Study on special type of a weakly symmetric Kähler manifold

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Abstract. We establish several results related to weakly symmetric Kähler manifolds satisfying the condition of having a special type of semi-symmetric metric connection.

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1 Introduction

An n-dimensional Riemannian manifold is said to be weakly symmetric if the following condition is satisfied:

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) =& A(X)R(Y,Z,U,V)\\ &+ B(Y)R(X,Z,U,V) + C(Z)R(Y,X,U,V)\\ &+ D(U)R(Y,Z,X,V) + E(V)R(Y,Z,U,X),\end{aligned} \tag{1.1}$$

and if the Ricci tensor S of the manifold satisfies

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + C(Z)S(Y,X), \tag{1.2}$$

where A, B, C, D, E are simultaneously non-vanishing 1-forms and X, Y, Z, U, V are vector fields, then the manifold is called a *weakly Ricci symmetric manifold.* The notions of weakly symmetric and weakly Ricci symmetric manifolds were introduced by L. Tamassy and T. Q. Binh [1, 7]. Chaki and Koley [2] in 1994 replaced A by $2A$ in (1.1), and called them *generalized pseudo symmetric manifolds.* In 1999, U. C. De and J. Sengupta [5] studied weakly symmetric Riemannian manifolds admitting a special type of semi-symmetric metric connection. L. Tamassy, U. C. De and T. Q. Binh [8] found interesting results on weakly symmetric Kähler manifolds and weakly Ricci symmetric Kähler manifolds in 2000. Also, U. C. De and B. K. De [4] discussed conformally flat generalized pseudo Ricci symmetric manifolds. Then, in 2008, B. B. Chaturvedi and P. N. Pandey [3] considered semi-symmetric non-metric connections and studied the Kähler manifolds.

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In 1995 M. Prvanovic [6] proved that if the manifold M is a weakly symmetric manifold satisfying (1.1), then $B = C = D = E$. In this paper we consider $B = C = D = E = \omega$ and then (1.1) and (1.2) become

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) =& A(X)R(Y,Z,U,V)\\ &+ \omega(Y)R(X,Z,U,V) + \omega(Z)R(Y,X,U,V)\\ &+ \omega(U)R(Y,Z,X,V) + \omega(V)R(Y,Z,U,X),\end{aligned} \tag{1.3}$$

and respectively

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + \omega(Y)S(X,Z) + \omega(Z)S(Y,X),$$

where $g(X,\rho) = \omega(X)$, and $g(X,\alpha) = A(X)$.

Let further ∇ be the Levi-Civita connection on M. Then the connection defined by

$$\nabla^*_X Y = \nabla_X Y + \omega(Y)X - g(X,Y)\rho, \tag{1.4}$$

is called a *special type of semi-symmetric metric connection* if the torsion tensor T and the curvature tensor R^* of the connection ∇^* satisfy the following conditions:

$$T(X,Y) = \omega(Y)X - \omega(X)Y, \quad (\nabla^*_X T)(Y,Z) = \omega(X)T(Y,Z),$$

and

$$R^*(X,Y)Z = 0. \tag{1.5}$$

Yano [9] proved in 1970 that the curvature tensor R^* of the semi-symmetric metric connection ∇^* defined by (1.4) is given by

$$\begin{aligned}R^*(X,Y,Z,U) =& R(X,Y,Z,U) - \theta(Y,Z)g(X,U) + \theta(X,Z)g(Y,U)\\ &- \theta(X,U)g(Y,Z) + \theta(Y,U)g(X,Z),\end{aligned} \tag{1.6}$$

where θ is a tensor field of type (0,2) defined by

$$\theta(X,Y) = (\nabla_X \omega)Y - \omega(X)\omega(Y) + \frac{1}{2}\omega(\rho)g(X,Y).$$

It was also proved by U. C. De and J. Sengupta [5] that the equation (1.6) has the form

$$\begin{aligned}R^*(X,Y,Z,U) =& R(X,Y,Z,U) + \omega(X)[\omega(Z)g(Y,U) - \omega(U)g(Y,Z)]\\ &+ \omega(Y)[\omega(U)g(X,Z) - \omega(Z)g(X,U)]\\ &+ \omega(\rho)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].\end{aligned} \tag{1.7}$$

Using (1.5) in (1.7), we infer

$$\begin{aligned}R(X,Y,Z,U) =& \omega(X)[\omega(U)g(Y,Z) - \omega(Z)g(Y,U)]\\ &+ \omega(Y)[\omega(Z)g(X,U) - \omega(U)g(X,Z)]\\ &+ \omega(\rho)[g(X,Z)g(Y,U) - g(Y,Z)g(X,U)].\end{aligned} \tag{1.8}$$

Putting $X = U = e_i, 1 \leq i \leq n$ in (1.8) and taking summation over i, we get

$$S(Y,Z) = (n-2)[\omega(Y)\omega(Z) - \omega(\rho)g(Y,Z)]. \tag{1.9}$$

Again, replacing $Y = Z = e_i, 1 \leq i \leq n$ in (1.9) and taking summation over i, we get

$$r = -(n-1)(n-2)\omega(\rho). \tag{1.10}$$

Further, from (1.9), we have

$$S(Y,\rho) = 0, \tag{1.11}$$

and equation (1.8) infers $R(X,Y,Z,\rho) = 0$.

2 Weakly symmetric Kähler manifolds

It is well known that in an n (even)-dimensional Kähler manifold, the following relations hold:

$$F^2(X) = -X, \quad g(\bar{X},\bar{Y}) = g(X,Y), \qquad (\nabla_X F)(Y) = 0,$$

where F is a tensor field of type (1,1) such that $F(X) = \bar{X}$, g is a Riemannian metric and ∇ is the Levi-Civita connection.

If the manifold M is a weakly symmetric Kähler manifold, then we can readily write

$$R(\bar{Y},\bar{Z},U,V) = R(Y,Z,U,V). \tag{2.1}$$

Taking the covariant derivative of (2.1), we get

$$(\nabla_X R)(\bar{Y},\bar{Z},U,V) = (\nabla_X R)(Y,Z,U,V). \tag{2.2}$$

Using (1.3) in (2.2), we obtain

$$\begin{aligned}\omega(Y)R(X,Z,U,V) + \omega(Z)R(Y,X,U,V) =& \omega(\bar{Y})R(X,\bar{Z},U,V)\\ &+ \omega(\bar{Z})R(\bar{Y},X,U,V).\end{aligned} \tag{2.3}$$

By substituting $Z = U = e_i, 1 \leq i \leq n$ in (2.3) and summing over i, we get

$$\omega(Y)S(X,V) - R(Y,X,V,\rho) = \omega(\bar{Y})S(X,\bar{V}) + R(\bar{Y},X,V,\bar{\rho}). \tag{2.4}$$

and by replacing $X = V = e_i, 1 \leq i \leq n$ in (2.4) and summing over i, we infer

$$r\omega(Y) - S(Y,\rho) = S(Y,\rho). \tag{2.5}$$

Using (1.11) in (2.5), we have

$$r\omega(Y) = 0, \tag{2.6}$$

which implies either $r = 0$, or $\omega(Y) = 0$. Now if we take $\omega(Y) = 0$ then the connection ∇^* defined by (1.4) will change. Hence $\omega(Y)$ never vanishes. Hence we can state the following result:

Theorem 2.1. *If M is a weakly symmetric Kähler manifold equipped with a special type of semi-symmetric metric connection ∇^*, then M is a manifold of zero scalar curvature with respect to the Riemannian connection ∇.*

From (1.3), we can easily write

$$(\nabla_X R)(Y,Z)U = A(X)R(Y,Z)U + \omega(Y)R(X,Z)U + \omega(Z)R(Y,X)U + \omega(U)R(Y,Z)X + R(Y,Z,U,X)\rho. \tag{2.7}$$

Contracting this with respect to X, we get

$$(divR)(Y,Z)U = A(R(Y,Z)U) + \omega(Y)S(U,Z) - \omega(Z)S(U,Y) + R(Y,Z,U,\rho). \tag{2.8}$$

Using the second Bianchi identity, we infer

$$(divR)(Y,Z)U = (\nabla_Y S)(Z,U) - (\nabla_Z S)(Y,U) \tag{2.9}$$

and

$$(divL)\,Y = \frac{1}{2}Y(r), \tag{2.10}$$

whence we obtain

$$(\nabla_Y S)(Z,U) - (\nabla_Z S)(Y,U) = A(R(Y,Z)U) + \omega(Y)S(Z,U) - \omega(Z)S(Y,U) + R(Y,Z,U,\rho).$$

By replacing in this relation $Y = U = e_i, 1 \le i \le n$, we have

$$(divL)Z - Z(r) = -S(Z,\alpha) + S(Z,\rho) - r\omega(Z) - S(Z,\rho),$$

whence, by using (2.6), (2.10) and $Z(r) = 0$ as $r = 0$, we get $S(Z,\alpha) = 0$. The equation from above can also be written as

$$S(Z,\alpha) = 0.g(Z,\alpha), \tag{2.11}$$

which, by replacing α by $\bar{\alpha}$, leads to

$$S(Z,\bar{\alpha}) = 0.g(Z,\bar{\alpha}). \tag{2.12}$$

Hence, from equations (2.11) and (2.12), we conclude:

Theorem 2.2. *If M is a weakly symmetric Kähler manifold equipped with a special type of semi-symmetric metric connection ∇^* then α and $\bar{\alpha}$ are eigenvectors of the Ricci tensor S with respect to the zero eigenvalue.*

3 Conformally flat weakly symmetric Kähler manifolds

Putting $X = U = e_i, 1 \le i \le n$ in (1.7) and taking summation over i, we get

$$S^*(Y,Z) = S(Y,Z) + (n-2)[\omega(\rho)g(Y,Z) - \omega(Y)\omega(Z)]. \tag{3.1}$$

Again Putting $Y = Z = e_i, 1 \leq i \leq n$ in (3.1) and taking summation over i, we have

$$r^* = r + (n-1)(n-2)\omega(\rho) \tag{3.2}$$

Now the Weyl conformal curvature tensor C^* of connection ∇^* is given by

$$\begin{aligned} C^*(X,Y,Z,U) =& R^*(X,Y,Z,U) - \frac{1}{(n-2)}[S^*(Y,Z)g(X,U) \\ & - S^*(X,Z)g(Y,U) + S^*(X,U)g(Y,Z) \\ & - S^*(Y,U)g(X,Z)] \\ & + \frac{r^*}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{3.3}$$

Using (1.7), (3.1) and (3.2) in (3.3), we get

$$C^*(X,Y,Z,U) = C(X,Y,Z,U). \tag{3.4}$$

If the manifold is conformally flat with respect to ∇^* then the manifold will be conformally flat with respect to the connection ∇. *i.e.*

$$C^*(X,Y,Z,U) = 0 \implies C(X,Y,Z,U) = 0. \tag{3.5}$$

Now equation (3.5) implies

$$\begin{aligned} R(X,Y,Z,U) =& \frac{1}{(n-2)}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ & + S(X,U)g(Y,Z) - S(Y,U)g(X,Z)] \\ & - \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{3.6}$$

Putting the values of S and r from (1.9) and (1.10) in (3.6), we have

$$\begin{aligned} R(X,Y,Z,U) =& [g(X,U)\omega(Y)\omega(Z) - g(Y,U)\omega(X)\omega(Z) \\ & + g(Y,Z)\omega(X)\omega(U) - g(X,Z)\omega(Y)\omega(U)] \\ & + \omega(\rho)[g(X,Z)g(Y,U) - g(Y,Z)g(X,U)]. \end{aligned} \tag{3.7}$$

Hence from the above equation, we conclude:

Theorem 3.1. *If M be a conformally flat Riemannian manifold equipped with a special type of semi-symmetric metric connection ∇^* then it is the manifold of quasi-constant curvature.*

Equations (1.8) and (3.7) are identical, and therefore Theorems 2.1 and 2.2 can be stated also as follows:

Theorem 3.2. *If M be a weakly symmetric conformally flat kähler manifold equipped with a special type of semi-symmetric metric connection ∇^* then M is a manifold of zero scalar curvature with respect to ∇.*

Theorem 3.3. *If M be a weakly symmetric conformally flat kähler manifold equipped with a special type of semi-symmetric metric connection ∇^* then α and $\bar{\alpha}$ are eigen-vectors of the Ricci tensor S with respect to zero eigenvalue.*

4 Conclusions

In the present paper we discussed the eigenvalues and eigenvectors of the Ricci tensor S, and proved that in a weakly symmetric Kähler manifold equipped with a special type of semi-symmetric metric connection ∇^*, the manifold is of zero scalar curvature with respect to the Riemannian connection ∇. In such a manifold, the the zero eigenvalue provides eigenvectors α and $\bar{\alpha}$. Also, it was shown that a conformally flat Riemannian manifold equipped with a special type of semi-symmetric metric connection ∇^* is manifold of quasi-constant curvature.

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