Geometric structure for the tangent bundle of direct limit manifolds

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Abstract. We equip the direct limit of tangent bundles of paracompact finite dimensional manifolds with a structure of convenient vector bundle with structural group \( GL(\infty, \mathbb{R}) = \lim_{\rightarrow} GL(\mathbb{R}^n) \).

On munit la limite directe des fibrés tangents à des variétés paracompactes de dimensions finies d’une structure de fibré vectoriel ‘convenient’ (au sens de Kriegel et Michor) de groupe structural \( GL(\infty, \mathbb{R}) = \lim_{\rightarrow} GL(\mathbb{R}^n) \).

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1 Introduction

G. Galanis proved in [4] that the tangent bundle of a projective limit of Banach manifolds can be equipped with a Fréchet vector bundle structure with structural group a topological subgroup of the general linear group of the fiber type. Various problems were studied in this framework: connections, ordinary differential equations, ... ([1], [2], ...).

Here we consider the situation for direct (or inductive) limit of tangent bundles \( TM_i \), where \( M_i \) is a finite dimensional manifold: we first have (Proposition 4.1) that \( M = \lim_{\rightarrow} M_i \) can be endowed with a structure of convenient manifold modelled on the convenient vector space \( \mathbb{R}^\infty = \lim_{\rightarrow} \mathbb{R}^n \) of finite sequences, equipped with the finite topology (cf [7]). We then prove (Theorem 4.2) that \( TM \) can be endowed with a convenient structure of vector bundle whose structural group is \( GL(\infty, \mathbb{R}) = \lim_{\rightarrow} GL(\mathbb{R}^n) \) (the group of invertible matrices of countable size, differing from the identity matrix at only finitely many places, first described by Milnor in [10]). As an example we consider the tangent bundle to \( S^\infty \). Other examples can be found in the framework of manifolds for algebraic topology, such as Grassmannians ([8]) or Lie groups ([5], [6]).

The paper is organized as follows: We first recall the framework of convenient calculus (part 2). In part 3, we review direct limit in different categories. We obtain the main result (Theorem 4.2) in the last part.
2 Convenient calculus

Classical differential calculus is perfectly adapted to finite dimensional or even Banach manifolds (cf. [9]).

On the other hand, convenient analysis, developed in [8], provides a satisfactory solution of the question how to do analysis on a large class of locally convex spaces and in particular on strict inductive limits of Banach manifolds or fiber bundles.

In order to endow some locally convex vector spaces (l.c.v.s.) $E$, which will be assumed Hausdorff, with a differentiable structure we first use the notion of smooth curves $c : \mathbb{R} \to E$, which poses no problems.

We denote the space $C^\infty(\mathbb{R}, E)$ by $C^\infty$; the set of continuous linear functionals is denoted by $E'$.

We then have the following characterization: a subset $B$ of $E$ is bounded iff $l(B)$ is bounded for any $l \in E'$.

**Definition 2.1.** A sequence $(x_n)$ in $E$ is called Mackey-Cauchy if there exists a bounded absolutely convex set $B$ and for every $\varepsilon > 0$ an integer $n_\varepsilon \in \mathbb{N}$ s.t. $a_n - a_m \in \varepsilon B$ whenever $n > m > n_\varepsilon$.

**Definition 2.2.** A locally convex vector space is said to be $c^\infty$-complete or convenient if one of the following (equivalent) conditions is satisfied:

1. if $c : \mathbb{R} \to E$ is a curve such that $l \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all continuous linear functional $l$, then $c$ is smooth.

2. Any Mackey-Cauchy sequence converges\(^1\) (i.e. $E$ is Mackey complete)

3. For any $c \in C^\infty$ there exists $\gamma \in C^\infty$ such that $\gamma' = c$.

The $c^\infty$-topology on a l.c.v.s. is the final topology with respect to all smooth curves $\mathbb{R} \to E$; it is denoted by $c^\infty E$. Its open sets will be called $c^\infty$-open.

Note that the $c^\infty$-topology is finer than the original topology. For Fréchet spaces, this topology coincides with the given locally convex topology.

In general, $c^\infty E$ is not a topological vector space.

The following theorem gives some constructions inheriting of $c^\infty$-completeness.

**Theorem 2.1.** The following constructions preserve $c^\infty$-completeness: limits, direct sums, strict inductive limits of sequences of closed embeddings.

The category $\text{CON}$ of convenient vector spaces and continuous linear maps forms a symmetric monoidal closed category.

Let $E$ and $F$ be two convenient spaces and let $U \subset E$ be a $c^\infty$-open. A map $f : E \ni U \to F$ is said to be smooth if $f \circ c \in C^\infty(\mathbb{R}, F)$ for any $c \in C^\infty(\mathbb{R}, U)$. Moreover, the space $C^\infty(U, F)$ may be endowed with a structure of convenient vector space.

\(^1\)This condition is equivalent to:

For every absolutely convex closed bounded set $B$ the linear span $E_B$ of $B$ in $E$, equipped with the Minkowski functional $p_B(v) = \inf \{ \lambda > 0 : v \in \lambda B \}$, is complete.
Let \( L(E, F) \) be the space of all bounded linear mappings. We can define the differential operator

\[
d f(x) v = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}
\]

which is linear and bounded (and so smooth).

3 Direct (or inductive) limits

3.1 Direct limit in a category

The references are [3] and [6].

**Definition 3.1.** A direct sequence in a category \( \mathcal{A} \) is a pair \( S = (X_i, \varepsilon_{ij})_{(i,j) \in \mathbb{N}^2, i \leq j} \) where \( X_i \) is an object of \( \mathcal{A} \) and each \( \varepsilon_{ij} : X_i \to X_j \) is a morphism, called bonding map, such that:

- \( \varepsilon_{ii} = \text{Id}_{X_i} \)
- \( \varepsilon_{jk} \circ \varepsilon_{ij} = \varepsilon_{ik} \) if \( i \leq j \leq k \)

**Definition 3.2.** A cone over \( S \) is a pair \( (X, \varepsilon_i)_{i \in \mathbb{N}} \) where \( X \) is an object of \( \mathcal{A} \) and \( \varepsilon_i : X_i \to X \) is a morphism of this category such that

\[ \varepsilon_j \circ \varepsilon_{ij} = \varepsilon_i \] if \( i \leq j \)

A cone \( (X, \varepsilon_i)_{i \in \mathbb{N}} \) is a direct limit cone over \( S \) in the category \( \mathcal{A} \) if for every cone \( (Y, \psi_i) \) over \( S \) there exists a unique morphism \( \psi : X \to Y \) such that \( \psi \circ \varepsilon_i = \psi_i \) for each \( i \).

We then write \( X = \lim_{\to} X_i \) and we call \( X \) the direct limit of \( S \).

3.2 Direct limit of sets

Let \( S = (X_i, \varepsilon_{ij})_{(i,j) \in \mathbb{N}^2, i \leq j} \) be a direct sequence of sets.

The direct sum \( \bigoplus_{n \in \mathbb{N}} X_n \) also called the coproduct \( \coprod_{n \in \mathbb{N}} X_n \) is the subspace of the cartesian product \( \prod_{n \in \mathbb{N}} X_n \) formed by all the points with only finitely many non-vanishing coordinates.

In this space we introduce the following binary relation (where \( x \in X_i \) and \( y \in Y_j \))

\[
(i, x) \sim (j, y) \iff \begin{cases} y = \varepsilon_{ij}(x) & \text{if } i \leq j \\ x = \varepsilon_{ji}(y) & \text{if } i \geq j \end{cases}
\]

which is an equivalence relation.
Then the set \( X = \coprod_{n \in \mathbb{N}} X_n / \sim \) together with the maps
\[
\varepsilon_i : X_i \longrightarrow X \quad x \mapsto (i, x)
\]
where \((i, x)\) is the equivalence class of \((i, x)\), is the \textit{direct limit} of \( S \) in the category \( \text{SET} \).

We have \( X = \bigcup_{i \in \mathbb{N}} \varepsilon_i (X_i) \). If each \( \varepsilon_{ij} \) is injective then so is \( \varepsilon_i \). \( S \) is then equivalent to the sequence of the subsets \( \varepsilon_i (X_i) \subset X \) with the inclusion maps.

### 3.3 Direct limit of topological spaces

Let \( S = (X_i, \varepsilon_{ij})_{(i,j) \in \mathbb{N}^2, i \leq j} \) be a direct sequence of topological spaces where the bonding maps are continuous.

We then endow \( X \) with the direct sum topology, i.e. is the final topology with respect to the family \((\varepsilon_i)_{i \in \mathbb{N}}\) which is the finest topology for which the maps \( \varepsilon_i \) are continuous. Then \( U \subset X \) is open if and only if \((\varepsilon_i) \end{equation} is open in \( X_i \) for each \( i \).

If the bonding maps are topological embeddings we call \( S \) \textit{strict direct limit}. For any \( i \in \mathbb{N} \), \( \varepsilon_i \) is then a topological embedding.

### 3.4 Fundamental example of \( \mathbb{R}^\infty \)

The space \( \mathbb{R}^\infty \) also denoted by \( \mathbb{R}^{(\mathbb{N})} \) of all finite sequences is the direct limit of \( (\mathbb{R}, \varepsilon_{ij})_{(i,j) \in \mathbb{N}^2, i \leq j} \) where \( \varepsilon_{ij} : (x_1, \ldots, x_i) \mapsto (x_1, \ldots, x_i, 0, \ldots, 0) \).

It is a convenient vector space ([8], 47.1).

### 3.5 Direct limit of finite dimensional manifolds

Let \( M = (M_i, \phi_{ij})_{i \leq j} \) be a direct sequence of paracompact finite dimensional smooth real manifolds where the bonding maps \( \phi_{ij} : M_i \longrightarrow M_j \) are injective smooth immersions and \( \sup \{ \dim \mathbb{R} M_i \} = \infty \). Adapting a result of Glöckner ([6], Theorem 3.1) to the convenient framework (using Proposition 3.6) we have:

**Theorem 3.1.** There exists a uniquely determined \( c^\infty \)-manifold structure on the direct limit \( M \) of \( M \) modelled on the convenient vector space \( \mathbb{R}^\infty \).

**Example 3.3.** The sphere \( S^\infty \) ([8], 47.2). The convenient vector space \( \mathbb{R}^\infty \) is equipped with the weak inner product given by the finite sum \( \langle x, y \rangle = \sum_i x_i y_i \) and is bilinear and bounded, therefore smooth. The topological inductive limit of \( S^1 \subset S^2 \subset \cdots \) is the closed subset \( S^\infty = \{ x \in \mathbb{R}^\infty : \langle x, x \rangle = 1 \} \) of \( \mathbb{R}^\infty \).

Choose \( a \in S^\infty \). We can define the stereographic atlas corresponding to the equivalence class of the two charts \( \{(U_+, u_+), (U_-, u_-)\} \) where \( U_+ = S^\infty \setminus \{a\} \) (resp. \( U_- = S^\infty \setminus \{-a\} \)) and
\[
u_+ : U_+ \longrightarrow \{a\}^\perp \quad u_- : U_- \longrightarrow \{a\}^\perp \quad x \mapsto \frac{x - \langle x, a \rangle a}{1 - \langle x, a \rangle} \quad x \mapsto \frac{x - \langle x, a \rangle a}{1 + \langle x, a \rangle}.
\]

Then \( S^\infty \) is a convenient manifold modelled on \( \mathbb{R}^\infty \).
4 Tangent bundle of direct limit of manifolds

4.1 Structure of manifold on direct limit of \( \{TM_i\}_{i \in \mathbb{N}} \)

Let \( p \geq 4 \) and \( \{M_i, \phi_{ij}\}_{i \leq j} \) be a direct sequence of \( C^p \) paracompact finite dimensional manifolds for which the connecting morphisms are \( C^p \) embeddings with closed image. Without loss of generality (cf. 3.2) we may assume that \( M_1 \subseteq M_1 \subseteq \cdots \subseteq M \) where \( \{M_i, \phi_i\} \) is the direct limit of \( \{M_i, \phi_{ij}\}_{i \leq j} \) in the category of topological spaces and the maps \( \phi_i : M_i \to M \) are inclusions \([6]\). Suppose that \( \dim M_i = d_i \) and consider for \( i \leq j \),

\[
\lambda_{ij} : \mathbb{R}^{d_i} \to \mathbb{R}^{d_j} \quad (x_1, \ldots, x_{d_i}) \mapsto (x_1, \ldots, x_{d_i}, 0, \ldots, 0)
\]

For \( x \in M \) there exists \( n \in \mathbb{N} \) such that \( x = \phi_n(x) \). Using tubular neighborhoods Glöckner proved that there exists an open neighborhood \( O_x \) of \( x \) in \( M \) and a sequence of \( C^{p-2} \) diffeomorphisms \( \{h_i^{(x)} : \mathbb{R}^{d_i} \to U_i\}_{i \geq n} \) (inverse of chart mappings) where \( U_i = \phi_i^{-1}(O_x) \). Moreover for \( j \geq i \geq n \) the compatibility condition

\[
h_j^{(x)} \circ \lambda_{ij} = \phi_{ij}|_{U_i} \circ h_i^{(x)}
\]

holds true ([5], Lemma 4.1).

Our first aim is to introduce appropriate connecting morphisms, say \( \{\Phi_{ij}\}_{i \leq j} \), such that \( \{TM_i, \Phi_{ij}\} \) form a direct system of manifolds in the sense of Glöckner.

For \( i \leq j \) define

\[
\Phi_{ij} : TM_i \to TM_j
\]

\[
[\alpha_i, x_i] \mapsto [\phi_{ij} \circ \alpha_i, \phi_{ij}(x_i)]
\]

where the bracket \( [.,.] \) stands for the equivalence classes of \( TM_i \) with respect to the classical equivalence relations between paths

\[
\alpha \sim_x \beta \Leftrightarrow \begin{cases} \alpha(0) = \beta(0) = x \\ \alpha'(0) = \beta'(0) \end{cases}
\]

where \( \alpha'(t) = [d \alpha(t)](1) \). Clearly \( \Phi_{ii} = \text{Id}_{TM_i} \) and \( \Phi_{ij} \circ \Phi_{ik} = \Phi_{ik} \), for \( i \leq j \leq k \), and \( \{TM_i\} \) is a sequence of \( C^{p-1} \) finite dimensional paracompact manifolds. Moreover \( \Phi_{ij}(TM_i) \) is diffeomorphic to a closed submanifold of \( TM_j \).

**Proposition 4.1.** Let \( p \geq 4 \) and \( \{M_i, \phi_{ij}\}_{i \leq j} \) be a direct sequence of \( C^p \) paracompact finite dimensional manifolds for which the connecting morphisms are \( C^p \) embeddings with closed image.

Then \( \lim_{i} TM_i \) is a \( C^{p-3} \) manifold modelled on \( \mathbb{R}^\infty \times \mathbb{R}^\infty = \lim_{i}(\mathbb{R}^i \times \mathbb{R}^i) \).

**Proof.** Let \( [f, x] \in \lim_{i} TM_i \). Then for some \( n \in \mathbb{N} \), \( [f, x] = \phi_n([f_n, x_n]) \in TM_n \).

Without loss of generality suppose that \( TM_1 \subseteq TM_2 \subseteq \cdots \subseteq TM \) and \( [f, x] \in TM_n(x) \). This means that \( x \) belongs to \( M_n \) and \( f : (-\epsilon, \epsilon) \to M_n(x) \) is a smooth curve passing through \( x \). Since \( \{M_i, \phi_{ij}\}_{i \leq j} \) is a directed system of manifolds satisfying Lemma 4.1. of [5], then there exists an open neighbourhood \( O_x \) of \( x \) in \( M \) and a
family of $C^{p-2}$ diffeomorphisms \{h_i^{(x)} : \mathbb{R}^d_1 \rightarrow U_i\}_{i \geq n(x)}$ where $U_i = \phi_i^{-1}(O_x)$ and (4.1) holds true. For $i \geq n(x)$ define

$$Th_i^{(x)} : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow TU_i \subseteq TM_i$$

$$(\bar{y}, \bar{v}) \longmapsto [\gamma, y]$$

where $(h_i^{(x)})^{-1} \circ \gamma(t) = \bar{y} + t\bar{v}$. For $i \leq j$ we get

$$\Phi_{ij} \circ Th_i^{(x)}(\bar{y}, \bar{v}) = \Phi_{ij}([\gamma, y]) = [\phi_{ij} \circ \gamma, \phi_{ij}(y)].$$

On the other hand,

$$Th_j^{(x)} \circ (\lambda_{ij} \times \lambda_{ij}) (y, v) = Th_j^{(x)}((\bar{y}, 0), (\bar{v}, 0)) = [\gamma', y']$$

for which $(h_j^{(x)})^{-1} \circ \gamma'(t) = (y, 0) + t(v, 0) = \lambda_{ij}(\bar{y} + t\bar{v})$. We claim that $[\phi_{ij} \circ \gamma, \phi_{ij}(y)] = [\gamma', y']$.

Using (4.1) we observe that

$$h_j^{(x)}(\phi_{ij} \circ \gamma(t)) = (h_j^{(x)})^{-1} \circ \phi_{ij} \circ \gamma(t) = (\lambda_{ij} \circ h_i^{(x)})^{-1} \circ \gamma(t)$$

$$= \lambda_{ij} \circ (h_i^{(x)})^{-1} \circ \gamma(t) = \lambda_{ij}(\bar{y} + t\bar{v}),$$

which proves the assertion.

Roughly speaking for any $[f, x] \in TM$, we constructed a family of $C^{p-3}$ diffeomorphisms

$$\{Th_i^{(x)} : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow TU_i \subseteq TM_i\}_{i \geq n(x)}$$

which satisfy the compatibility conditions

$$\Phi_{ij} \circ h_i^{(x)} = h_j^{(x)} \circ (\lambda_{ij} \times \lambda_{ij}) ; \quad j \geq i \geq n(x).$$

As a consequence the limit map $Th^{(x)} = \lim_{i \geq n(x)} Th_i^{(x)} : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow TU^{(x)} := \bigcup_{i \geq n(x)} TU_i$ can be defined. The map $Th^{(x)}$ denotes the diffeomorphism whose restriction to $\mathbb{R}^d \times \mathbb{R}^d$ is $Th_i^{(x)}$.

The next step is to establish that the family $\mathcal{B} = \{Th^{(x)}_{i}; x \in M\}$ is an atlas for $TM$. For $[f, x]$ and $[f', x']$ in $TM$ define $n = \max\{n(x), n(x')\}$. Set $\tau := Th^{(x')} \circ Th^{(x)}$. Since for $i \geq n$

$$\tau \circ \lambda_i = \lambda_i \circ Th^{(x')} \circ Th^{(x)} \circ \lambda_i$$

it follows that $\tau$ is a $C^{p-3}$ diffeomorphism too. Moreover for every natural number $i$, $TM_i$ is a locally compact topological space. This last means that $\lim_{i \to \infty} TM_i$ is Hausdorff ([7], [6]) which completes the proof. □
4.2 The Lie group $Gl(\infty, \mathbb{R})$

In the situation described in [4] (tangent bundle of projective limit of Banach manifolds), the general linear group $GL(F)$ cannot play the rôle of structural group and is replaced by $H_0(F)$ which is a projective limit of Banach Lie groups.

In our framework we are going to use the convenient Lie group $GL(\infty, \mathbb{R})$ as structural group. It is defined as follows. The canonical embeddings $\mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ induce injections $GL(\mathbb{R}^n) \rightarrow GL(\mathbb{R}^{n+1})$. The inductive limit is given by

$$GL(\infty, \mathbb{R}) = \lim_{\rightarrow} GL(\mathbb{R}^n)$$

and can be endowed with a real analytic regular Lie group modeled on $\mathbb{R}^\infty$ (cf [8], Theorem 47.8).

4.3 Convenient vector bundle structure on $TM$

**Theorem 4.2.** $TM$ over $M$ admits a convenient vector bundle structure with the structure group $GL(\infty, \mathbb{R})$.

*Proof.* For any $i \in \mathbb{N}$ consider the natural projection $\pi_i : TM_i \rightarrow M_i$ which maps $[\gamma, y]$ onto $y$. As a first step we show that the limit map $\pi := \lim \pi_i$ exists. For $j \geq i$ and $[\gamma, x] \in TM_i$ we have

$$\phi_{ij} \circ \pi_i[\gamma, y] = \phi_{ij}(y)$$

On the other hand

$$\pi_j \circ \Phi_{ij}[\gamma, y] = \pi_j[\phi_{ij} \circ \gamma, \phi_{ij}(y)]$$

The compatibility condition $\phi_{ij} \circ \pi_i = \pi_j \circ \Phi_{ij}$ leads us to the limit (differentiable) map

$$\pi := \lim_{\rightarrow} \pi_i : \lim_{\rightarrow} TM_i \rightarrow \lim_{\rightarrow} M_i$$

whose restriction to $TM_i$ is given by $\phi_i \circ \pi_i = \pi \circ \Phi_i$.

For $[f, x] \in \lim_{\rightarrow} TM_i$ consider the family of diffeomorphisms $\{h_i^{(x)} : \mathbb{R}^{d_i} \rightarrow U_i^{(x)}\}_{i \geq n(x)}$ as before. For any $i \geq n(x)$ define

$$\Psi_i : \pi_i^{-1}(U_i^{(x)}) \rightarrow U_i^{(x)} \times \mathbb{R}^{d_i}$$

$$[\gamma, y] \mapsto \left(y, (h_i^{(x)} )^{-1} \circ \gamma)'(0)\right).$$

With the standard calculation for the finite dimensional manifolds it is known that $\Psi_i, i \in \mathbb{N}$, is a diffeomorphism. For $j \geq i \geq n(x)$, we claim that the following diagram is commutative

$$\begin{array}{ccc}
\pi_i^{-1}(U_i^{(x)}) & \xrightarrow{\Psi_i} & U_i^{(x)} \times \mathbb{R}^{d_i} \\
\Phi_{ij} \downarrow & & \downarrow \phi_{ij} \times \lambda_{ij} \\
\pi_j^{-1}(U_j^{(x)}) & \xrightarrow{\Psi_j} & U_j^{(x)} \times \mathbb{R}^{d_j}
\end{array}$$

Geometric structure for the tangent bundle
To see that we argue as follows.

\[
(\phi_{ij} \times \lambda_{ij}) \circ \Psi_j([\gamma, y]) = (\phi_{ij} \times \lambda_{ij}) \left( y, (h_i(x))^{-1} \circ \gamma)'(0) \right)
\]

\[
= (\phi_{ij}(y), \lambda_{ij} \circ (h_i(x))^{-1} \circ \gamma)'(0)
\]

\[
= (\phi_{ij}(y), (\lambda_{ij} \circ h_i(x))^{-1} \circ \gamma)'(0)
\]

\[
= \Psi_j([\phi_{ij} \circ \gamma, \phi_{ij}(y)] = (\lambda_{ij} \circ \Psi_{ij})(\gamma, y)
\]

For \((**)\) we used the equation \((4.1)\) and for \((*)\) using the linearity of \(\lambda_{ij}\) we get

\[
\lambda_{ij} \circ (h_i(x))^{-1} \circ \gamma)'(0) = \lambda_{ij} \left( \lim_{t \to 0} \frac{(h_i(x))^{-1} \circ \gamma)(t) - (h_i(x))^{-1} \circ \gamma)(0)}{t} \right)
\]

\[
= \lim_{t \to 0} \frac{(\lambda_{ij} \circ h_i(x))^{-1} \circ \gamma)(t) - (\lambda_{ij} \circ h_i(x))^{-1} \circ \gamma)(0)}{t}
\]

\[
= (\lambda_{ij} \circ h_i(x))^{-1} \circ \gamma)'(0).
\]

Since \(\pi^{-1}_i(U_j(x))\), \(i \geq n(x)\), is open and since \(\pi^{-1}_i(U) = \lim_{t \to 0} \pi^{-1}_i(U_j(x))\), it follows that \(\pi^{-1}_i(U) \subseteq TM\) is open. Furthermore \(\Psi_x := \lim_{t \to 0} \Psi_i : \pi^{-1}_i(U) \to U \times \mathbb{R}^\infty\) exists and, as a direct limit of \(C^p\) diffeomorphisms, is a \(C^p\) diffeomorphism. On the other hand

\[
\Psi_x|_{\pi^{-1}_i(y)} : \pi^{-1}_i(y) \to \{\gamma\} \times \mathbb{R}^\infty
\]

is linear and \(pr_1 \circ \Psi_x\) coincides with \(\pi\). \((pr_1 \text{ stands for projection to the first factor.})\)

Suppose that \([f, x], [g, y] \in TM\), \(n = \max\{n(x), n(y)\}\) and the intersection \(U_{xy} := U(x) \cap U(y)\) is not empty. Then

\[
(\Psi_y)^{-1} |_{U_{xy} \times \mathbb{R}^\infty} \circ \Psi_x|_{U_{xy} \times \mathbb{R}^\infty} : U_{xy} \times \mathbb{R}^\infty \to U_{xy} \times \mathbb{R}^\infty
\]

arises as the inductive limit of the family

\[
(\Psi_y)^{-1} |_{U_{xy} \times \mathbb{R}^n} \circ \Psi_x|_{U_{xy} \times \mathbb{R}^n} : U_{xy} \to GL(\mathbb{R}^d), \quad \tilde{y} \mapsto T_{xy}(\tilde{y}).
\]

Finally the family of maps \(\{T_{xy}^i := (\Psi_y)^{-1}|_{U_{xy} \times \mathbb{R}^d} \circ \Psi_x|_{U_{xy} \times \mathbb{R}^d}, \}_{i \geq n}\), satisfy the required compatibility condition and their limit \(T_{xy} := \lim \sum_{i \geq n} T_{xy}^i\) belongs to \(\lim GL(\mathbb{R}^d) = GL(\mathbb{R}^\infty, \mathbb{R})\). Consequently \(\lim TM_i\) becomes a (convenient) vector bundle with the fibres of type \(\mathbb{R}^\infty\) and the structure group \(GL(\mathbb{R}^\infty, \mathbb{R})\).

\( \square \)

**Example 4.1.** Tangent bundle to \(S^\infty\). The tangent bundle \(TS^\infty\) to the sphere \(S^\infty\) is diffeomorphic to \(\{(x, v) \in S^\infty \times \mathbb{R}^\infty : \langle x, v \rangle = 0\}\).

**Proposition 4.3.** \(\lim TM_i\) as a set is isomorphic to \(TM\).
Proof. Arguing as before, let \([f, x] \in \lim_{\to} TM_i\). Then there exists \(n(x) \in \mathbb{N}\) such that, for \(i \geq n(x)\), \([f, x]\) belongs to \(TM_i\) which means that \(x \in M_i\) and \(f : (-\epsilon, \epsilon) \rightarrow M_i\) for some \(\epsilon > 0\). This last means that \(f : (-\epsilon, \epsilon) \rightarrow \lim_{\to} M_i\) and consequently \([f, x]\) belongs to \(TM\).

Conversely, suppose that \([f, x]\) belongs to \(TM\) that is \(x \in M\) and \(f\) is a curve in \(M = \lim_{\to} M_i\). Again there exists \(n(x)\) such that \(x \in M_i\) and \(f : (-\epsilon, \epsilon) \rightarrow M_i\) is a smooth curve for \(i \geq n(x)\). Since \([f, x] \in TM_i, i \geq n(x)\), then \([f, x] \in \lim_{\to} TM_i\) which completes the proof. \(\square\)

References


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