized $(\kappa, \mu, 0)$-space is called a contact generalized $(\kappa, \mu)$-space.

**Definition 4.2.** Let $M$ be a contact generalized $(\kappa, \mu)$-space. If both the functions $\kappa$ and $\mu$ are constants, then $M$ is called a contact $(\kappa, \mu)$-space (see [18], [23], [24]). A generalized $(\kappa, \mu)$-space is said to be proper if $|db|^2 + |d\mu|^2 \neq 0$.

One can see that Sasakian manifolds are contact $(\kappa, \mu)$-spaces with $\kappa = 1$ and $h = 0$.

**Remark 4.3.** Contact generalized $(\kappa, \mu, \nu)$-spaces are of particular interest in dimension 3. In fact the following results are known.

**Theorem 4.2** ([22]). Let $M$ be a non-Sasakian contact generalized $(\kappa, \mu, \nu)$-space. If $\dim M > 3$, then $\kappa$ and $\mu$ are constants and $\nu = 0$.

**Corollary 4.3** ([23]). Let $M$ be a non-Sasakian contact generalized $(\kappa, \mu)$-space. If $\dim M > 3$, then $\kappa$ and $\mu$ are constants.

Perrone gave the following variational characterization of generalized $(\kappa, \mu)$-property.

**Theorem 4.4** ([34]). On a contact Riemannian 3-manifold $M$, its Reeb vector field $\xi$ is a harmonic map from $(M, g)$ into the unit tangent sphere bundle $T_1 M$ equipped with the Sasaki-lift metric of $g$ if and only if $M$ satisfies the generalized $(\kappa, \mu)$-condition on an open dense subset of $M$.

Koufogiorgos, Markellos and Papantoniou generalized Perrone’s theorem as follows.
Theorem 4.5 ([22]). Let $M$ be a contact Riemannian 3-manifold. If $M$ is a contact generalized $(\kappa, \mu, \nu)$-space then $M$ is an $H$-contact manifold, that is, its Reeb vector field $\xi$ is a critical point of the energy functional on the space of all unit vector fields on $M$. Conversely, if $M$ is $H$-contact, then $M$ satisfies the generalized $(\kappa, \mu, \nu)$-condition on an open dense subset of $M$.

Proposition 4.6 ([22]). Let $M$ be a non-Sasakian 3-dimensional contact generalized $(\kappa, \mu, \nu)$-space. Then there exits a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3 = \xi\}$ such that

\[ he_1 = \lambda e_1, \quad he_2 = -\lambda e_2, \quad e_2 = \varphi e_1, \]

where $\lambda = \sqrt{1 - \kappa} > 0$. The Ricci operator $Q$ is given by

\[ Q = \alpha I + \beta \eta \otimes \xi + \mu h + \nu \varphi h \]

with

\[ \alpha = \frac{1}{2} (s - 2\kappa) = \frac{s}{2} - (1 - \lambda^2) = g(\nabla_\xi e_1, e_2), \]
\[ \beta = \frac{1}{2} (6\kappa - s) = -\frac{s}{2} + 3(1 - \lambda^2), \]
\[ \mu = -2\alpha, \quad \nu = \frac{\xi(\lambda)}{\lambda}. \]

More specifically we have

\[ Qe_1 = \frac{1}{2} (s - 2\kappa + 2\mu \sqrt{1 - \kappa}) e_1 + \nu \sqrt{1 - \kappa} e_2, \]
\[ Qe_2 = \nu \sqrt{1 - \kappa} e_1 + \frac{1}{2} (s - 2\kappa - 2\mu \sqrt{1 - \kappa}) e_2, \]
\[ Qe_3 = 2\kappa e_3. \]

The covariant derivative $\nabla_\xi h$ of $h$ is given by $\nabla_\xi h = \mu \varphi h + \nu h$.

4.2

Now we study pseudo-symmetry of 3-dimensional contact generalized $(\kappa, \mu, \nu)$-spaces.

Let $M$ be a 3-dimensional contact generalized $(\kappa, \mu, \nu)$-spaces. Then the characteristic polynomial $\Psi(t) = \det(t \delta_{ij} - \rho_{ij})$ for $Q$ is given by

\[ \Psi(t) = (t - 2\kappa) F(t), \]

where

\[ F(t) = t^2 - (s - 2\kappa)t + \frac{1}{4} (s - 2\kappa)^2 - (1 - \kappa)(\mu^2 + \nu^2). \]

Now we investigate pseudo-symmetry of generalized $(\kappa, \mu, \nu)$-spaces.

- Case 1: $\rho_{33} = 2\kappa$ solves $F(t) = 0$: Direct computation shows that $F(2\kappa) = 0$ if and only if

\[ (1 - \kappa)(\mu^2 + \nu^2) = \left(3\kappa - \frac{s}{2}\right)^2 \]
In this case we have
\[ F(t) = t^2 - (s - 2\kappa)t + 2\kappa s - 8\kappa^2 = (t - 2\kappa)(t + 4\kappa - s). \]
Thus the principal Ricci curvatures are
\[ 2\kappa, 2\kappa, -4\kappa + s. \]

- Case 2: \( F(t) = 0 \) has double roots: The discriminant \( D \) of \( F(t) = 0 \) is
\[ D = 4(1 - \kappa)(\mu^2 + \nu^2). \]
Hence \( F(t) = 0 \) has double roots if and only if \( \mu = \nu = 0 \). Since \( \mu = -2\alpha \) and \( \nu = \xi(\lambda)/\lambda \), \( M \) is a generalized \((\kappa, 0)\)-space with constant \( \xi \)-sectional curvature.
Hence \( \lambda \) is constant and so is \( \kappa \). The principal Ricci curvatures are
\[ s/2 - \kappa, s/2 - \kappa, 2\kappa. \]

**Theorem 4.7.** Let \( M \) be a non-Sasakian 3-dimensional contact generalized \((\kappa, \mu, \nu)\)-space. Then \( M \) is pseudo-symmetric if and only if \( \mu = 0 \) or \( \mu^2 + \nu^2 = |(3\kappa - s/2)/\sqrt{1 - \kappa}|. \) In the former case, \( M \) is a contact \((\kappa, 0)\)-space.

### 4.3

Next let us consider Ricci \(*\)-tensor of contact generalized \((\kappa, \mu, \nu)\)-spaces.

Take a local orthonormal frame field \( \mathcal{E} = \{e_1, e_2, e_3\} \) as before. Then we have
\[ R(e_i, \varphi e_i)\xi = 0, \quad i = 1, 2, 3. \]
Thus, for any tangent vector field \( Y \), we have
\[ \rho^*(\xi, Y) = \frac{1}{2} \sum_{i=1}^2 g(R(\xi, \varphi Y)\varphi e_i, e_i) \]
\[ = g(R(\varphi e_1, e_1)\xi, \varphi Y) = 0. \]

**Proposition 4.8.** Let \( M \) be a 3-dimensional contact generalized \((\kappa, \mu, \nu)\)-space. Then \( \rho^*(\xi, \cdot) = 0. \) Hence \( M \) is weakly \(*\)-Einstein.

**Corollary 4.9.** Let \( M \) be a 3-dimensional contact generalized \((\kappa, \mu, \nu)\)-space. If its \(*\)-scalar curvature is constant, then \( M \) is \( \varphi \)-Einstein.

### 5 \( M_\ell \)-manifolds

#### 5.1

Let \( M \) be a contact Riemannian manifold. Then \( M \) is said to be an \( M_\ell \)-manifold if its characteristic Jacobi field \( \ell \) vanishes. We study pseudo-symmetry of 3-dimensional \( M_\ell \)-manifolds.
Assume that $M$ is a 3-dimensional non-Sasakian contact Riemannian manifold. Take a local orthonormal frame field $\mathcal{E} = (e_1, e_2, e_3)$ as in Lemma 3.2, then we have

$$
\ell(e_1) = -(\lambda^2 + 2\alpha \lambda - 1)e_1 + d\lambda(\xi)e_2,
$$

$$
\ell(e_2) = d\lambda(\xi)e_1 - (\lambda^2 - 2\alpha \lambda - 1)e_2.
$$

Thus $\ell = 0$ if and only if $a = 0$ and $\lambda^2 = 1$. In such a case, the Ricci operator $Q$ is given by

$$
Qe_1 = \frac{s}{2}e_1 + 2b\xi, \quad Qe_2 = \frac{s}{2}e_2 + 2c\xi, \quad Q\xi = 2be_1 + 2ce_2.
$$

**Proposition 5.1.** A non-Sasakian contact Riemannian 3-manifold is a $M\ell$-manifold if and only if $Q\xi = 0$ and $\rho_{11} = \rho_{22} = s/2$.

Now let $M$ be a 3-dimensional $M\ell$-manifold. Then the characteristic polynomial for $Q$ is

$$
\det(tI - Q) = (t - s/2)F(t), \quad F(t) = t^2 - \frac{s}{2}t^4(4b^2 + c^2).
$$

Thus Ricci eigenvalues are $\{\rho_1, \rho_+, \rho_-\},$

$$
\rho_1 = \rho_{11} = \rho_{22} = \frac{s}{2}, \quad \rho_\pm = \frac{s}{4} \pm \frac{1}{4}\sqrt{s^2 + 16|Q\xi|^2}.
$$

- $\rho_1$ solves $F(t)$ if and only if $Q\xi = 0,$ i.e., $b = c = 0.$ Thus we have Ricci eigenvalues $s/2, s/2$ and 0. On the other hand, since $Q\xi = 0$, we have $s = 0.$ Thus $Q = 0$ so $M$ is flat.

- $\rho_+ = \rho_- \ $if and only if $s = 0$ and $Q\xi = 0$, so $M$ is flat.

**Proposition 5.2.** A 3-dimensional $M\ell$ manifold is pseudo-symmetric if and only if it is flat.

### 5.2
Koufogiorgos and Tsichlias obtained the following result.

**Theorem 5.3 ([25]).** Let $M$ be a complete, simply connected 3-dimensional contact Riemannian manifold with $\ell = 0$. Assume that the norm $|Q\xi|$ is a constant, say $q$ on $M$.

- If $q = 0$ then $M$ is flat.

- If $q > 0$, then for each point $p \in M$, there exits a unique Lie group structure such that $p$ is the unit element, the orthonomal frame field $\{Q\xi/q, -\varphi Q\xi/q, \xi\}$ and the metric $g$ are left invariant with respect to it. Moreover $M$ has constant negative scalar curvature.

This theorem motives us to study homogeneous contact Riemannian 3-manifolds satisfying $\ell = 0$. 
6 Homogeneous contact Riemannian 3-manifolds

6.1
In this section we collect some fundamental facts on contact Riemannian 3-manifolds which have isometric actions of Lie groups which preserve contact structure.

Definition 6.1. A diffeomorphism \( f \) on a contact 3-manifold \((M, \eta)\) is said to be a contact transformation if \( f \) preserves the contact structure \( \mathcal{D} = \text{Ker} \eta \). In particular, a contact transformation \( f \) is said to be a strictly contact transformation if \( f \) preserves \( \eta \), i.e., \( f^* \eta = \eta \).

Definition 6.2. A contact Riemannian 3-manifold \( M = (M, \varphi, \xi, \eta, g) \) is said to be a homogeneous contact Riemannian 3-manifold if there exists a Lie group \( H \) of isometries which acts transitively on \( M \) such that every element of \( H \) is a strictly contact transformation.

Here we recall the following result due to Tanno [37].

Lemma 6.1. Let \( M \) be a contact Riemannian 3-manifold and \( f \) a diffeomorphism on \( M \). If \( f \) is \( \varphi \)-holomorphic, i.e., \( df \circ \varphi = \varphi \circ df \), then there exists a positive constant \( a \) such that \( f_* \xi = a \xi, \ f^* \eta = a \eta, \ f^* g = a g + a(a - 1) \eta \otimes \eta \).

This Lemma implies that every \( \varphi \)-holomorphic isometry is a strict contact transformation.

By virtue of a result of Sekigawa [36], Perrone obtained the following classification.

Theorem 6.2 ([32]). Let \( M \) be a simply connected homogeneous contact Riemannian 3-manifold, then \( M \) is a Lie group equipped with left invariant contact Riemannian structure.

7 Unimodular Lie groups

7.1
Let \( G \) be a 3-dimensional unimodular Lie group with a left invariant metric \( \langle \cdot, \cdot \rangle \). Then there exists an orthonormal basis \( \{e_1, e_2, e_3\} \) of the Lie algebra \( g \) such that

\[
\begin{align*}
[e_1, e_2] &= c_3 e_3, & e_2, e_3 &= c_1 e_1, & e_3, e_1 &= c_2 e_2, & c_i &\in \mathbb{R}.
\end{align*}
\]

Three-dimensional unimodular Lie groups are classified by Milnor as follows:

<table>
<thead>
<tr>
<th>Signature of ((c_1, c_2, c_3))</th>
<th>Simply connected Lie group</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>((+, +, +))</td>
<td>(SU(2))</td>
<td>compact and simple</td>
</tr>
<tr>
<td>((- , - , +))</td>
<td>(\mathbb{SL}_2 \mathbb{R})</td>
<td>non-compact and simple</td>
</tr>
<tr>
<td>((0, +, +))</td>
<td>(E(2))</td>
<td>solvable</td>
</tr>
<tr>
<td>((0, - , +))</td>
<td>(E(1, 1))</td>
<td>solvable</td>
</tr>
<tr>
<td>((0, 0, +))</td>
<td>Heisenberg group (\text{Nil}_3)</td>
<td>nilpotent</td>
</tr>
<tr>
<td>((0, 0, 0))</td>
<td>((\mathbb{R}^3, +))</td>
<td>Abelian</td>
</tr>
</tbody>
</table>
To describe the Levi-Civita connection $\nabla$ of $G$, we introduce the following constants:

$$\mu_1 = \frac{1}{2}(c_1 + c_2 + c_3) - c_1.$$ 

**Proposition 7.1.** The Levi-Civita connection is given by

$$\nabla_{e_1}e_1 = 0, \quad \nabla_{e_1}e_2 = \mu_1 e_3, \quad \nabla_{e_1}e_3 = -\mu_1 e_2$$
$$\nabla_{e_2}e_1 = -\mu_2 e_3, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_3 = \mu_2 e_1$$
$$\nabla_{e_3}e_1 = \mu_3 e_2, \quad \nabla_{e_3}e_2 = -\mu_3 e_1, \quad \nabla_{e_3}e_3 = 0.$$  

The Riemannian curvature $R$ is given by

$$R(e_1, e_2)e_1 = (\mu_1 \mu_2 - c_3 \mu_3) e_2, \quad R(e_1, e_2)e_2 = -(\mu_1 \mu_2 - c_3 \mu_3) e_1,$$
$$R(e_2, e_3)e_2 = (\mu_2 \mu_3 - c_1 \mu_1) e_3, \quad R(e_2, e_3)e_3 = -(\mu_2 \mu_3 - c_1 \mu_1) e_2,$$
$$R(e_1, e_3)e_1 = (\mu_3 \mu_1 - c_2 \mu_2) e_3, \quad R(e_1, e_3)e_3 = -(\mu_3 \mu_1 - c_2 \mu_2) e_1.$$ 

The basis $\{e_1, e_2, e_3\}$ diagonalises the Ricci tensor. The principal Ricci curvatures are given by

$$\rho_1 = 2\mu_2 \mu_3, \quad \rho_2 = 2\mu_3 \mu_1, \quad \rho_3 = 2\mu_1 \mu_2.$$ 

The natural-reducibility obstruction $U$ is given by

$$U(e_1, e_2) = \frac{1}{2}(-c_1 + c_2)e_3, \quad U(e_1, e_3) = \frac{1}{2}(c_1 - c_3)e_2, \quad U(e_2, e_3) = \frac{1}{2}(-c_2 + c_3)e_1.$$ 

**7.2**

According to a result due to Perrone, simply connected homogeneous contact Riemannian 3-manifolds are classified by the Webster scalar curvature $W$ and the torsion invariant $|\tau|^2$ as follows:

**Theorem 7.2.** Let $(M^3, \varphi, \xi, \eta, g)$ be a simply connected homogeneous contact Riemannian 3-manifold. Then $M$ is a Lie group $G$ together with a left invariant contact Riemannian structure $(\varphi, \xi, \eta, g)$. If $G$ is unimodular, then $G$ is one of the following:

1. the Heisenberg group $\text{Nil}_3$ if $W = |\tau| = 0$.
2. $\text{SU}(2)$ if $4\sqrt{2}W > |\tau|$.
3. $\tilde{E}(2)$ if $4\sqrt{2}W = |\tau| > 0$.
4. $\text{SL}_2\mathbb{R}$ if $|\tau| \neq 4\sqrt{2}W < |\tau|$.
5. $E(1,1)$ if $4\sqrt{2}W = -|\tau| < 0$.

The Lie algebra $\mathfrak{g}$ of $G$ is generated by an orthonormal basis $\{e_1, e_2, e_3\}$ as in (7.1) with $c_3 = 2$. The left invariant contact Riemannian structure is determined by

$$\xi = e_3, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi \xi = 0.$$
Proposition 7.3. The endomorphism field $h$, the Webster scalar curvature and the torsion invariant of a unimodular Lie group $G$ equipped with a left invariant homogeneous contact Riemannian structure are given by

$$he_1 = -\frac{1}{2}(c_1 - c_2)e_1, \quad he_2 = \frac{1}{2}(c_1 - c_2)e_2.$$  

$$W = \frac{1}{4}(c_1 + c_2), \quad |\tau|^2 = (c_1 - c_2)^2.$$  

The holomorphic sectional curvature of $G$ is

$$H = -3 + \frac{1}{4}(c_1 - c_2)^2 + c_1 + c_2.$$  

Corollary 7.4. If a unimodular Lie group $G$ is non-Sasakian, i.e., $c_1 \neq c_2$, then $G$ is a $(\kappa, \mu)$-space with

$$\kappa = 1 - \frac{1}{4}(c_1 - c_2)^2, \quad \mu = 2 - (c_1 + c_2).$$  

Proposition 7.5. The $\phi$-Ricci tensor field of a unimodular Lie group $G$ is given by

$$\rho_{11}^\phi = \rho_{22}^\phi = H, \quad \rho_{ij}^\phi = 0 \text{ for other } i, j.$$  

Hence $G$ is $\phi$-Einstein.

7.3

The characteristic Jacobi operator $\ell$ of a unimodular Lie group $G$ is computed as

$$\ell(e_1) = \ell_1 e_1, \quad \ell(e_2) = \ell_2 e_2,$$

where

$$\ell_1 = \frac{1}{4}(c_1 - c_2)(c_1 + 3c_2 - 4) + 1, \quad \ell_2 = -\frac{1}{4}(c_1 - c_2)(3c_1 + c_2 - 4) + 1.$$  

Hence $\ell = 0$ if and only if $c_1 + c_2 = 0$ and $c_1 - c_2 = \pm 2$.

$$\begin{cases} (c_1, c_2) = (0, 2) & \lambda = 1, \\ (c_1, c_2) = (2, 0) & \lambda = -1, \end{cases}$$

Thus the only possible Lie algebra is $\mathfrak{e}(2)$ and the metric is flat.

Remark 7.1. Let us denote by $\tilde{E}(2)$ the universal covering of $E(2)$ equipped with the flat associated metric. Then $\tilde{E}(2)$ is realised as $\mathbb{R}^3(x, y, z)$ with contact form $\eta = (dz - ydx)/2$ with

$$g = \begin{pmatrix} 1 + y^2 + z^2 & z & -y \\ z & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}.$$
This example can be generalized to arbitrary odd-dimension as follows (cf. [4, p. 121]):

\[
g = \frac{1}{4} \begin{pmatrix}
\delta_{ij} + y_i y_j + \delta_{ij} z & \delta_{ij} z & -y_i \\
\delta_{ij} z & \delta_{ij} & 0 \\
-y_j & 0 & 1
\end{pmatrix}, \quad \eta = \frac{1}{2} \left( dz = \sum_{i=1}^{n} y_i dx_i \right)
\]

on \( \mathbb{R}^{2n+1} \). The resulting contact Riemannian manifold satisfies \( \ell = 0 \) but not contact \((0,0)-space for \( n > 1 \). In particular this contact Riemannian manifold is non-flat for \( n > 1 \).

8 Non-unimodular Lie groups

8.1

Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \). Denote by \( \text{ad} \) the adjoint representation of \( \mathfrak{g} \),

\( \text{ad}: \mathfrak{g} \to \text{End}(\mathfrak{g}); \quad \text{ad}(X)Y = [X,Y]. \)

Then one can see that \( \text{tr} \text{ad} \);

\[ X \mapsto \text{tr} \text{ad}(X) \]

is a Lie algebra homomorphism into the commutative Lie algebra \( \mathbb{R} \). The kernel

\[ u = \{ X \in \mathfrak{g} \mid \text{tr} \text{ad}(X) = 0 \} \]

of \( \text{tr} \text{ad} \) is an ideal of \( \mathfrak{g} \) which contains the ideal \( [\mathfrak{g}, \mathfrak{g}] \).

Now we equip a left invariant Riemannian metric \( \langle , \rangle \) on \( G \). Denote by \( u \) the orthogonal complement of \( u \) in \( \mathfrak{g} \) with respect to \( \langle , \rangle \). Then the homomorphism theorem implies that \( \dim u^\perp = \dim \mathfrak{g}/u \leq 1 \).

The following criterion for unimodularity is known (see [27, p. 317]).

**Lemma 8.1.** A Lie group \( G \) with a left invariant metric is unimodular if and only if \( u = \mathfrak{g} \).

Based on this criterion, the ideal \( u \) is called the unimodular kernel of \( \mathfrak{g} \). In particular, for a 3-dimensional non-unimodular Lie group \( G \), its unimodular kernel \( u \) is commutative and of 2-dimension.

8.2

Now let us consider 3-dimensional non-unimodular Lie groups equipped with left invariant contact Riemannian structure. Here we recall Perrone's construction [32].

Let \( G \) be a 3-dimensional non-unimodular homogeneous contact Riemannian manifold. Then one can easily check that \( \xi \in u \). We take an orthonormal basis \( \{ e_2, e_3 = \xi \} \) of \( u \). Then \( e_1 = -\varphi e_2 \in u^\perp \) and hence \( \text{ad}(e_1) \) preserves \( u \). Express \( \text{ad}(e_1) \) as

\[ [e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3 \]

over \( u \). The compatibility condition \( d\xi = \Phi \) implies that \( \beta = 2 \). Next, \( \nabla_\xi \xi = 0 \) implies that \( \delta = 0 \). Moreover one can deduce that \( [e_2, e_3] = 0 \) from the Jacobi identity.
Remark 8.1. Milnor [27] chose the following orthonormal basis \( \{u_1, u_2, u_3\} \) for a non-unimodular Lie group \( G \) with left invariant Riemannian metric.

\[
u_1 \in u^\perp, \quad \langle \text{ad}(u_1)u_2, \text{ad}(u_1)u_3 \rangle = 0.
\]

This orthonormal basis \( \{u_1, u_2, u_3\} \) satisfies

\[
[u_1, u_2] = \alpha v_2 + \beta u_3, \quad [u_2, u_3] = 0, \quad [u_1, u_4] = \gamma u_2 + \delta u_3
\]

with \( \alpha + \delta \neq 0 \) and \( \alpha \gamma + \beta \delta = 0 \). Moreover \( \{u_1, u_2, u_3\} \) diagonalises the Ricci tensor. On the other hand, the basis \( \{e_1, e_2, e_3\} \) constructed for a non-unimodular homogeneous contact Riemannian 3-manifold \( G \) does not satisfy the orthogonality condition \( \langle \text{ad}(u_1)u_2, \text{ad}(u_1)u_3 \rangle = 0 \). In fact, \( \{e_1, e_2, e_3\} \) satisfies this orthogonality condition if and only if \( \gamma = 0 \).

Theorem 8.2 ([32]). Let \( G \) be a 3-dimensional non-unimodular Lie group equipped with a left invariant contact Riemannian structure. Then the Lie algebra \( g \) satisfies the commutation relations

\[
[e_1, e_2] = \alpha e_2 + 2e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = -\gamma e_2,
\]

with \( e_3 = \xi, e_1 = -\varphi e_2 \in u^\perp \) and \( \alpha \neq 0 \). The Webster scalar curvature and the torsion invariant satisfy the relation:

\[
4\sqrt{2}W < |\tau|.
\]

The Lie algebra \( g = g(\alpha, \gamma) \) is given explicitly by

\[
g(\alpha, \gamma) = \left\{ \begin{pmatrix} (1 + \alpha)x & \gamma x & y \\ 2x & x & z \\ 0 & 0 & x \end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\}
\]

with basis

\[
e_1 = \begin{pmatrix} (1 + \alpha) & \gamma & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The corresponding simply connected Lie group \( G(\alpha, \gamma) = \exp g(\alpha, \gamma) \) is realized as \( \mathbb{R}^3(x, y, z) \) with left invariant metric \( \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \eta \otimes \eta \), where

\[
\omega^1 = \frac{1}{2}dx, \quad \omega^2 = \frac{1}{2}(\alpha y + \gamma z)dx + dy, \quad \eta = dz + ydx.
\]

The Levi-Civita connection of \( G \) is given by the following table:

<table>
<thead>
<tr>
<th>( \nabla )</th>
<th>( e_1 )</th>
<th>( e_2 )</th>
<th>( e_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nabla_{e_1} e_1 )</td>
<td>0</td>
<td>(-\frac{1}{2}(\gamma + 2)e_3)</td>
<td>( \frac{1}{2}(\gamma + 2)e_1 )</td>
</tr>
<tr>
<td>( \nabla_{e_2} e_1 )</td>
<td>(-\alpha e_2 - \frac{1}{2}(\gamma + 2)e_3)</td>
<td>( \alpha e_1 )</td>
<td>( \frac{1}{2}(\gamma + 2)e_1 )</td>
</tr>
<tr>
<td>( \nabla_{e_3} e_1 )</td>
<td>(-\frac{1}{2}(\gamma + 2)e_2)</td>
<td>( \frac{1}{2}(\gamma + 2)e_1 )</td>
<td>0</td>
</tr>
<tr>
<td>( \nabla_{e_1} e_2 )</td>
<td>( -\frac{1}{2}(\gamma - 2)e_3)</td>
<td>( \frac{1}{2}(\gamma - 2)e_2 )</td>
<td>0</td>
</tr>
<tr>
<td>( \nabla_{e_2} e_2 )</td>
<td>( \alpha e_1 )</td>
<td>( \alpha e_1 )</td>
<td>0</td>
</tr>
<tr>
<td>( \nabla_{e_3} e_2 )</td>
<td>( \frac{1}{2}(\gamma + 2)e_1 )</td>
<td>( \frac{1}{2}(\gamma + 2)e_1 )</td>
<td>0</td>
</tr>
</tbody>
</table>
The endomorphism field $h$ is given by

$$he_1 = -\frac{1}{2}\gamma e_1, \quad he_2 = \frac{1}{2}\gamma e_2.$$  

The Riemannian curvature $R$ is given by

$$R(e_1, e_2)e_1 = -\left\{ \frac{1}{4}(\gamma^2 - 4\gamma - 12) - \alpha^2 \right\} e_2 + \alpha\gamma e_3,$$

$$R(e_1, e_2)e_2 = \left\{ \frac{1}{4}(\gamma^2 - 4\gamma - 12) - \alpha^2 \right\} e_1,$$

$$R(e_1, e_3)e_1 = \alpha\gamma e_2 + \frac{1}{4}(3\gamma^2 + 4\gamma - 4)e_3,$$

$$R(e_1, e_3)e_3 = -\frac{1}{4}(3\gamma^2 + 4\gamma - 4)e_1,$$

$$R(e_2, e_3)e_2 = -\frac{1}{4}(\gamma + 2)^2 e_3,$$

$$R(e_2, e_3)e_3 = \frac{1}{4}(\gamma + 2)^2 e_2,$$

$$R(e_1, e_2)e_3 = -\alpha\gamma e_1.$$  

$$H = K_{12} = \frac{1}{4}(\gamma^2 - 4\gamma - 12) - \alpha^2, \quad K_{13} = -\frac{1}{4}(3\gamma^2 + 4\gamma - 4), \quad K_{23} = \frac{1}{4}(\gamma + 2)^2.$$  

The Ricci curvatures are given by

$$\rho_{11} = -\alpha^2 - 2 - 2\gamma - \frac{\gamma^2}{2}, \quad \rho_{22} = -\alpha^2 - 2 + \frac{\gamma^2}{2}, \quad \rho_{33} = 2 - \frac{\gamma^2}{2}, \quad \rho_{23} = -\alpha\gamma.$$  

The natural-reducibility obstruction $U$ is given by

$$U(e_1, e_2) = -\frac{1}{2}(\alpha e_2 + \gamma e_3), \quad U(e_1, e_3) = -e_2,$$

$$U(e_2, e_2) = \alpha e_1, \quad U(e_2, e_3) = \frac{1}{2}(\gamma + 2)e_1.$$  

The Lie algebra $\mathfrak{g}$ is classified by the Milnor’s invariant $\mathcal{D} = -8\gamma/\alpha^2$.  

By using this table, the $\varphi$-Ricci tensor field is computed as

$$\rho_{11}^{\varphi} = \rho_{22}^{\varphi} = H = \frac{1}{4}(\gamma^2 - 4\gamma - 12) - \alpha^2, \quad \rho_{33}^{\varphi} = 0, \quad \rho_{23}^{\varphi} = -\frac{1}{2}\alpha\gamma.$$  

Hence $G$ is $\varphi$-Einstein. In particular, $G$ is $\varphi$-Einstein if and only if $\gamma = 0$. As we saw in [32], $G$ satisfies $\gamma = 0$ if and only if it is isometric to a Sasaki space form $\tilde{\text{SL}}_2\mathbb{R}$ of constant holomorphic sectional curvature $-3 - \alpha^2 < -3$. Note that $G$ with $\gamma = 0$ is not isomorphic to $\tilde{\text{SL}}_2\mathbb{R}$ as a Lie group.

$$\ell(e_1) = -\frac{1}{4}(\gamma + 2)(3\gamma - 2)e_1, \quad \ell(e_2) = \frac{1}{4}(\gamma + 2)^2 e_2.$$  

(8.1)
Thus $\ell = 0$ if and only if $\gamma = -2$. In this case,

$$\lambda_1 = 1, \quad \lambda_2 = -1.$$ 

The scalar curvature is $s = -2\alpha^2 < 0$. $|Q\xi| = \beta |\alpha| > 0$. This $G(\alpha, -2)$ has constant $\xi$-sectional curvature 0 and constant holomorphic sectional curvature $H = -\alpha^2$. This space is neither pseudo-symmetric and nor $\varphi$-Einstein.

**Proposition 8.4.** Let $G = G(\alpha, \gamma)$ be a simply connected non-unimodular Lie group corresponding to $g(\alpha, \gamma)$ equipped with a left invariant contact Riemannian structure. Then $G(\alpha, \gamma)$ is an $M_\ell$-manifold if and only if $\gamma = -2$.

**Remark 8.2.** In our previous works [9], [10], [13], we studied pseudo-symmetry of contact 3-manifolds. In particular it is shown that a non-unimodular Lie group $G$ is pseudo-symmetric if and only if $\gamma = 0$ [13]. In [20], 3-dimensional pseudo-symmetric Lie groups are investigated. In [8], 3-dimensional pseudo-symmetric real hypersurfaces in complex space forms are investigated.

The Lie group $G(\alpha, 0)$ is characterized as follows.

**Proposition 8.5.** Let $G = G(\alpha, \gamma)$ be a simply connected non-unimodular Lie group corresponding to $g(\alpha, \gamma)$ equipped with a left invariant contact Riemannian structure. Then the following three conditions are mutually equivalent:

- $G$ satisfies $\gamma = 0$.
- $G$ is Sasakian. In this case, $G$ is a Sasakian space form of constant holomorphic sectional curvature $-3 - \alpha^2 < -3$.
- $G$ is pseudo-symmetric, that is, at least two of principal Ricci curvatures coincide.
- $G$ is $\varphi$-Einstein.

According to [5], a contact Riemannian 3-manifold is said to be

- **strongly locally $\varphi$-symmetric** if its characteristic reflections are isometric.
- **locally $\varphi$-symmetric** if

$$g((\nabla_X R)(Y, Z)V, W) = 0$$

for all $X, Y, Z, V, W \perp \xi$.

One can see that every strongly locally $\varphi$-symmetric contact Riemannian 3-manifold is locally $\varphi$-symmetric. Strong local $\varphi$-symmetry under local homogeneity is characterized as follows:

**Proposition 8.6** ([5]). A locally homogeneous contact Riemannian 3-manifold is strongly locally $\varphi$-symmetric if and only if its Reeb vector field is an eigenvector field of the Ricci operator.

**Corollary 8.7.** Let $M$ be a locally homogeneous contact Riemannian 3-manifold. Then the following conditions are mutually equivalent:
• \( \xi \) is an eigenvector field of \( Q \).
• \( M \) is a generalized \((\kappa, \mu, \nu)\)-space.
• \( M \) is strongly locally \( \varphi \)-symmetric.

Every 3-dimensional unimodular Lie group equipped with left invariant contact Riemannian structure is strongly locally \( \varphi \)-symmetric. On the other hand, non-unimodular Lie groups, the following result was obtained.

**Theorem 8.8** (\cite{5}). Let \( G(\alpha, \gamma) \) be a 3-dimensional non-unimodular Lie group equipped with a left invariant contact Riemannian structure as before, then

- if \( \gamma < 0 \), none of \( G(\alpha, \gamma) \) is locally \( \varphi \)-symmetric,
- if \( \gamma = 0 \), then \( G(\alpha, \gamma) \) is a Sasakian locally \( \varphi \)-symmetric,
- if \( \gamma > 0 \), then none of \( G(\alpha, \gamma) \) is strongly \( \varphi \)-symmetric, but there exists one which is locally \( \varphi \)-symmetric. The Lie group \( G(\alpha, \gamma) \) is locally \( \varphi \)-symmetric only when \( \gamma = 2 \).

### 8.3

Ghosh and Sharma \cite{15} introduced the notion of Jacobi \((\kappa, \mu)\)-contact space as a generalization of a contact \((\kappa, \mu)\)-space and \( K \)-contact manifold.

**Definition 8.3.** A contact Riemannian manifold \((M, \varphi, \xi, \eta, g)\) is said to be a Jacobi \((\kappa, \mu)\)-contact space if \( M \) satisfies

\[
\ell = -\kappa \varphi^2 + \mu h
\]

for real constants \( \kappa \) and \( \mu \).

One can see that every contact \((\kappa, \mu)\)-space is a Jacobi \((\kappa, \mu)\)-contact space. Indeed in the \((\kappa, \mu, \nu)\)-contact condition (4.1) with constant \( \kappa, \mu \) and \( \nu = 0 \), substitution \( Y = \xi \) implies (8.2).

Here we give new proper examples of Jacobi \((\kappa, \mu)\)-contact 3-spaces.

**Proposition 8.9.** Every 3-dimensional non-unimodular Lie group with left invariant contact Riemannian structure is a Jacobi \((\kappa, \mu)\)-contact space. Except the Sasakian case \( G(\alpha, 0) \), \( G(\alpha, \gamma) \) is not a contact \((\kappa, \mu)\)-space.

**Proof.** Let \( G = G(\alpha, \gamma) \) be a non-unimodular Lie group equipped with left invariant contact Riemannian structure as before. Then the characteristic Jacobi operator \( \ell \) is given by (8.1). On the other hand, for any constants \( \kappa \) and \( \mu \), the operator \(-\kappa \varphi^2 + \mu h\) is computed as

\[
( -\kappa \varphi^2 + \mu h ) e_1 = (\kappa - \gamma \mu / 2) e_1, \quad ( -\kappa \varphi^2 + \mu h ) e_2 = (\kappa + \gamma \mu / 2) e_2.
\]

First we assume that \( \gamma \neq 0 \). In this case, Comparing (8.3) with (8.1), we obtain that \( G(\alpha, \gamma) \) is Jacobi \((\kappa, \mu)\)-contact space with

\[
\kappa = -\frac{1}{4} (\gamma^2 - 4), \quad \mu = \gamma + 2.
\]

Next suppose that \( \gamma = 0 \). In this case \( G(\alpha, 0) \) is Sasakian and hence it is Jacobi \((\kappa, \mu)\)-contact space.

\[\square\]
In addition here we give a non-homogeneous example of Jacobi \((\kappa, \mu)\)-contact space.

**Example 8.4.** In [33], Perrone gave the following example of 3-dimensional weakly \(\varphi\)-symmetric space which is neither homogeneous nor strongly \(\varphi\)-symmetric. Let \(M\) be the open submanifold \(\{(x, y, z) \in \mathbb{R}^3 \mid x \neq 0\}\) of Cartesian 3-space \(\mathbb{R}^3\) together with a contact form \(\eta = xydx + dz\). The Reeb vector field of this contact 3-manifold is \(\xi = \partial / \partial z\). Take a global frame field

\[
e_1 = -\frac{2}{x} \frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial x} - \frac{4z}{x} \frac{\partial}{\partial y} - xy \frac{\partial}{\partial z}, \quad e_3 = \xi
\]

and define a Riemannian metric \(g\) by the condition \(\{e_1, e_2, e_3\}\) is orthonormal with respect to it. Moreover, define an endomorphism field \(\varphi\) by \(\varphi e_1 = e_2, \varphi e_2 = -e_1\) and \(\varphi \xi = 0\). Then \((\varphi, \xi, \eta, g)\) is the associated almost contact metric structure of \((M, \eta)\). The endomorphism field \(h\) satisfies \(he_1 = e_1, he_2 = -e_2\). Hence \(M\) is non-Sasakian.

Perrone showed that this contact Riemannian 3-manifold is non-homogeneous. We obtain

\[\ell = 4h.\]

Namely, \((M, \varphi, \xi, \eta, g)\) is a Jacobi \((\kappa, \mu)\)-contact manifold with \(\kappa = 0\) and \(\mu = -4\).

**Proposition 8.10.** The example due to Perrone satisfies

- Jacobi \((\kappa, \mu)\)-contact manifold, but non contact \((\kappa, \mu)\)-space,
- neither pseudo-symmetric, nor \(H\)-contact,
- weakly \(\varphi\)-symmetric, but not strongly \(\varphi\)-symmetric,
- non-homogeneous.

**Remark 8.5.** In a separate publication [12], contact Riemannian 3-manifolds whose characteristic Jacobi operator is invariant under the Reeb flows are investigated.

### 8.4

Here we introduce the following notion which is a generalization of Jacobi \((\kappa, \mu)\)-contact condition:

**Definition 8.6.** A contact Riemannian manifold \(M\) is said to be a generalized Jacobi \((\kappa, \mu)\)-contact space if

\[\ell = -\kappa \varphi^2 + \mu h\]

for some functions \(\kappa\) and \(\mu\).

We would like to point out the existence of generalized Jacobi \((\kappa, \mu)\)-contact spaces. To this end we recall critical metric conditions for contact 3-manifolds.

For a contact 3-manifold \((M, \eta)\), we denote by \(\mathcal{M}(\eta)\) the space of all Riemannian metrics associated to \(\eta\). Since the volume element \(dv_\eta\) of an associated metric is \(d\eta \wedge \eta / 2\) (see eg. [4]), all the metrics in \(\mathcal{M}(\eta)\) has same volume element.
Theorem 8.11 ([30]). Let \((M, \varphi, \xi, \eta, g)\) be a compact contact Riemannian 3-manifold. Then the metric \(g\) is a critical point of the total scalar curvature functional on the space of all associated metric to \(\eta\) if and only if \(\nabla_\xi h = 0\).

On the other hand, Chern and Hamilton studied the following functional on \(M(\eta)\).

\[
E_{\text{CH}}(g) = \int_M \frac{1}{2} |\tau|^2 \, dv_g.
\]

The functional \(E_{\text{CH}}\) is called the Chern-Hamilton energy.

Theorem 8.12 ([7]). An associated metric \(g\) of a compact contact 3-manifold \((M, \eta)\) is a critical point of the Chern-Hamilton energy if and only if \(\nabla_\xi h = 2h\varphi\).

It is known that every contact Riemannian manifold satisfies (cf. [4, p. 112]):

\[
\ell = -h^2 - \varphi^2 + \varphi(\nabla_\xi h).
\]

Now assume that \(M\) is non-Sasakian contact Riemannian 3-manifold and denote by \(\lambda\) the positive eigenvalue of \(h\). Then one can see that

\[
\ell = (\lambda^2 - 1)\varphi^2 + \varphi(\nabla_\xi h).
\]

From this formula we obtain the following fact.

Proposition 8.13. Let \(M\) be a contact Riemannian 3-manifold. If \(M\) satisfies \(\nabla_\xi h = 0\), then the characteristic Jacobi operator satisfies

\[
\ell = -\kappa \varphi^2,
\]

where \(\kappa\) is a smooth function. In case \(\kappa = 1\), \(M\) is Sasakian.

On contact Riemannian 3-manifolds satisfying \(Q\varphi = \varphi Q\), the equation \(\nabla_\xi h = 0\) holds. The following example is due to Blair [3].

Example 8.7. Let \(f(x, y)\) be a smooth non-constant function on \(\mathbb{R}^3(x, y, z)\) bounded below by a positive constant \(c\). Take a contact form \(\eta = (dz - ydx)/2\) and consider a compatible metric

\[
g = \frac{1}{4} \begin{pmatrix}
(e^{zf} + 1 + f^2)e^{-zf} - 2f^2 + y^2 & (e^{zf} - 1)/f & -y \\
(e^{zf} - 1)/f & e^{zf} & 0 \\
y & 0 & 1
\end{pmatrix}.
\]

We equip an almost contact structure associated to \((\eta, g)\) (for more detail, see [3] or [4, p. 226]). Then the resulting contact Riemannian 3-manifold satisfies \(\nabla_\xi h = 0\) and \(Q\varphi \neq \varphi Q\). The characteristic Jacobi operator satisfies \(\ell = -\kappa \varphi^2\) with \(\kappa = f(x, y)^2\).

Now let \(M\) be a contact Riemannian 3-manifold satisfying \(\nabla_\xi h = 2h\varphi\). Then we have

\[
\varphi \nabla_\xi h = 2h.
\]

Hence the characteristic Jacobi operator satisfies

\[
\ell = (\lambda^2 - 1)\varphi^2 + 2h.
\]
Proposition 8.14. Let $M$ be a contact Riemannian 3-manifold satisfying $\nabla_\xi h = 2h\phi$. Then $M$ is a generalized Jacobi $(\kappa, 2)$-contact space.

Let $M^2(c)$ be a Riemannian 2-manifold of constant curvature $c = \pm 1$, then its unit tangent sphere bundle $M := T_1M^2(c)$ equipped with standard contact Riemannian structure satisfies $\nabla_\xi h = 2h\phi$ and constant eigenvalue $\lambda = 0$ for $c = 1$ and $\lambda = 2$ for $c = -1$ (see [4, p. 209]). In case $c = 1$, as is well known, $M$ is Sasakian. In case $c = -1$, $M$ is non-Sasakian and Jacobi $(\kappa, \mu)$-contact space with $\kappa = -3$ and $\mu = 2$.

More strongly, $M$ is a contact $(-3, 2)$-space.

We conclude this paper with the following problem.

Problem 8.8. Classify all 3-dimensional Jacobi $(\kappa, \mu)$-contact spaces.

Acknowledgments. The second named author was partially supported by Kakenhi 2454063, 15K04834.

References


[23] Th. Koufogiorgos and Ch. Tsichlias, *Generalized $(\kappa, \mu)$-contact metric manifolds with $|\text{grad} \kappa| = \text{constant}$*, J. Geom. 78 (2003), 83-91.


Author’s address:

Jong Taek Cho  
Department of Mathematics, Chonnam National University,  
CNU, The Institute of Basic Science, Kwangju, 500–757, Korea.  
E-mail: jtcho@chonnam.ac.kr

Jun-ichi Inoguchi  
Department of Mathematical Sciences, Faculty of Science,  
Yamagata University, Yamagata, 990-8560, Japan.  
E-mail: inoguchi@sci.kj.yamagata-u.ac.jp  
*Current address*: Institute of Mathematics,  
University of Tsukuba, Tsukuba 305-8571, Japan.  
E-mail: inoguchi@math.tsukuba.ac.jp