Some basic inequalities for submanifolds of nearly quasi-constant curvature manifolds

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Abstract. Certain basic inequalities involving the squared mean curvature and one of the Ricci curvature, the scalar curvature and the sectional curvature for a submanifold of quasi-constant curvature manifolds and nearly quasi-constant curvature manifolds are obtained. Equality cases are also discussed.

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1 Introduction

In [2], B.-Y. Chen recalled that one of the basic interests of submanifold theory is to establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. The main extrinsic invariant is the squared mean curvature and the main intrinsic invariants include the classical curvature invariants namely the Ricci curvature and the scalar curvature. There are also other important modern intrinsic invariants of (sub)manifolds introduced by B.-Y. Chen [6].

In the literature, we find a lot of work done in establishing basic inequalities involving the squared mean curvature and one of the classical curvature invariants namely the scalar curvature, the sectional curvature and the Ricci curvature for different kind of submanifolds of real space forms and complex space forms. First results in these directions were proved by B.-Y. Chen in [2], [3] and [4]. To prove these kind of results, one needs an extra condition on the Riemann curvature tensor of the ambient manifold, like its constancy in the case of real space forms and the constancy of holomorphic sectional curvature in the case of complex space forms.

In [4], B.-Y. Chen extends the notion of Ricci curvature to k-Ricci curvature ($2 \leq k \leq n$) in an n-dimensional Riemannian manifold. Since the notion of k-Ricci curvatures involves curvature functions that "interpolate" between the sectional curvature ($k = 2$) and the Ricci curvature ($k = n - 1$), it is natural to ask to study the role of k-Ricci curvatures in finding such inequalities for any Riemannian submanifolds. Motivated by a result of B.-Y. Chen [4], a basic inequality, involving the

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Ricci curvature and the squared mean curvature of the submanifold of any Riemannian manifolds, was proved recently [12]. The goal was achieved by use of the concept of k-Ricci curvature. Similarly, in [11] the authors obtained inequalities for Riemannian submanifolds, involving the squared mean curvature and one of scalar curvature and sectional curvature.

On the other hand, B.-Y. Chen and K. Yano [8] generalized the notion of real space forms to quasi-constant curvature manifolds, which was further generalized by U.C De and A.K. Gazi to nearly quasi-constant curvature manifolds. Then in [15] and [16], several inequalities were obtained for submanifolds of quasi-constant curvature manifolds and nearly quasi-constant curvature manifolds.

In this paper, we obtain basic inequalities for a submanifold of nearly quasi-constant curvature manifold, and in particular, of quasi-constant curvature manifold involving the squared mean curvature and one of the intrinsic invariants namely the Ricci curvature, the scalar curvature and the sectional curvature of the submanifold. The paper is organized as follows. In section 2, we recall the definitions of the Ricci curvature, the k-Ricci curvature, the scalar curvature, and the normalized scalar curvature. Then we give basic equations and definitions for a submanifolds. Section 3 contains a brief account of quasi-constant curvature manifolds and nearly quasi constant curvature manifolds. In section 4, first we recall Chen-Ricci inequality for submanifolds of a Riemannian manifold, which involves the Ricci curvature and the squared mean curvature of the submanifold. Then applying this, we obtain Chen-Ricci inequalities for submanifolds of nearly quasi-constant curvature manifold and quasi-constant curvature manifold. In section 5, first we recall scalar inequalities for submanifolds of Riemannian manifolds, which involve the squared mean curvature and the scalar curvature of the submanifold. Then applying this, we obtain scalar inequalities for submanifolds of nearly quasi-constant curvature manifold and quasi-constant curvature manifold. In section 6, first we recall a basic inequality involving sectional curvatures and the squared mean curvature for submanifolds of a Riemannian manifold, then by applying this inequality we find a similar inequality for submanifolds of nearly quasi-constant curvature manifold and quasi-constant curvature manifold.

2 Preliminaries

Let M be an n-dimensional Riemannian manifold equipped with a Riemannian metric g. The inner product of the metric g is denoted by $\langle , \rangle$. We denote the set of unit vectors in T_pM by T_p^1M; thus

$$T_p^1M = \{X \in T_pM \mid \langle X, X \rangle = 1\}.$$

Let $\{e_1, \ldots, e_n\}$ be any orthonormal basis for T_p^1M. For a fixed $i \in \{1, \ldots, n\}$, the *Ricci curvature* of e_i, denoted $\mathrm{Ric}\,(e_i)$, is defined by

$$\mathrm{Ric}\,(e_i) = \sum_{j \neq i}^{n} K_{ij}. \tag{2.1}$$

where K_{ij} denotes the sectional curvature of the plane section spanned by e_i and e_j.

Let Π_k be a k-plane section of T_pM and X a unit vector in Π_k. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of Π_k such that $e_1 = X$. The Ricci curvature Ric_{Π_k} of Π_k at X is defined by [4]

$$\mathrm{Ric}_{\Pi_k}(X) = K_{12} + K_{13} + \cdots + K_{1k}. \tag{2.2}$$

$\mathrm{Ric}_{\Pi_k}(X)$ is called a *k-Ricci curvature*. The scalar curvature $\tau(\Pi_k)$ of the k-plane section Π_k is given by

$$\tau(\Pi_k) = \sum_{1\leq i<j\leq k} K_{ij}, \tag{2.3}$$

where $\{e_1, \ldots, e_k\}$ is any orthonormal basis of the k-plane section Π_k. The scalar curvature $\tau(p)$ of M at p is identical with the scalar curvature of the tangent space T_pM of M at p, that is, $\tau(p) = \tau(T_pM)$. If Π_2 is a plane section, $\tau(\Pi_2)$ is simply the sectional curvature $K(\Pi_2)$ of Π_2. Geometrically, $\tau(\Pi_k)$ is the scalar curvature of the image $\exp_p(\Pi_k)$ of Π_k at p under the exponential map at p. We define the normalized scalar curvature $\tau_N(\Pi_k)$ of Π_k by

$$\tau_N(\Pi_k) = \frac{2\tau(\Pi_k)}{k(k-1)}. \tag{2.4}$$

The normalized scalar curvature at p is defined as [3]

$$\tau_N(p) = \frac{2\tau(p)}{n(n-1)}. \tag{2.5}$$

Then, we see that

$$\tau_N(p) = \tau_N(T_pM).$$

Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$ equipped with a Riemannian metric $\widetilde{g}$. We use the inner product notation $\langle , \rangle$ for both the metrics $\widetilde{g}$ of $\widetilde{M}$ and the induced metric g on the submanifold M.

The Gauss and Weingarten formulas are given by, respectively,

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \qquad \text{and} \qquad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are, respectively, the Riemannian, induced Riemannian and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M, respectively, and σ is the second fundamental form related to the shape operator A by $\langle \sigma(X, Y), N\rangle = \langle A_N X, Y\rangle$. The equation of Gauss is given by

$$\begin{aligned} R(X,Y,Z,W) &= \widetilde{R}(X,Y,Z,W) + \langle \sigma(X,W), \sigma(Y,Z)\rangle \\ &\quad - \langle \sigma(X,Z), \sigma(Y,W)\rangle \end{aligned} \tag{2.6}$$

for all $X, Y, Z, W \in TM$, where $\widetilde{R}$ and R are the curvature tensors of $\widetilde{M}$ and M, respectively.

The mean curvature vector H is given by $H = \frac{1}{n}\,\mathrm{trace}(\sigma)$. The submanifold M is *totally geodesic* in $\widetilde{M}$ if $\sigma = 0$, and *minimal* if $H = 0$. If $\sigma(X, Y) = g(X, Y)\, H$ for all $X, Y \in TM$, then M is *totally umbilical.*

The *relative null space* of M at p is defined by [4]

$$\mathcal{N}_p = \{X \in T_pM \mid \sigma(X,Y) = 0 \text{ for all } Y \in T_pM\},$$

which is also known as the *kernel of the second fundamental form* at p [5].

Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space T_pM and e_r $(r = n+1, \dots, m)$ belongs to an orthonormal basis $\{e_{n+1}, \dots, e_m\}$ of the normal space $T_p^{\perp}M$. We put

$$\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r\rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} \langle \sigma(e_i, e_j), \sigma(e_i, e_j)\rangle .$$

Let K_{ij} and $\widetilde{K}_{ij}$ denotes the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\widetilde{M}$, respectively. Thus, K_{ij} and $\widetilde{K}_{ij}$ are the intrinsic and extrinsic sectional curvature of the $\mathrm{Span}\{e_i, e_j\}$ at p. In view of the equation (2.6) of Gauss, we have

$$K_{ij} = \widetilde{K}_{ij} + \sum_{r=n+1}^{m} \left(\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2\right). \tag{2.7}$$

From (2.7) it follows that

$$2\tau(p) = 2\widetilde{\tau}(T_pM) + n^2 \|H\|^2 - \|\sigma\|^2 , \tag{2.8}$$

where

$$\widetilde{\tau}(T_pM) = \sum_{1 \le i < j \le n} \widetilde{K}_{ij}$$

denote the scalar curvature of the n-plane section T_pM in the ambient manifold $\widetilde{M}$. Thus, $\tau(p)$ and $\widetilde{\tau}(T_pM)$ are the intrinsic and extrinsic scalar curvature of the submanifold at p, respectively.

3 Nearly quasi constant curvature manifolds

An m-dimensional Riemannian manifold with constant sectional curvature c, denoted $R^m(c)$, is called a real space form, and its Riemann curvature tensor is then given by

$$R(X,Y,Z,W) = c\,\{\langle Y, Z\rangle \langle X, W\rangle - \langle X, Z\rangle \langle Y, W\rangle\}, \tag{3.1}$$

for all $X, Y, Z, W \in TR^m(c)$. The model spaces for real space forms are the Euclidean spaces $(c = 0)$, the spheres $(c > 0)$, and the hyperbolic spaces $(c < 0)$.

In 1972, B.-Y. Chen and K. Yano [8] introduced the notion of a Riemannian manifold of quasi-constant curvature. A conformally flat Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is of quasi-constant curvature if its curvature $\widetilde{R}$ satisfies

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = c\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\} + d\{\langle X,W\rangle A(Y)A(Z)\\ - \langle X,Z\rangle A(Y)A(W) + A(X)A(W)\langle Y,Z\rangle - A(X)A(Z)\langle Y,W\rangle\},\end{aligned} \tag{3.2}$$

for all $X,Y,Z,W \in T\widetilde{M}$, where c, d are C^∞ functions and A is a 1-form.

In 1968, Gh. Vranceanu [17] defined the notion of almost constant curvature by the same expression as (3.2) without assuming the condition of being conformally flat. Later, in 1986, A.L. Mocanu [13] showed that the manifold introduced by Chen and Yano coincide with the manifolds introduced by Gh. Vranceanu. Thus a Riemannian manifold is of quasi-constant curvature if its curvature satisfies the relation (3.2), and it becomes conformally flat.

It is known that a quasi-constant curvature manifold $(\widetilde{M}, \widetilde{g})$ is a quasi-Einstein manifold ([1], [14, Eq. (4.7)]) as its Ricci tensor satisfies

$$\widetilde{\mathrm{Ric}}(X,Y) = c\,g\,(X,Y) + d\,A\,(X)\,A\,(Y)\,, \qquad X,Y \in T\widetilde{M}.$$

If $d = 0$ then a quasi-constant curvature manifold reduces to a real space form and a quasi-Einstein manifold becomes merely an Einstein manifold.

In [10], U.C. De and A.K. Gazi generalized the notion of quasi-constant curvature manifold to a nearly quasi-constant curvature manifold, where the curvature tensor $\widetilde{R}$ satisfies

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = c\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\} + d\{\langle X,W\rangle B(Y,Z)\\ - \langle X,Z\rangle B(Y,W) + B(X,W)\langle Y,Z\rangle - B(X,Z)\langle Y,W\rangle\},\end{aligned} \tag{3.3}$$

for all $X,Y,Z,W \in T\widetilde{M}$, where c, d are C^∞ functions and B is a symmetric tensor field of type $(0,2)$. If $d = 0$ then the manifold reduces to a real space form. A non-flat Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of dimension > 2 is a nearly quasi-Einstein manifold [9] if its Ricci tensor $\widetilde{\mathrm{Ric}}$ satisfies

$$\widetilde{\mathrm{Ric}}(X,Y) = c\,\widetilde{g}\,(X,Y) + d\,B\,(X,Y)\,,$$

for all $X,Y \in T\widetilde{M}$, where c, d are some C^∞ functions such that $d \neq 0$ and B is a non-zero symmetric tensor field of type $(0,2)$. Also, it is known that a nearly quasi-constant curvature manifold is always nearly quasi-Einstein. We note that if $B = A \otimes A$ for some 1-form A, then a nearly quasi-constant curvature manifold reduces to a quasi-constant curvature manifold.

4 Chen-Ricci inequalities

First, we recall the Chen-Ricci inequality (4.1) in the following:

Theorem 4.1. ([12, Theorem 3.1],[11, Theorem 6.1]) *Let M be an n-dimensional submanifold of a Riemannian manifold. Then, the following statements are true.*

(a) *For $X \in T_p^1 M$, it follows that*

$$\operatorname{Ric}(X) \leq \frac{1}{4} n^2 \|H\|^2 + \widetilde{\operatorname{Ric}}_{(T_p M)}(X), \tag{4.1}$$

where $\widetilde{\operatorname{Ric}}_{(T_p M)}(X)$ is the n-Ricci curvature of $T_p M$ at $X \in T_p^1 M$ with respect to the ambient manifold $\widetilde{M}$.

(b) *The equality case of* (4.1) *is satisfied by $X \in T_p^1 M$ if and only if*

$$\begin{cases} \sigma(X, Y) = 0, & \text{for all } Y \in T_p M \text{ orthogonal to } X, \\ 2\sigma(X, X) = nH(p). \end{cases} \tag{4.2}$$

(c) *The equality case of* (4.1) *holds for all $X \in T_p^1 M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

We immediately have the following

Corollary 4.2. ([11, Corollary 6.2]) *Let M be an n-dimensional submanifold of a Riemannian manifold. Then for $X \in T_p^1 M$ any two of the following three statements imply the remaining one.*

(a) *X satisfies the equality case of* (4.1).

(b) *$H(p) = 0$.*

(c) *X belongs to the relative null space of M at p defined by* [4]

$$\mathcal{N}_p = \{X \in T_p M \mid \sigma(X, Y) = 0 \text{ for all } Y \in T_p M\},$$

which is also known as the kernel of the second fundamental form at p [5].

We need the following Lemma.

Lemma 4.3. *Let M be an n-dimensional submanifold of an m-dimensional manifold $\widetilde{M}$ of nearly quasi-constant curvature. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space $T_p M$ and e_r belongs to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space $T_p^\perp M$. Then*

$$\widetilde{K}_{ij} = c + d\,\{B(e_i, e_i) + B(e_j, e_j)\}, \tag{4.3}$$

$$\widetilde{\operatorname{Ric}}_{(T_p M)}(e_j) = (n-1)\,c + d \operatorname{trace}(B|_M) + (n-2) d\, B(e_j, e_j), \tag{4.4}$$

$$\widetilde{\tau}(T_p M) = \frac{1}{2} n(n-1)\,c + (n-1)\,d \operatorname{trace}(B|_M). \tag{4.5}$$

Proof. Equation (4.3) follows from (3.3). Using $\widetilde{\operatorname{Ric}}_{(T_p M)}(e_i) = \sum_{j \neq i}^{n} \widetilde{K}_{ij}$, we obtain (4.4) from (4.3). Next, using $2\widetilde{\tau}(T_p M) = \sum_{i=1}^{n} \widetilde{\operatorname{Ric}}_{(T_p M)}(e_i)$, we get (4.5) from (4.4). □

Now, we establish the Chen-Ricci inequality for a submanifold of a nearly quasi-constant curvature manifold.

Theorem 4.4. *Let M be an n-dimensional submanifold of an m-dimensional manifold $\widetilde{M}$ of nearly quasi-constant curvature. Then the following statements are true.*

(a) *If $X \in T_p^1 M$, then*

$$\operatorname{Ric}(X) \leq \frac{1}{4} n^2 \|H\|^2 + (n-1)c + d\left(\operatorname{trace}\left(B|_M\right) + (n-2)B(X,X)\right). \tag{4.6}$$

(b) *If $H(p) = 0$, then $X \in T_p^1 M$ satisfies the equality cases of the inequality* (4.6) *if and only if $X \in \mathcal{N}_p$.*

(c) *The equality cases of the inequality* (4.6) *is satisfied for all $X \in T_p^1 M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. Using (4.4) in the Chen-Ricci inequality (4.1), we find the inequality (4.6). Rest of the proof is straightforward. □

Using $B = A \otimes A$ in (4.6), we get the following

Corollary 4.5. *Let M be an n-dimensional submanifold of an m-dimensional manifold $\widetilde{M}$ of quasi-constant curvature. Then the following statements are true.*

(a) *If $X \in T_p^1 M$, then*

$$\begin{aligned} \operatorname{Ric}(X) \ \leq \ & \frac{1}{4} n^2 \|H\|^2 + (n-1)c \\ & + d\ \left(\operatorname{trace}(A \otimes A) + (n-2)A(X)\,A(X)\right). \end{aligned} \tag{4.7}$$

(b) *If $H(p) = 0$, then $X \in T_p^1 M$ satisfies the equality cases of the inequality* (4.7) *if and only if $X \in \mathcal{N}_p$.*

(c) *The equality cases of the inequality* (4.7) *is satisfied for all $X \in T_p^1 M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Using $d = 0$ in (4.6), we get the following Chen-Ricci inequality for submanifolds of a real space form.

Corollary 4.6. [4, Theorem 4] *Let M be an n-dimensional submanifold of a real space form $R^m(c)$. Then,*

(a) *For $X \in T_p^1 M$, we have*

$$\|H\|^2 \geq \frac{4}{n^2} \left\{\operatorname{Ric}(X) - (n-1)\,c\right\}. \tag{4.8}$$

(b) *If $H(p) = 0$, then $X \in T_p^1 M$ satisfies the equality case of* (4.8) *if and only if $X \in \mathcal{N}_p$.*

(c) *The equality case of* (4.8) *holds for all* $X \in T_p^1 M$ *if and only if either* p *is a totally geodesic point or* $n = 2$ *and* p *is a totally umbilical point.*

For an immersion $x : M^n \to E^m$ from an n-dimensional manifold into a Euclidean m-space, the Beltramy formula is $\triangle x = -nH$; thus M is minimal if and only if all the coordinate functions are harmonic functions relative to the induced metric on M. If $x : M^n \to E^m$ is a minimal immersion, then M is non-compact. From Corollary 4.6, we see that if M is a minimal submanifold of a Euclidean space, then the Ricci tensor of M is negative semi-definite. The spherical hypercylinder $S^2(r) \times E^1$ and round hypercone in the Euclidean 4-space E^4 satisfy the equality case of the equation corresponding to (4.8) [4]. We also have the following

Example 4.1. *([12, Example 4.1])* Let N^{n-1} be a $(n-1)$-dimensional minimal submanifold of a Euclidean $(m-1)$-space E^{m-1}. Then the product submanifold $M^n = N^{n-1} \times E^1$ is a minimal submanifold of E^m satisfying $\mathrm{Ric} = 0$.

5 Scalar curvature

In view of (2.8) it follows that for an n-dimensional submanifold M of a Riemannian manifold

$$\tau(p) \le \frac{1}{2} n^2 \|H\|^2 + \widetilde{\tau}(T_p M) \tag{5.1}$$

with equality if and only if M is totally geodesic. As an improved version of the inequality (5.1), we have the following

Theorem 5.1. [11, Theorem 4.2] *For an n-dimensional submanifold M in a Riemannian manifold, at each point $p \in M$, we have*

$$\tau(p) \le \frac{n(n-1)}{2} \|H\|^2 + \widetilde{\tau}(T_p M) \tag{5.2}$$

with equality if and only if p is a totally umbilical point.

In view of Theorem 5.1, we have

Theorem 5.2. [11, Theorem 4.4] *For an n-dimensional submanifold M of a Riemannian manifold, at each point $p \in M$, we have*

$$\tau_N(p) \le \|H\|^2 + \widetilde{\tau}_N(T_p M), \tag{5.3}$$

where τ_N is the normalized scalar curvature of M at p, and $\widetilde{\tau}_N(T_p M)$ denotes the normalized scalar curvature of $T_p M$ in the ambient manifold $\widetilde{M}$. The equality in (5.3) *holds if and only if p is a totally umbilical point.*

Theorems 5.1 and 5.2 provide the following obstructions for a minimal immersion into a Riemannian manifold.

Theorem 5.3. [11, Theorem 4.5] *Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$. If the intrinsic scalar curvature (resp. intrinsic normalized scalar curvature) of M is greater than the extrinsic scalar curvature (resp. extrinsic normalized scalar curvature), then M admits no minimal immersion into $\widetilde{M}$.*

Now, we obtain the scalar inequalities for a submanifold of a nearly quasi-constant curvature manifold.

Theorem 5.4. *For an n-dimensional submanifold M of a nearly quasi-constant curvature manifold $\widetilde{M}$, at each point $p \in M$, we have*

$$\tau(p) \leq \frac{n(n-1)}{2}\|H\|^2 + \frac{1}{2}n(n-1)c + (n-1)d\operatorname{trace}(B|_M), \tag{5.4}$$

$$\tau_N(p) \leq \|H\|^2 + c + \frac{2}{n}d\operatorname{trace}(B|_M), \tag{5.5}$$

with equality if and only if p is a totally umbilical point.

Proof. Using (4.5) in (5.2), we get (5.4). From (5.4) we get (5.5). □

Remark 5.1. In [16], the authors proved an inequality (4.1) in Theorem 4.1 without any information about equality case. However, one can see that the inequality (5.5) is exactly the inequality (4.1) in Theorem 4.1 of [16].

Using $B = A \otimes A$ in (5.4) and (5.5), we get the following

Corollary 5.5. *Let M be an n-dimensional submanifold of an m-dimensional manifold $\widetilde{M}$ of quasi-constant curvature. Then for each point $p \in M$, we have*

$$\tau(p) \leq \frac{n(n-1)}{2}\|H\|^2 + \frac{1}{2}n(n-1)c + (n-1)d\operatorname{trace}((A\otimes A)|_M), \tag{5.6}$$

$$\tau_N(p) \leq \|H\|^2 + c + \frac{2}{n}d\operatorname{trace}((A\otimes A)|_M), \tag{5.7}$$

with equality if and only if p is a totally umbilical point.

Remark 5.2. In [15], the author proved an inequality (4.1) in Theorem 4.1 without any information about equality case. However, one can see that the inequality (5.7) is exactly the inequality (4.1) in Theorem 4.1 of [15].

Using $d = 0$ in (5.5), we get the following scalar inequality for submanifolds of a real space form.

Corollary 5.6. [3, Lemma 1] *Let M be an n-dimensional submanifold of a real space form $R^m(c)$. Then at each point $p \in M$, the normalized scalar curvature τ_N of M satisfies*

$$\tau_N(p) \leq \|H\|^2 + c,$$

with equality holding if and only if p is a totally umbilical point. Consequently, if the normalized scalar curvature of M is greater than c, then M admits no minimal immersion into the real space form $R^m(c)$.

6 Sectional curvature

We recall the following Theorem, where it is established an inequality for submanifolds M of a Riemannian manifold involving intrinsic invariants, namely the sectional curvature and the scalar curvature of M and the main extrinsic invariant, namely the squared mean curvature.

Theorem 6.1. ([11, Theorem 5.2]) *Let M be an n-dimensional $(n \geq 3)$ submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$. Then, for each point $p \in M$ and each plane section $\Pi_2 \subset T_pM$, we have*

$$\tau - K(\Pi_2) \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \widetilde{\tau}(T_pM) - \widetilde{K}(\Pi_2). \tag{6.1}$$

The equality in (6.1) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \dots, e_m\}$ of $T_p^{\perp}M$ such that **(a)** $\Pi_2 = \mathrm{Span}\{e_1, e_2\}$ *and* **(b)** *the forms of shape operators $A_r \equiv A_{e_r}$, $r = n+1, \dots, m$, become*

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)\, I_{n-2} \end{pmatrix}, \tag{6.2}$$

$$A_r = \begin{pmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \qquad r \in \{n+2, \dots, m\}. \tag{6.3}$$

Now, we obtain an inequality for section curvatures for a submanifold of a nearly quasi-constant curvature manifold.

Theorem 6.2. *Let M be an n-dimensional $(n \geq 3)$ submanifold of a nearly quasi-constant curvature manifold $\widetilde{M}$. Then, for each point $p \in M$ and each plane section $\Pi_2 = \mathrm{Span}\{e_1, e_2\} \subset T_pM$, we have*

$$\begin{aligned} \tau - K(\Pi_2) \quad &\leq \quad \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}\,(n+1)(n-2)c \\ &\quad + d\, \left((n-1)\,\mathrm{trace}\,(B|_M) - \mathrm{trace}\,(B|_{\Pi_2})\right). \end{aligned} \tag{6.4}$$

The equality in (6.4) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \dots, e_m\}$ of $T_p^{\perp}M$ such that the shape operators $A_r \equiv A_{e_r}$, $r = n+1, \dots, m$, become of forms (6.2) and (6.3).

Proof. Using (4.5) and (4.3) in (6.1), we get (6.4). □

Remark 6.1. The inequality (6.4) is same as the inequality (3.1) in Theorem 3.1 of [16]. Here the proof is very short as compare to that of Theorem 3.1 of [16].

Using $B = A \otimes A$ in (6.4) we get the following

Theorem 6.3. *Let M be an n-dimensional $(n \geq 3)$ submanifold of a quasi-constant curvature manifold $\widetilde{M}$. Then, for each point $p \in M$ and each plane section $\Pi_2 = \operatorname{Span}\{e_1, e_2\} \subset T_pM$, we have*

$$\begin{aligned} \tau - K(\Pi_2) \quad \leq \quad & \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}\,(n+1)(n-2)c \\ & + d\,((n-1)\operatorname{trace}((A \otimes A)|_M) - \operatorname{trace}((A \otimes A)|_{\Pi_2}))\,. \end{aligned} \tag{6.5}$$

The equality in (6.5) *holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of $T_p^{\perp}M$ such that the shape operators $A_r \equiv A_{e_r}$, $r = n+1, \ldots, m$, become of forms* (6.2) *and* (6.3).

Remark 6.2. The inequality (6.5) is exactly the inequality (3.2) in Theorem 3.1 of [15].

In 1993, B.-Y. Chen [2] introduced a new type Riemannian invariant known as *Chen invariant* for a Riemannian manifold M as follows:

$$\delta_M(p) = \tau(p) - (\inf K)(p).$$

One can find many results concerning the δ-invariants, inequalities and their applications on submanifolds of a (pseudo) Riemannian manifold in [7].

Now, using $d = 0$ in (6.4), we get the following sharp inequality for submanifolds M in a real space form involving intrinsic invariant, namely Chen invariant of M and the main extrinsic invariant, namely the squared mean curvature as follows:

Corollary 6.4. ([2, Lemma 3.2]) *Let M be an n-dimensional $(n \geq 3)$ submanifold of a real space form $R^m(c)$. Then*

$$\delta_M \equiv \tau - \inf K \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}\,(n+1)(n-2)c.$$

Equality holds if and only if, with respect to suitable orthonormal frame fields $e_1, \ldots, e_n$, $e_{n+1}, \ldots, e_m$, the forms of the shape operators $A_r = A_{e_r}$, $r = n+1, \ldots, m$ become (6.2) *and* (6.3).

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