Characterizations of real hypersurfaces with structure Lie operator in a nonflat complex space form

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Abstract. In this paper, we prove that a real hypersurface $M$ in a complex space form $M_n(c)$, whose structure Lie operator and structure tensor symmetric and skew-symmetric, is a Hopf hypersurface. We characterize such Hopf hypersurface of $M_n(c)$.

Key words: real hypersurface; structure Lie operator; Hopf hypersurface; model spaces of type $A$.

1 Introduction

A complex $n$-dimensional Kaehlerian manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. As is well-known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n(C)$, a complex Euclidean space $C^n$ or a complex hyperbolic space $H_n(C)$, according to $c > 0$, $c = 0$ or $c < 0$.

We consider a real hypersurface $M$ in a complex space form $M_n(c)$, $c \neq 0$. Then $M$ has an almost contact metric structure $(\phi, g, \xi, \eta)$ induced from the Kaehler metric and complex structure $J$ on $M_n(c)$. The structure vector field $\xi$ is said to be principal if $A\xi = \alpha \xi$ is satisfied, where $A$ is the shape operator of $M$ and $\alpha = \eta(A\xi)$. In this case, it is known that $\alpha$ is locally constant ([3]) and that $M$ is called a Hopf hypersurface.

Typical examples of Hopf hypersurfaces in $P_n(C)$ are homogeneous ones, namely those real hypersurfaces are given as orbits under subgroup of the projective unitary group $PU(n + 1)$. Takagi [11] completely classified such hypersurfaces as six model spaces which are said to be $A_1$, $A_2$, $B$, $C$, $D$ and $E$. On the other hand, real hypersurfaces in $H_n(C)$ have been investigated by Berndt [1], Montiel and Romero [6] and so on. Berndt [1] classified all homogeneous Hopf hypersurfaces in $H_n(C)$ as four model spaces which are said to be $A_0$, $A_1$, $A_2$ and $B$. If $M$ is a real hypersurface of type $A_1$ or $A_2$ in $P_n(C)$ or type $A_0$, $A_1$ or $A_2$ in $H_n(C)$, then $M$ is said to be of type $A$ for simplicity.

As a typical characterization of real hypersurfaces of type $A$, the following is due to Okumura [8] for $c > 0$ and Montiel and Romero [6] for $c < 0$. 

**Theorem 1.1.** ([6],[8]) Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on $M$ if and only if $M$ is locally congruent to one of the model spaces of type $A$.

The induced operator $L_\xi$ on real hypersurface $M$ from the 2-form $L_\xi g$ is defined by $(L_\xi g)(X, Y) = g(L_\xi X, Y)$ for any vector field $X$ and $Y$ on $M$, where $L_\xi$ denotes the operator of the Lie derivative with respect to the structure vector field $\xi$. This operator $L_\xi$ is given

$$L_\xi = \phi A - A\phi$$

on $M$, and call it structure Lie operator of $M$. One of the most interesting problems in the study of real hypersurfaces $M$ with commuting operators and anti-commuting operators in $M_n(c)$ is to investigate a geometric characterization of these model spaces. With respect to the above conditions, some characterizations of real hypersurfaces in $M_n(c)$ are determined by the above conditions of real hypersurfaces and many important results on them have been obtained by many differential geometers ([2], [3], [5] and [9] etc).

As for the commuting Ricci operator, Kimura ([9]) for $c > 0$ and Ki and Suh ([3]) showed the following.

**Theorem 1.2.** ([9]) Let $M$ be a real hypersurface of $P_n(\mathbb{C})$, $n \geq 3$. Then $M$ satisfies $\phi S = S\phi$ and $\xi$ is principal if and only if $M$ lies on a tube of radius $r$ over one of the following kähler submanifolds:

- (a) totally geodesic $P^k(c)$ ($1 \leq k \leq n - 1$), where $0 < r < \frac{\pi}{2}$,
- (b) complex quadric $Q^{n-1}(c)$, where $0 < r < \frac{\pi}{4}$ and $\cot^22r = n - 2$,
- (c) $P^1(c) \times P^{n-2}$, where $0 < r < \frac{\pi}{4}$ and $\cot^22r = \frac{1}{(n-2)}$ and $n(\geq 5)$ is odd,
- (d) complex Grassmann $G_{2,5}(c)$, where $0 < r < \frac{\pi}{4}$, $\cot^22r = \frac{3}{5}$ and $n = 9$,
- (e) Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$, $\cot^22r = \frac{5}{9}$ and $n = 15$,
- (f) $k$-dimensional Kähler submanifold on which the rank of each shape operator is not greater than 2 and non zero principal curvature $\neq \pm((2k-1)/(2n-2k-1))^{1/2}$ with $\cot^2r = (2k-1)/(2n-2k-1)$.

**Theorem 1.3.** ([3]) Let $M$ be a connected real hypersurface of $H_n(\mathbb{C})$, $n \geq 3$. Then the Ricci tensor of $M$ commutes with the almost contact structure of $M$ induced from $H_n(\mathbb{C})$ if and only if $M$ is of type $A_1$, $A_2$.

On the other hand, Cho and Ki ([2]) and Sohn and Lim ([4]) have proved the followings.

**Theorem 1.4.** ([2]) Let $M$ be a real hypersurface of $P_n(\mathbb{C})$, $c \neq 0$. It satisfies $R_\xi \phi A = A\phi R_\xi$ on $M$, then $M$ is locally congruent to one of the model space of type $A$. 

Theorem 1.5. ([4]) Let $M$ be a real hypersurface of $P_n(\mathbb{C})$, $c \neq 0$. Then $M$ satisfies $R_\xi \phi A + A \phi R_\xi = 0$ if and only if $M$ is locally congruent to one of the model space of type $A$.

With respect to the structure Lie operator, the present author and W.H. Sohn ([5]) have proved the following.

Theorem 1.6. ([5]) Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$. If $M$ has $L_\xi \phi A = A \phi L_\xi$, then $M$ is locally congruent to a real hypersurface of type $A$.

In this paper, we shall study a real hypersurface in a nonflat complex space form $M_n(c)$ with symmetric or skew-symmetric operators of $L_\xi$ and $\phi$ and give some characterizations of such a real hypersurface in $M_n(c)$. Namely, we shall prove the following theorems

**Theorem A** Let $M$ be a real hypersurface satisfying $L_\xi \phi = \phi L_\xi$ in a complex space form $M_n(c)$, $c \neq 0$. Then $M$ is a Hopf hypersurface in $M_n(c)$.

**Theorem B** Let $M$ be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Then it satisfies $L_\xi \phi = \phi L_\xi$ on $M$ if and only if $M$ is locally congruent to one of the model spaces of type $A$.

**Theorem C** Let $M$ be a real hypersurface satisfying $L_\xi \phi + \phi L_\xi = 0$ in a complex space form $M_n(c)$, $c \neq 0$. Then $M$ is a Hopf hypersurface in $M_n(c)$.

All manifolds in the present paper are assumed to be connected and of class $C^\infty$ and the real hypersurfaces supposed to be orientable.

2 Preliminaries

Let $M$ be a real hypersurface immersed in a complex space form $M_n(c)$, and $N$ be a unit normal vector field of $M$. By $\nabla$ we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\tilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\nabla_X Y = \nabla_X Y + g(AX, Y)N, \quad \nabla_X N = -AX,$$

for any vector fields $X$ and $Y$ tangent to $M$, where $g$ denotes the Riemannian metric tensor of $M$ induced from $\tilde{g}$, and $A$ is the shape operator of $M$ in $M_n(c)$. For any vector field $X$ on $M$ we put

$$JX = \phi X + \eta(X)\xi, \quad JN = -\xi,$$

where $J$ is the almost complex structure of $M_n(c)$. Then we see that $M$ induces an almost contact metric structure $(\phi, g, \xi, \eta)$, that is,

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi \xi = 0, \quad \eta(\xi) = 1,$$

(2.1) $$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi).$$
for any vector fields $X$ and $Y$ on $M$. Since the almost complex structure $J$ is parallel, we can verify from the Gauss and Weingarten formulas the followings:

\[ \nabla_X \xi = \phi A X, \tag{2.2} \]

\[ (\nabla_X \phi) Y = \eta(Y) AX - g(AX, Y) \xi. \tag{2.3} \]

Since the ambient manifold is of constant holomorphic sectional curvature $c$, we have the following Gauss and Codazzi equations respectively:

\[ R(X, Y) Z = \frac{c}{4} \{ g(Y, Z) X - g(X, Z) Y + g(\phi Y, Z) \phi X - g(\phi X, Z) \phi Y \]

\[ -2g(\phi X, Y) \phi Z \} + g(AY, Z) AX - g(AX, Z) AY, \tag{2.4} \]

\[ (\nabla_X A) Y - (\nabla_Y A) X = \frac{c}{4} \{ \eta(X) \phi Y - \eta(Y) \phi X - 2g(\phi X, Y) \xi \}, \tag{2.5} \]

for any vector fields $X$, $Y$ and $Z$ on $M$, where $R$ denotes the Riemannian curvature tensor of $M$. By the use of (2), we have $(L_\xi g)(X, Y) = g((\phi A - A \phi) X, Y)$, for any vector fields $X$ and $Y$ on $M$, and hence the induced operator $L_\xi$ from $L_\xi g$ is given

\[ L_\xi X = (\phi A - A \phi) X. \tag{2.6} \]

Let $W$ be a unit vector field on $M$ with the same direction of the vector field $-\phi \nabla_\xi \xi$, and let $\mu$ be the length of the vector field $-\phi \nabla_\xi \xi$ if it does not vanish, and zero (constant function) if it vanishes. Then it is easily seen from (2) that

\[ A \xi = \alpha \xi + \mu W, \tag{2.7} \]

where $\alpha = \eta(A \xi)$. We notice here that $W$ is orthogonal to $\xi$.

We put

\[ \Omega = \{ p \in M \mid \mu(p) \neq 0 \}. \tag{2.8} \]

Then $\Omega$ is an open subset of $M$.

### 3 Proof of Theorems

In this section, we shall prove Theorems A, B and C. Now we need the following lemmas in order to prove the our main results.

**Lemma 3.1**([3], [7]) If $\xi$ is a principal curvature vector, then the corresponding principal curvature $\alpha$ is locally constant.

**Lemma 3.2**([7]) Assume that $\xi$ is a principal curvature vector and the corresponding principal is $\alpha$. Then

\[ A \phi A - \frac{\alpha}{2} (A \phi + \phi A) - \frac{c}{4} = 0. \tag{3.1} \]
Proof of Theorem A.
We assume that the Lie structure operator \( L_\xi \) and structure tensor field \( \phi \) are commute, that is, \( L_\xi \phi = \phi L_\xi \). Hence, we have
\[
2(\phi A \phi + A)X = (2\alpha \eta(X) + \mu w(X))\xi + \mu \eta(X)W,
\]
for any vector field \( X \) on \( \Omega \), where \( w \) is the dual 1-form of the unit vector field \( W \).

If we put \( X = \xi \) into (3.2), then we have
\[
2A \xi = 2\alpha \xi + \mu W.
\]
(3.3)
If we substitute (2.7) into (3.3) then we obtains
\[
\mu = 0 \text{ on } \Omega, \text{ and it is a contradiction. Thus the set } \Omega \text{ is empty and hence } M \text{ is a Hopf hypersurface.} \quad \square
\]

Proof of Theorem B.
By Theorem A, \( M \) is a Hopf hypersurface in \( M_n(c) \), that is \( A \xi = \alpha \xi \). Therefore the assumption \( L_\xi \phi = \phi L_\xi \) is equivalent to
\[
(\phi A \phi + A)X = \alpha \eta(X)\xi.
\]
(3.4)
For any vector field \( X \) on \( M \) such that \( AX = \lambda X \), it follows from (3.1) that
\[
\left( \lambda - \frac{\alpha}{2} \right) A\phi X = \frac{1}{2} \left( \alpha \lambda + \frac{c}{2} \right) \phi X.
\]
(3.5)
We can choose an orthonormal frame field \( \{X_0 = \xi, X_1, X_2, \ldots, X_{2(n-1)}\} \) on \( M \) such that \( AX_i = \lambda_i X_i \) for \( 1 \leq i \leq 2(n-1) \). If \( \lambda_i \neq \frac{\alpha}{2} \) for \( 1 \leq i \leq p \leq 2(n-1) \), then we see from (3.5) that \( \phi X_i \) is also a principal direction, say \( A\phi X_i = \mu_i \phi X_i \). From (3.4), we have \( \mu_i = \lambda_i \) and hence \( A\phi X_i = \phi AX_i \) for \( 1 \leq i \leq p \). If \( \lambda_i \neq \frac{\alpha}{2} \) and \( \lambda_j = \frac{\alpha}{2} \) for \( 1 \leq i \leq p \) and \( p+1 \leq j \leq 2(n-1) \) respectively, then we gets \( c = -\alpha^2 \) and it follows from (3.4) that
\[
\phi A\phi X_j + AX_j = 0.
\]
(3.6)
Taking inner product of (3.6) with \( X_i \), we obtain \( (\lambda_i - \mu_i)g(X_i, X_j) = 0 \) for \( 1 \leq i \leq p \).
If \( \lambda_i = \mu_i \) for \( 1 \leq i \leq p \), then we have
\[
\lambda_i = \mu_i = \frac{\alpha \pm \sqrt{\alpha^2 + c}}{2}.
\]
(3.7)
Since \( c = -\alpha^2 \), it follows that \( \lambda_i = \frac{\alpha}{2} \), and hence a contradiction. If \( \lambda_j = \frac{\alpha}{2} \) for \( 1 \leq j \leq 2(n-1) \), then it is easily seen that \( A\phi X_j = \phi AX_j \) for all \( j \). Therefore we have \( L_\xi = \phi A - A\phi = 0 \) on \( M \). Theorem B follows Theorem 1. \quad \square

Proof of Theorem C.
Assume that the Lie structure operator \( L_\xi \) and structure tensor field \( \phi \) are skew-symmetric, that is, \( L_\xi \phi + \phi L_\xi = 0 \). Hence, we have
\[
\eta(X)A\xi - \eta(AX)\xi = 0.
\]
(3.8)
If we put $X = \xi$ into (3.8), then we have

\[(3.9) \quad A\xi = \alpha \xi.\]

Comparing (2.7) with (3.9), we get $\mu = 0$ and hence a contradiction. Thus the set $\Omega$ is empty and hence $M$ is a Hopf hypersurface. \(\Box\)

References


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