Semi-slant warped product submanifolds of a nearly Kaehler manifold

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Abstract. Many differential geometric properties of a submanifold of a Kaehler manifold are conceived via the canonical structure tensors P and F on the submanifold. For instance, a CR-submanifold of a Kaehler manifold is a CR-product if and only if P is parallel on the submanifold (cf. [6]). Warped product submanifolds are generalized versions of CR-product submanifolds. Therefore, it is natural to see how the non-triviality of the covariant derivatives of P and F gives rise to warped product submanifolds. In the present article, we have worked out characterizations in terms of P and F under which a semi-slant submanifold of a nearly Kaehler manifold reduces to a warped product submanifold.

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1 Introduction

On a submanifold M of an almost Hermitian manifold $(\bar{M}, J, g)$, for any vector field U on M, JU decomposes into tangential and normal parts respectively, as $JU = PU + FU$. This defines a $(1,1)-$tensor field P and a normal valued one-form F on M. B. Y. Chen described many extrinsic geometric properties of submanifolds in terms of the tensors P and F. For instance, "*a CR- submanifold M of a Kaehler manifold is a CR-product if and only if P is parallel on M*" (cf. [6]); more generally, "*P is parallel on a submanifold M of a Kaehler manifold if and only if M is a Riemannian product $N_1 \times N_2 \times \ldots \times N_k$, where each N_i is either a Kaehler submanifold, a totally real submanifold or a Kaehlerian slant submanifold*" (cf. [8]). Similar characterizations are available in terms of the one-form F. Later, such characterizations were extended and generalized to the settings of nearly Kaehler and locally conformal Kaehler manifolds, as well as for CR-warped product submanifolds of Kaehler manifolds (cf. [18]). Since a warped product manifold is a generalized version of the Riemannian product, and provides a natural framework for time dependent mechanical systems, it is expected that the covariant derivatives of P and F on these manifolds (as submanifolds) are non-zero. However, explicit formulas for ∇P and ∇F need to be worked out, which could turn the submanifold to a warped product submanifold. In the present article,

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we have studied warped product spaces from extrinsic geometric point of view, that is, as submanifolds of nearly Kaehler manifolds. More specifically, we have investigated characterizations (in terms of the tensor fields P and F), under which a semi-slant submanifold of a nearly Kaehler manifold reduces to a warped product submanifold. The results obtained in the paper generalize and extend the conditions of Chen [6], Nadia et.al [1] and of V. A.Khan et.al [18].

2 Preliminaries

Let $(\bar{M}, J, g)$ be an almost Hermitian manifold i.e., $\bar{M}$ is a smooth manifold endowed with an almost complex structure J and a Hermitian metric g, such that

$$J^2 = -I \quad \text{and} \quad g(JU, JV) = g(U, V) \tag{2.1}$$

for all vector fields U, V on $\bar{M}$. If J is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$, then $\bar{M}$ is called a *Kaehler manifold*. A more general structure on $\bar{M}$, known as *nearly Kaehler structure* can be defined by a weaker condition namely

$$(\bar{\nabla}_U J)U = 0, \tag{2.2}$$

or, equivalently,

$$(\bar{\nabla}_U J)V + (\bar{\nabla}_V J)U = 0.$$

A necessary and sufficient condition for a nearly Kaehler manifold to be a Kaehler manifold is the vanishing of the Nijenhuis tensor of J. Any four dimensional nearly Kaehler manifold is a Kaehler manifold. A typical example of a nearly Kaehler, non-Kaehler manifold is the six dimensional sphere S^6, which has an almost complex structure J defined by the vector cross product in the space of purely imaginary Cayley numbers (cf. [11]). This almost complex structure is not integrable and satisfies (2.2). Let M be an isometrically immersed submanifold of a Riemannian manifold $\bar{M}$. If ∇ and $\nabla^{\perp}$ are the induced Levi-Civita connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, then the Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_U V = \nabla_U V + h(U, V) \tag{2.3}$$

and

$$\bar{\nabla}_U \xi = -A_{\xi} U + \nabla^{\perp}_U \xi \tag{2.4}$$

for all $U, V \in TM$ and $\xi \in T^{\perp}M$, where h is the second fundamental form and A_{ξ} is the shape operator (corresponding to the normal vector field ξ) of the immersion of M into $\bar{M}$. These are related via

$$g(A_{\xi} U, V) = g(h(U, V), \xi), \tag{2.5}$$

where g denotes both the Riemannian metric on $\bar{M}$ and the one induced on M.

Let the ambient manifold $\bar{M}$ be an almost Hermitian manifold with an almost complex structure J and Hermitian metric g. Then for any $U \in TM$, we put

$$PU = tan(JU) \quad and \quad FU = nor(JU), \tag{2.6}$$

where $\tan_x$ and nor_x are the natural projections associated to the direct sum decomposition

$$T_x\bar{M} = T_xM \oplus T_x^{\perp}M, \quad x \in M$$

Similarly, for $\xi \in T^{\perp}M$, we put

$$t\xi = \tan(J\xi) \;\; and \;\; f\xi = nor(J\xi). \tag{2.7}$$

Thus, P (resp. f) is an $(1,1)-$tensor field on TM (resp. $T^{\perp}M$), whereas t (resp. F) is a tangential (resp. normal) valued 1-form on $T^{\perp}M$ (resp. TM).

The covariant derivatives of the tensor fields P, F, t and f are defined respectively as

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{2.8}$$

$$(\bar{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V, \tag{2.9}$$

$$(\bar{\nabla}_U t)\xi = \nabla_U t\xi - t\nabla^{\perp}_U \xi, \tag{2.10}$$

$$(\bar{\nabla}_U f)\xi = \nabla^{\perp}_U f\xi - f\nabla^{\perp}_U \xi. \tag{2.11}$$

A differentiable distribution D^{θ} on a submanifold M of an almost Hermitian manifold $(\bar{M}, J, g)$ is called a *slant distribution* if the Wirtinger angle $\theta(X) \in [0, \pi/2]$ between JX and D^{θ}_x has the same value θ for each $x \in M$ and $X \in T_x(M)$, $X \neq 0$. A submanifold M is called a *slant submanifold* if the tangent bundle TM is slant. Holomorphic and totally real submanifolds are special cases of slant submanifolds with wirtinger angle 0 and $\pi/2$ respectively. A slant submanifold is called *proper slant* if it is neither holomorphic nor totally real (cf. [8]).

If M is a slant submanifold of an almost Hermitian manifold $\bar{M}$, then we have

$$P^2 = -cos^2\theta I, \tag{2.12}$$

where θ is the wirtinger angle of M in $\bar{M}$. This gives

$$g(PU, PV) = cos^2\theta g(U, V), \tag{2.13}$$

and

$$g(FU, FV) = sin^2\theta g(U, V), \tag{2.14}$$

for U, V tangent to M [8].

Let M be a submanifold of an almost Hermitian manifold $\bar{M}$ and $D_x = T_x(M) \cap JT_x(M)$ be the maximal complex subspace of $T_x(M)$. If D defines a smooth distribution on M, then M is called a *generic submanifold.* B.Y.Chen [7] introduced the notion of generic submanifolds as a generalized version of CR-submanifolds (cf. [3]). In fact, a generic submanifold is a CR-submanifold if the orthogonal complement of D in TM is totally real.

A special class of generic submanifolds (but more general than the class of CR-submanifolds) is the class of *semi-slant submanifolds.* For a semi-slant submanifold the orthogonal complement of the holomorphic distribution is a slant distribution. That means in this case $TM = D \oplus D^\theta$, where D^θ is a slant distribution on M with Wirtinger angle θ (cf. [19]). A semi-slant submanifold M is a CR-submanifold if the slant distribution D^θ on M is totally real, i.e., $\theta = \pi/2$ whereas a semi-slant submanifold reduces to a slant submanifold if $D = \{0\}$. A semi-slant submanifold is said to be *proper* if $\theta \neq \pi/2$.

Since the tangent bundle of a semi-slant submanifold M of an almost Hermitian manifold admits the orthogonal complementary sub-bundles D and D^θ, these induce two canonical projectors B and C on TM, such that for any $U \in TM$ we have

$$U = BU + CU, \tag{2.15}$$

where $BU \in D$ and $CU \in D^\theta$. Similarly, the normal bundle $T^\perp M$ is decomposed as

$$T^\perp M = FD^\theta \oplus \nu \tag{2.16}$$

where ν is the orthogonal complementary distribution to FD^θ in $T^\perp M$ and is invariant under J. This means $J\xi = f\xi$ for each $\xi \in \nu$. It can also be seen that $f\xi \in FD^\theta$ for each $\xi \in FD^\theta$.

On a semi-slant submanifold of an almost Hermitian manifold, the following are straightforward observations

$$\begin{cases} (a)\ FD = \{0\}, & (b)\ PD = D, \\ (c)\ PD^\theta \subseteq D^\theta, & (d)\ t(T^\perp M) = D^\theta. \end{cases} \tag{2.17}$$

Moreover, we also have

$$\begin{cases} (e)\ P^2 + tF = -I, & (f)\ f^2 + Ft = -I, \\ (g)\ FP + fF = 0, & (h)\ tf + Pt = 0. \end{cases} \tag{2.18}$$

R. L.Bishop and B. O'Neill [4], while investigating manifolds of negative sectional curvatures, introduced the warped product metric by generalizing the notion of (Riemannian) product metric, which is obtained by homothetically warping the product metric on to the fibers. Warped product manifolds provide an excellent setting to model space-time near black holes or bodies with high gravitational fields (cf. [2], [12], [14]). The study of warped product manifolds from extrinsic geometric point of view was intensified after B. Y. Chen's work on CR-warped product submanifolds of Kaehler manifolds (cf. [9], [10]).

Since our aim is to study semi-slant submanifolds which are warped product submanifolds of an almost Hermitian manifold, we recall in the following paragraphs the notion of warped product manifolds and some intrinsic geometric properties of these manifolds.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively and let f be a positive differentiable function on M_1. Then the *warped product* $M_1 \times_f M_2$ is the manifold $M_1 \times M_2$ endowed with the Riemannian metric g given by

$$g = \pi_1^*(g_1) + (f \circ \pi_1)^2 \pi_2^*(g_2), \tag{2.19}$$

where $\pi_i (i = 1, 2)$ are the projection maps of M onto M_1 and M_2 respectively. The function f, in this case, is known as the *warping function* (cf. [4]). If the warping function f is just a constant, then the warped product is simply a Riemannian product, known as a *trivial warped product*.

Several important observations and formulas revealing geometric aspects of a warped product manifold were obtained by R. L. Bishop and B. O'Neill:

Proposition 2.1. *([4]). Let $M = M_1 \times_f M_2$ be a warped product manifold. If $X, Y \in TM_1$ and $Z, W \in TM_2$, then*

(i) $\nabla_X Y \in TM_1$,

(ii) $\nabla_X Z = \nabla_Z X = (Xf/f)Z$,

(iii) $nor(\nabla_Z W) = -(g(Z, W)/f)\nabla f$,

where $nor(\nabla_Z W)$ denotes the component of $\nabla_Z W$ in TM_1 and ∇f is the gradient of f defined as

$$g(\nabla f, U) = Uf \tag{2.20}$$

for any $U \in TM$.

Corollary 2.2. *On a warped product manifold $M = M_1 \times_f M_2$,*

(i) M_1 is totally geodesic in M

(ii) M_2 is totally umbilical in M.

A warped product manifold isometrically immersed into a Riemannian manifold is known as *warped product submanifold*.

3 Semi-slant submanifolds of a nearly Kaehler manifold

The study of semi-slant submanifolds in Kaehlerian manifolds was initiated by N. Papaghiuc [19] and was later extended to the settings of contact manifolds by J. L. Cabrerizo et.al [5]. In view of the interesting geometric features of nearly Kaehler manifolds (cf. [11]), it is natural to extend and study the notion of semi-slant submanifolds to the setting of nearly Kaehler manifolds. Some initial results of this setting are worked out in [17]. Further, as proper semi-slant warped product submanifolds of a Kaehler manifold are trivial (cf. [20]), our aim in this section is to investigate semi-slant submanifolds as warped products in a nearly Kaehler manifold.

We further denote by M_T, a holomorphic submanifold and by M_θ, a slant submanifold (with Wirtinger angle θ) of an almost Hermitian manifold $\bar{M}$. With these factors, there are two possible warped product submanifolds of $\bar{M}$, namely (i) $M_\theta \times_f M_T$ and (ii) $M_T \times_f M_\theta$. The existence of warped product submanifolds of type (i) in nearly Kaehler manifolds is ruled out in view of the following theorem.

Theorem 3.1. *([16]). Let $\bar{M}$ be a nearly Kaehler manifold and $M = N \times_f N_T$ a warped product submanifold of $\bar{M}$ with N and N_T a Riemannian and a holomorphic submanifolds respectively of $\bar{M}$. Then M is a trivial warped product submanifold.*

However, semi-slant warped product submanifolds (namely the warped product submanifolds of type (ii)) of a nearly Kaehler manifold are known to exist and are studied by many geometers with various geometric stand point (cf. [15], [16], [21], etc).

The following theorems ensure the existence of holomorphic and slant factors on a semi-slant submanifold of a nearly Kaehler manifold.

Theorem 3.2. *([17]). Let M be a semi-slant submanifold of a nearly Kaehler manifold $\bar{M}$. Then the holomorphic distribution D is involutive if and only if*

$$h(X, JY) = h(JX, Y) \quad \text{and} \quad nor[(\bar{\nabla}_X J)Y] = 0.$$

for each $X, Y \in D$.

Theorem 3.3. *([15]). The slant distribution D^θ on a semi-slant submanifold M of a nearly Kaehler manifold is involutive if and only if*

$$2g(\nabla_W Z, X) = g(\nabla_Z PW + \nabla_W PZ - A_{FZ}W - A_{FW}Z, PX), \tag{3.1}$$

for each $X \in D$ and $Z, W \in D^\theta$.

Let $M = M_T \times_f M_\theta$ be a semi-slant warped product submanifold of an almost Hermitian manifold $\bar{M}$. Then by Proposition 2.1, we have

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z, \tag{3.2}$$

for each $X \in TM_T$ and $Z \in TM_\theta$. It follows from (3.2) and (2.8) that

$$(\bar{\nabla}_X P)Z = 0 \tag{3.3}$$

and

$$(\bar{\nabla}_Z P)X = (PX \ln f)Z - (X \ln f)PZ. \tag{3.4}$$

More generally, we have

Lemma 3.4. *On a semi-slant warped product submanifold of an almost Hermitian manifold $\bar{M}$,*

(a) $(\bar{\nabla}_{CU} P)BV = C(\bar{\nabla}_U P)BV = (PBV \ln f)CU - (BV \ln f)PCU$,

(b) $B(\bar{\nabla}_U P)CV = g(CU, CV)P\nabla \ln f - g(CU, PCV)\nabla \ln f$,

(c) $C(\bar{\nabla}_U P)CV = C(\bar{\nabla}_{CU} P)CV$,

for all $U, V \in TM$.

Proof. By formulas (2.15) and (3.3), we may deduce that

$$(\bar{\nabla}_U P)CV = (\bar{\nabla}_{CU} P)CV, \tag{3.5}$$

and by (2.15) and (3.4),

$$(\bar{\nabla}_U P)BV = (\bar{\nabla}_{BU} P)BV + (PBV \ln f)CU - (BV \ln f)PCU. \tag{3.6}$$

Taking account of the facts that $(\bar{\nabla}_X P)Y \in TM_T$ for each $X, Y \in TM_T$ and $PZ \in TM_\theta$ for any $Z \in TM_\theta$, it is easy to deduce from the above equation that

$$B(\bar{\nabla}_U P)BV = (\bar{\nabla}_{BU} P)BV, \tag{3.7}$$

and

$$C(\bar{\nabla}_U P)BV = (PBV \ln f)CU - (BV \ln f)PCU. \tag{3.8}$$

It is easy to notice from (3.7) that

$$B(\bar{\nabla}_{CU} P)BV = 0,$$

and thus, from (3.6), (3.7) and (3.8) we have:

$$(\bar{\nabla}_{CU} P)BV = C(\bar{\nabla}_U P)BV = (PBV \ln f)CU - (BV \ln f)PCU. \tag{3.9}$$

This proves part (a) of the lemma.

With regard to D and D^θ components of $(\bar{\nabla}_U P)CV$, let us first consider $g((\bar{\nabla}_U P)CV, X)$ for $X \in D$. By virtue of (2.8) and (3.5), we have

$$g((\bar{\nabla}_U P)CV, X) = g((\bar{\nabla}_{CU} P)CV, X) = g(\nabla_{CU} PCV, X) - g(P\nabla_{CU} CV, X).$$

Now, using formulas (3.2), (2.20) and the fact that $g(PU, V) = -g(U, PV)$, the above equation yields

$$B(\bar{\nabla}_U P)CV = B(\bar{\nabla}_{CU} P)CV = g(CU, CV)P\nabla \ln f - g(CU, PCV)\nabla \ln f.$$

This proves part (b) of the lemma.

Now, taking product of $(\bar{\nabla}_U P)CV$ with $W \in D^\theta$, we have

$$g((\bar{\nabla}_U P)CV, W) = g((\bar{\nabla}_{BU} P)CV, W) + g((\bar{\nabla}_{CU} P)CV, W).$$

The first term in the right hand side of the above equation vanishes by virtue of formula (3.3) and consequently, we obtain that

$$C(\bar{\nabla}_U P)CV = C(\bar{\nabla}_{CU} P)CV. \tag{3.10}$$

Equations (3.7) and (3.10) prove part (c) of the lemma . □

Lemma 3.5. *If M is a semi-slant warped product submanifold of a nearly Kaehler manifold $\bar{M}$, then*

(a) $(\bar{\nabla}_X P)X = 0$,

(b) $B(\bar{\nabla}_Z P)Z = BA_{FZ}Z = -\|Z\|^2 P(\nabla \ln f)$,

(c) $C(\bar{\nabla}_Z P)Z = CA_{FZ}Z + th(Z, Z)$,

for each $X \in TM_T$ and $Z \in TM_\theta$.

Proof. Making use of (2.3), (2.4) and (2.6) - (2.11), we may obtain the tangential and normal parts of $(\bar{\nabla}_U J)V$ as:

$$tan[(\bar{\nabla}_U J)V] = (\bar{\nabla}_U P)V - A_{FV}U - th(U, V), \tag{3.11}$$

$$nor[(\bar{\nabla}_U J)V] = (\bar{\nabla}_U F)V + h(U, PV) - fh(U, V). \tag{3.12}$$

Thus, on using (2.2), the above equations yield

$$(\bar{\nabla}_U P)U = A_{FU}U + th(U, U), \tag{3.13}$$

$$(\bar{\nabla}_U F)U = fh(U, U) - h(U, PU). \tag{3.14}$$

From (3.13), it is straightforward to see that

$$(\bar{\nabla}_X P)X = th(X, X). \tag{3.15}$$

Now, as $th(X, X) \in TM_\theta$ and $(\bar{\nabla}_X P)X \in TM_T$, it follows from (3.15) that $(\bar{\nabla}_X P)X = 0$ and $h(X, X) \in \nu$. This proves part (a).

Now, on comparing D and D^θ components in equation (3.13) while taking account of the fact that $tN \in D^\theta$ for each $N \in T^\perp M$, we get

$$B(\bar{\nabla}_Z P)Z = BA_{FZ}Z \tag{3.16}$$

and

$$C(\bar{\nabla}_Z P)Z = CA_{FZ}Z + th(Z, Z), \tag{3.17}$$

for each $Z \in TM_\theta$.

Let h^0 be the second fundamental form of the immersion of M_θ into M and ∇^0 the induced Riemannian connection on M_θ. Then by formulas (2.3) and (2.8),

$$(\bar{\nabla}_Z P)Z = \nabla^0_Z PZ + h^0(Z, PZ) - P\nabla^0_Z Z - Ph^0(Z, Z).$$

The second term in the right hand side of the above equation is zero by virtue of formula (*iii*) in Proposition 2.1 and the fact that Z and PZ are orthogonal whereas the last term of the equation reduces to $g(Z,Z)P\nabla \ln f$. Now, denoting $\nabla^0_Z PZ - P\nabla^0_Z Z$ by $(\nabla^0_Z P)Z$, the above equation takes the form

$$(\bar{\nabla}_Z P)Z = (\nabla^0_Z P)Z - g(Z,Z)P\nabla \ln f.$$

Therefore,

$$B(\bar{\nabla}_Z P)Z = -g(Z,Z)P\nabla \ln f \tag{3.18}$$

and

$$C(\bar{\nabla}_Z P)Z = (\nabla^0_Z P)Z$$

Part (b) of the lemma is proved by virtue of equations (3.18) and (3.16) whereas equation (3.17) establishes part (c) of the lemma. □

Lemma 3.6. *On a semi-slant warped product submanifold M of a nearly Kaehler manifold $\bar{M}$,*

$$g(h(X,Z),FW) = \frac{1}{3}(X \ln f)g(PZ,W) - (PX \ln f)g(Z,W),$$

for each $X \in TM_T$ and $Z, W \in TM_\theta$.

Proof. Using nearly Kaehler condition in (3.11), we get

$$0 = (\bar{\nabla}_X P)Z + (\bar{\nabla}_Z P)X - 2th(X,Z) - A_{FZ}X,$$

which on applying (3.3) and (3.4) takes the form

$$(PX \ln f)Z - (X \ln f)PZ = 2th(X,Z) + A_{FZ}X. \tag{3.19}$$

Taking product with Z on both sides of the above equation yields

$$g(h(PX,Z),FZ) = (X \ln f)\|Z\|^2.$$

The above equation can also be written as:

$$g(h(PX,Z),FW) + g(h(PX,W),FZ) = 2(X \ln f)g(Z,W), \tag{3.20}$$

for any $X \in D$ and $Z, W \in D^\theta$. On the other hand, taking product with $W \in D^\theta$ in (3.19) gives

$$(PX \ln f)g(Z,W) - (X \ln f)g(PZ,W) = -2g(h(X,Z),FW) + g(h(X,W),FZ).$$

Interchanging Z and W and subtracting the obtained equation from the above, we get

$$g(h(X,Z),FW) - g(h(X,W),FZ) = \frac{2}{3}(X \ln f)g(PZ,W). \tag{3.21}$$

From (3.20) and (3.21), it can be deduced that

$$g(h(X,Z),FW) = \frac{1}{3}(X \ln f)g(PZ,W) - (PX \ln f)g(Z,W). \tag{3.22}$$

This completes the proof of the lemma. □

Theorem 3.7. *Let M be a semi-slant submanifold of a nearly Kaehler manifold $\bar{M}$ with involutive distributions D and D^θ. Then M is locally a semi-slant warped product submanifold if and only if $\nabla_Z PZ \in D^\theta$ and*

$$(\bar{\nabla}_U P)U = \|CU\|^2 P\nabla\mu + (PBU\mu)CU - (BU\mu)PCU + CA_{FCU}CU + th(CU, CU) \tag{3.23}$$

for each $U \in TM$, $Z \in D^\theta$ and μ a C^∞-function on M satisfying $Z\mu = 0$.

Proof. Let $M = M_T \times_f M_\theta$ be a semi-slant warped product submanifold of a nearly Kaehler manifold $\bar{M}$, then using formula (ii) in Proposition 2.1, we obtain $\nabla_Z PZ \in D^\theta$.

For any $U \in TM$, we may write

$$\begin{aligned}(\bar{\nabla}_U P)U &= (\bar{\nabla}_U P)BU + (\bar{\nabla}_U P)CU \\ &= (\bar{\nabla}_U P)BU + B(\bar{\nabla}_U P)CU + C(\bar{\nabla}_U P)CU\end{aligned} \tag{3.24}$$

Making use of Lemma 3.4, the above equation takes the form,

$$(\bar{\nabla}_U P)U = (PBU \ln f)CU - (BU \ln f)PCU + \|CU\|^2 P\nabla\mu + C(\bar{\nabla}_U P)CU.$$

The last term in the right hand side of the above equation is equal to $C(\bar{\nabla}_{CU} P)CU$ by virtue of equation (3.5). Hence on applying part (c) of Lemma 3.5, we get,

$$(\bar{\nabla}_U P)U = \|CU\|^2 P\nabla\mu + (PBU\mu)CU - (BU\mu)PCU + CA_{FCU}CU + th(CU, CU).$$

This verifies (3.23).

Conversely, assume that M is a semi-slant submanifold of a nearly Kaehler manifold $\bar{M}$ with involutive distributions D and D^θ, such that (3.23) holds for a C^∞ function μ on M with $Z\mu = 0$ for each $Z \in D^\theta$. Equation (3.23) can also be written as:

$$\begin{aligned}(\bar{\nabla}_U P)V + (\bar{\nabla}_V P)U = {} & (PBU\mu)CV + (PBV\mu)CU - (BU\mu)PCV \\ & - (BV\mu)PCU + 2g(CU, CV)P(\nabla \ln f) \\ & + C(A_{FCU}CV + A_{FCV}CU) + 2th(CU, CV).\end{aligned} \tag{3.25}$$

On using the fact that $\bar{M}$ is nearly Kaehler in (3.11), it follows that

$$(\bar{\nabla}_X P)Y + (\bar{\nabla}_Y P)X = 2th(X, Y),$$

for all $X, Y \in D$. The left hand side of the above equation is zero by virtue of (3.25). Therefore, $th(X, Y) = 0$ which means that

$$h(X, Y) \in \nu.$$

Taking account of this observation and the nearly Kaehler condition in equation (3.12) we obtain that

$$\nabla_X Y + \nabla_Y X \in D$$

As D is assumed to be integrable, the above observation yields that $\nabla_X Y \in D$, i.e., D is parallel. In other words, the leaves of D are totally geodesic in M.

Now, for $X \in D$ and $Z \in D^\theta$, equation (3.11) together with nearly Kaehler condition gives that

$$(\bar{\nabla}_X P)Z + (\bar{\nabla}_Z P)X = A_{FZ}X + 2th(X, Z),$$

whereas from (3.25), we have

$$(\bar{\nabla}_X P)Z + (\bar{\nabla}_Z P)X = (PX\mu)Z - (X\mu)PZ.$$

That gives

$$A_{FZ}X + 2th(X, Z) = (PX\mu)Z - (X\mu)PZ.$$

Taking product with $W \in D^\theta$ in this equation, we get

$$g(h(X, W), FZ) - 2g(h(X, Z), FW) = (PX\mu)g(Z, W) - (X\mu)g(PZ, W),$$

which on simplifying yields

$$g(h(X, Z), FW) + g(h(X, W), FZ) = -2(PX\mu)g(Z, W).$$

As D^θ is involutive, using Theorem 3.3, the above equation gives

$$g(\nabla_Z PW + \nabla_W PZ, X) + 2g(\nabla_W Z, PX) = -2(PX\mu)g(Z, W).$$

Further, as $\nabla_Z PZ \in D^\theta$, the above equation reduces to

$$g(\nabla_W Z, X) = -(X\mu)g(Z, W). \tag{3.26}$$

Let M_T and M_θ denote the leaves of D and D^θ respectively. If h^0 denotes the second fundamental form of the immersion of M_θ into M, then (3.26) can be written as

$$g(h^0(Z, W), X) = -(X\mu)g(Z, W) = -g(\nabla\mu, X)g(Z, W).$$

Therefore, we obtain

$$h^0(Z, W) = -g(Z, W)\nabla\mu.$$

This implies that, each leaf M_θ of D^θ is totally umbilical in M. Also, as $Z\mu = 0$ for each $Z \in D^\theta$, the mean curvature vector $\nabla\mu$ is parallel on M_θ, i.e., M_θ is an extrinsic sphere in M. Hence, by virtue of a result in [13], which states that, "*If the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = E_1 \oplus E_0$ of non-trivial vector sub-bundles such that E_1 is auto-parallel and its orthogonal complement E_0 is spherical, then the manifold M is locally isometric to the warped product $N_1 \times_f N_0$*" we find that M is locally a semi-slant warped product submanifold of $\bar{M}$. $\square$

In particular if M is a CR-submanifold, then $PZ = 0$ for any $Z \in D^\perp$, therefore by (2.8), $(\bar{\nabla}_Z P)Z = -P\nabla_Z Z$ implying that $g((\bar{\nabla}_Z P)Z, W) = 0$, for all $Z, W \in D^\perp$. That shows that $C(\bar{\nabla}_Z P)Z = 0$. Hence by Lemma 3.5, $C(A_{FCU}CU + th(CU, CU)) = 0$. In view of these observations, for a CR-submanifold of a nearly Kaehler manifold, equation (3.23) reduces to

$$(\bar{\nabla}_U P)U = \|CU\|^2 P\nabla\mu + (PBU\mu)CU.$$

Hence, we deduce that

Corollary 3.8. *Let M be a CR-submanifold of a nearly Kaehler manifold with involutive distributions D and $D^{\perp}$. Then M is locally a CR-warped product submanifold if and only if there exists a C^{∞}-function μ on M with $Z\mu = 0$ for each $Z \in D^{\perp}$ satisfying*

$$(\bar{\nabla}_U P)U = \|CU\|^2 J\nabla\mu + (PU\mu)CU,$$

for each $U \in TM$.

The above result is proved in [1].

In terms of F, we have the following

Theorem 3.9. *Let M be a semi-slant submanifold of a nearly Kaehler manifold $\bar{M}$ with both the distributions D and D^{θ} being involutive and $(\bar{\nabla}_Z F)W$ lies in ν for each $Z, W \in D^{\theta}$. Then M is locally a semi-slant warped product submanifold if and only if*

$$[(\bar{\nabla}_U F)V]_{FD^{\theta}} = \frac{8}{3} cosec^2\theta(PBU\mu)FPCV - (BV\mu)FCU \tag{3.27}$$

for each $U, V \in TM$, where μ is a C^{∞} function on M such that $W\mu = 0$ for all $W \in D^{\theta}$.

Proof. For any $U, V \in TM$, we may write

$$(\bar{\nabla}_U F)V = (\bar{\nabla}_{BU} F)BV + (\bar{\nabla}_{BU} F)CV + (\bar{\nabla}_{CU} F)BV + (\bar{\nabla}_{CU} F)CV. \tag{3.28}$$

If $M = M_T \times_f M_\theta$ is a semi-slant warped product submanifold of a nearly Kaehler manifold $\bar{M}$, then taking account of the fact that M_T is totally geodesic in M, while using formula (2.9), we obtain that

$$(\bar{\nabla}_X F)Y = 0, \tag{3.29}$$

for each $X, Y \in TM$.

Now, by (2.2), (3.12) and (2.9), we have

$$(\bar{\nabla}_X F)Z = 2fh(X, Z) - h(X, PZ) - h(PX, Z) + F\nabla_Z X$$

for any $X \in TM_T$ and $Z \in TM_\theta$. Therefore, on using (3.12), (2.18) and (2.14), we get

$$\begin{aligned} g((\bar{\nabla}_X F)Z, FW) &= 2g(h(X, Z), FPW) - g(h(X, PZ), FW) \\ &\quad - g(h(PX, Z), FW) + Sin^2\theta g(\nabla_Z X, W), \end{aligned}$$

for any $W \in TM_\theta$. Now on applying Lemma 3.6, the above equation takes the form

$$g((\bar{\nabla}_X F)Z, FW) = \frac{8}{3}(PX \ln f)g(PZ, W).$$

Now, on taking account of (2.14), it follows from the above equation that

$$(\bar{\nabla}_X F)Z\Big|_{FD^{\theta}} = \frac{8}{3} cosec^2\theta(PX \ln f)FPZ. \tag{3.30}$$

Further,

$$g((\bar{\nabla}_Z F)X, FW) = -g(F\nabla_Z X, FW) \tag{3.31}$$
$$= -(Xlnf)g(FZ, FW)$$

As $(\bar{\nabla}_Z F)W$ lies in ν for any $Z, W \in TM_\theta$, we have

$$g((\bar{\nabla}_Z F)W, FW') = 0 \tag{3.32}$$

for any $W' \in TM_\theta$.

Substituting from (3.29), (3.30), (3.31) and (3.32) into (3.28), we obtain

$$[(\bar{\nabla}_U F)V]_{FD^\theta} = \frac{8}{3} cosec^2\theta(PBU \ln f)FPCV - (BV \ln f)FCU.$$

Conversely, suppose that M is a semi-slant submanifold of a nearly Kaehler manifold with canonical distributions D and D^θ being involutive on M such that (3.27) holds, then by (2.9) and (3.27)

$$(\bar{\nabla}_X F)X = 0,$$

for each $X \in D$. That means $\nabla_X Y + \nabla_Y X \in D$, for each $X, Y \in D$. Since, D is assumed to be involutive, the above observation yields that D is parallel i.e., the leaves of D are totally geodesic in M.

Now for $X \in D$ and $Z \in D^\theta$, by (3.27)

$$g((\bar{\nabla}_Z F)X, FW) = -X\mu g(FZ, FW)$$

for any $W \in D^\theta$. Thus, we get

$$g(F\nabla_Z X, FW) = X\mu g(FZ, FW)$$

or,

$$g(\nabla_Z X, W) = (X\mu)g(Z, W) \tag{3.33}$$

Let M_θ be a leaf of D^θ. If h^0 denotes the second fundamental form of M_θ into M, then in view of Gauss formula, (3.33) and (2.20) imply that

$$h^0(Z, W) = g(Z, W)\nabla\mu.$$

This shows that M_θ is totally umbilical in M with mean curvature vector $\nabla\mu$. Further as $W\mu = 0$ for each $W \in D^\theta$, it follows that leaves of D^θ are extrinsic spheres in M. Hence M is locally a semi-slant warped product submanifold $M_T \times_f M_\theta$ of $\bar{M}$ in view of the theorem by Heipko [13] as stated in the proof of Theorem 3.7. □

If M is a CR-submanifold of a nearly Kaehler manifold, then the first term in the right hand side of (3.27) vanishes as in this case $PCV = 0$ and we conclude that

Corollary 3.10. *If M is a CR-submanifold of a nearly Kaehler manifold with canonical distributions involutive and $(\bar{\nabla}_Z F)W$ lies in ν for each $Z, W \in D^\perp$, then M is locally a CR-warped product submanifold if and only if there exists a C^∞ function μ on M with $W\mu = 0$ for all $W \in D^\perp$ such that*

$$g((\bar{\nabla}_U F)V, JW) = (BV\mu)g(CU, W),$$

for each $U, V \in TM$.

The above result agrees with the characterization for a CR-warped product submanifold obtained in [1].

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