On the flag curvatures on projectivized tangent bundles deduced from contact metric structures

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Abstract. In [5] and [6], Sasaki type metric and more generalized Riemannian metric ($h\nu$ metric $\tilde{g}$ [7]) were considered as a Riemannian metric constructing a contact metric structure deduced from the contact structure on the projectivized tangent bundle $PTM$. In this paper, we consider $h\nu$ metric $\tilde{g}$ under the certain conditions and study the flag curvature on $PTM$.

Key words: Finsler manifold; projectivized tangent bundle; contact structure; contact metric structure.

1 Preliminaries

A Finsler manifold $M$ has a tangent bundle $\pi : TM \to M$. From $TM$ we obtain the projectivized tangent bundle of $M$, $PTM$, by identifying the non zero vectors differing from each other by a real factor. Geometrically $PTM$ is the space of line elements on $M$.

The $x^i, y^i$ are local coordinates on $TM$. They are also local coordinates on $PTM$ with $y^i$ being homogeneous coordinates (determined up to a real factor). We can consider $PTM$ as the base manifold of the vector bundle $P^*TM$, pulled back with the canonical projection map $p : PTM \to M$ defined by $p(x^i, y^i) = (x^i)$. The fibers of $P^*TM$ are the vector spaces of dimension $m$ and the base manifold $PTM$ is of dimension $2m - 1$.

A differential form on $PTM$ can be represented as one on $TM$ provided the latter is invariant under rescaling in the $y^i$ and yields zero when contracted with $y^i \frac{\partial}{\partial y^i}$. Our differential forms on $PTM$ will be so represented, and exterior differentiation on $PTM$ will be obtained formal differentiation on $TM$.

the Chern-Rund connection coefficients $N^i_j$ and the associated local dual adapted forms $\delta y^j$ are respectively defined as:

$$N^i_j = \frac{1}{2} \frac{\partial G^i}{\partial y^j}, \quad \delta y^j = dy^j + N^i_j dx^k,$$
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where

\[ G^i = g^i j \left( y^s \frac{\partial^2 (1/2 F^2)}{\partial y^l \partial x^s} - \frac{\partial (1/2 F^2)}{\partial x^l} \right). \]

Then the corresponding orthonormal vectors in \( TPTM \) and the dual orthonormal vectors in \( T^*PTM \) are given by

\[ \hat{e}_i = p^j_i \frac{\partial}{\partial x^j} \iff \omega^i = q^j_i dx^j \quad (i = 1, \ldots, m) \]

and

\[ \hat{e}_{m+\alpha} = p^j_\alpha \frac{\partial}{\partial y^j} \iff \omega^\alpha = q^j_\alpha dy^j \quad (\alpha = 1, \ldots, m - 1) \quad (\omega^m = 0), \]

where

\[ \frac{\partial}{\partial x^l} = \frac{\partial}{\partial x^l} - N^j_i \frac{\partial}{\partial y^l}, \quad \frac{\delta}{\delta y^l} = F \frac{\partial}{\partial y^l}. \]

The local components \( A_{ijk} \) of the Cartan tensor are given by

\[ A_{ijk} := \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k}. \]

The slash and the semicolon of a (2,0)-type tensor \( h \) (resp. (0,2)-type tensor) are defined by (see [1])

\[ h_{ij} |_s := \delta_{xs} h_{ij} - h_{kj} \Gamma^k_{is} - h_{ik} \Gamma^k_{js}, \quad h_{ij} ;_s := F \frac{\partial}{\partial y^s} h_{ij}, \]

respectively

\[ h^{ij} |_s := \delta_{xs} h^{ij} + h^{kj} \Gamma^i_{ks} + h^{ik} \Gamma^j_{ks}, \quad h^{ij} ;_s := F \frac{\partial}{\partial y^s} h^{ij}, \]

where

\[ \Gamma^i_{jk} = g^{is} \left( \frac{\delta g_{js}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j} \right). \]

Also, the slash and the semicolon of a (1,0)-type tensor \( \ell \) (resp. (0,1)-type tensor) are defined by

\[ \ell_{ij} |_s := \delta_{xs} \ell_{ij} - \ell_{kj} \Gamma^k_{is}, \quad \ell_{ij} ;_s := F \frac{\partial}{\partial y^s} \ell_{ij}, \]

(resp.)

\[ \ell^i |_s := \delta_{xs} \ell^i + \ell^k \Gamma^i_{ks}, \quad \ell^i ;_s := F \frac{\partial}{\partial y^s} \ell^i. \]

The following lemma is well known ([1]);
Lemma 1.1. The covariant derivatives of the fundamental tensor \( g \) are given by
\[
g_{ij;s} := 0, \quad g_{ij:s} := 2A_{ij}s, \quad g^{ij}_{|s} = 0, \quad g^{ij:s} = -2A^{ij}_s,
\]
where
\[
A^{ij}_{hk} = g^{si}g^{tj}A_{stk}.
\]
Moreover we have
\[
\ell_{i|s} = 0, \quad \ell_{i:s} = g_{is} - \ell_i\ell_s, \quad \ell_{i|s} = 0, \quad \ell_{i:s} = \delta^i_s - \ell^i_s.
\]
(1.3) shows that both the distinguished section \( \ell := \hat{\epsilon}_m \) and the Hilbert form \( \omega \) are covariantly constant along horizontal directions. Their vertical derivatives are equal to suitable configurations of the angular metric \( \tilde{h}_{ij} \) (see [1]), where the angular metric \( \tilde{h}_{ij} \) denotes
\[
\tilde{h}_{ij} := g_{ij} - \ell_i\ell_j.
\]

The following lemma is well known ([1]);

Lemma 1.2. Lie Brackets among the \( \frac{\delta}{\delta x} \) and the \( F \frac{\partial}{\partial y} \) are given by
\[
\left[ \frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^l} \right] = -\ell^i R^i_{jkl} \delta y^j,
\]
\[
\left[ \frac{\delta}{\delta x^k}, F \frac{\partial}{\partial y^l} \right] = \left\{ \hat{A}^{i}_{kl} + \frac{\ell^i}{F} (F\ell_k)_{xl} - \ell^i N_{kl} \right\} \delta y^i,
\]
\[
\left[ F \frac{\partial}{\partial y^k}, F \frac{\partial}{\partial y^l} \right] = \ell_k \delta y^l - \ell_l \delta y^k.
\]
where
\[
N_{ij} := N^k_{ji}g_{ki}, \quad \hat{A}^{i}_{kl} = \hat{a}^{h}_{kl} = \hat{g}^{h}_{kl}g_{hk|s}l^s.
\]
Moreover (1.5) (resp. (1.6)) is rewritten to
\[
\left[ \frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^l} \right] = \frac{1}{F} \left( \frac{\delta}{\delta x^k} N^i_{k} - \frac{\delta}{\delta x^l} N^i_{l} \right) \delta y^i,
\]
(resp.
\[
\left[ \frac{\delta}{\delta x^k}, F \frac{\partial}{\partial y^l} \right] = \frac{1}{2} (G^i_{y^k y^l} \delta y^i = \Gamma^i_{kl} \delta y^i + \hat{A}^i_{kl} \delta y^i.
\]

Generally a \((2n+1)\)-dimensional manifold \( \overline{M} \) is said to have a contact structure and is called a contact manifold if it carries a global 1-form \( \eta \) such that
\[
\eta \wedge (d\eta)^n \neq 0
\]
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where the exponent denotes the $n$th exterior power. We call $\eta$ a contact form of $M$. Also, a structure tensors $(\phi, \xi, \eta, \overline{g})$ on $(2n+1)$-dimensional manifold $M$ is said to be an almost contact metric structure if a tensor field of type $(1,1)$ $\phi$, a vector field $\xi$, a $1$-form $\eta$ and a Riemannian metric $\overline{g}$ satisfy

$$\eta(\xi) = 1, \quad \phi^2 = -I + \xi \otimes \eta, \quad \phi \xi = 0, \quad \eta(\phi X) = 0,$$

for any vector fields $X$ and $Y$ on $M$ ([2], [3]).

Let $M$ be a $(2n+1)$ dimensional manifold with contact form $\eta$. Then, it is well known that on $M$ there exists an almost contact metric structure $(\phi, \xi, \eta, \overline{g})$ such that

$$\overline{g}(\phi X, Y) = d\eta(X, Y)$$

for any vector fields $X$ and $Y$ on $M$. Then $(\phi, \xi, \eta, \overline{g})$ is said to be a contact metric structure on $M$ ([10]).

Taking the exterior derivative Hilbert form $\omega^m$ on $PTM$, we have ([4])

$$d\omega^m = \omega^\alpha \wedge \omega^m_\alpha \quad (\alpha = 1, \ldots, m-1).$$

where $\omega^m_\alpha$ is

$$\omega^m_\alpha = -p^i_\alpha \frac{\partial^2 F}{\partial y^i \partial y^j} dy^j + p^i_\alpha \frac{\partial F}{\partial y^i} - y^j \frac{\partial^2 F}{\partial y^i \partial x^j} \omega^m_\alpha$$

$$+ p^i_\alpha p^j_\beta \frac{\partial^2 F}{\partial x^i \partial y^j} \omega^\beta_\alpha + \lambda_{\alpha \beta} \omega^\beta$$

(see [4] about $\lambda_{\alpha \beta}$).

Then, the following theorem holds good ([4]).

**Theorem 1.3.** $PTM$ has a contact structure with respect to the Hilbert form $\omega$.

On the manifold $PTM$, we consider a natural Riemannian metric (a Sasaki type metric on $TM \setminus \{0\}$)

$$g^s = g_{ij} dx^i \otimes dx^j + g_{ij} \frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F}.$$ 

For $\{\hat{e}_i (\text{resp.} \omega^i), \hat{e}_{m+\alpha} (\text{resp.} \omega^\alpha_m)\}$ in $TPTM$ (resp.$T^*PTM$), we can rewrite it as

$$g^s = \delta_{ij} \omega^i \otimes \omega^j + \delta_{m+\alpha} \omega^\alpha_m \otimes \omega^\beta_m,$$

(see [1]).

Then it is known that $PTM$ has a contact metric structure $(\phi, e_m, \omega, g^s)$, where $\phi$ is defined as follows ([5]):

$$\phi \hat{e}_\alpha = -\hat{e}_{m+\alpha}, \quad \phi \hat{e}_{m+\alpha} = \hat{e}_\alpha.$$
2 A Riemannian metric constructing the contact metric structure on $PTM$

We use the following symbols simply

$$\partial_x^i := \frac{\partial}{\partial x^i}, \quad \partial_y^i := \frac{\partial}{\partial y^i},$$

$$\delta_x^i := \delta_x^i = \partial_x^i - N_j^i \partial_y^j, \quad \delta_y^i := \delta_y^i = F \partial_y^i.$$

Now we consider the following metric on $TM \setminus \{0\}$ which is called an $h$-$v$ metric on $TM \setminus \{0\}$,

$$\tilde{g} := h_{ij} dx^i \otimes dx^j + v_{ij} \delta_y^i \otimes \delta_y^j.$$  

(cf. [9]). We define $g_{PTM}$ as the metric on $P TM$:

$$g_{PTM} := h_{ij} p_i^k \omega^k \otimes \omega^j + v_{ij} p_i^\alpha \omega^\alpha \otimes \omega^\beta,$$

that is $g_{PTM}$ is an $h$-$v$ metric on $P TM$.

Moreover, from now on, we consider $g_{PTM}$ as a Riemannian metric constructing a contact metric structure deduced from the contact structure $\omega \wedge (d\omega)^{m-1}$ on $PTM$.

Then we have a contact metric structure $(\phi, \xi, \eta, g_{PTM})$ on $P TM$, that is,

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta(X) = g_{PTM}(X, \xi),$$

$$g_{PTM}(\phi X, \phi Y) = g_{PTM}(X, Y) - \eta(X) \eta(Y),$$

$$g_{PTM}(X, \phi Y) = d\eta(X,Y).$$

From now on, we describe $g_{PTM}$ as $\tilde{g}$ simply and denote the Levi-Civita connection in $(T P TM, \tilde{g})$ as $\tilde{\nabla}$. First, we have the following lemma.

**Lemma 2.1.** ([6]) $\tilde{\nabla}$ has the following formulas;

$$\tilde{\nabla}_{\delta_x^i} \delta_y^j = \frac{1}{2} \left( -v_{ijk} + \dot{A}^l_{ki} v_l + \dot{\Lambda}^l_{kj} v_l \right) h^{kl} \delta_x^j$$

$$+ \frac{1}{2} \left( v_{jki} + v_{kj} - v_{ijk} - 2 \ell_j v_{ik} + 2 \ell_k v_{ij} \right) v^{kl} \delta_y^j,$$  

$$\tilde{\nabla}_{\delta_x^i} \delta_x^j = \frac{1}{2} \left( h_{ijk} + \ell \bar{R}_{h}^{i}_{jk} v_i \right) h^{kl} \delta_x^j$$

$$+ \frac{1}{2} \left( v_{kl} - \left\{ \dot{A}^l_{ji} + \frac{\ell}{F} (F \ell_j) x^i - \ell \frac{N_{ji}}{F} - \Gamma^l_{ij} \right\} v_{kl}$$

$$- \left\{ \dot{\Lambda}^l_{jk} + \frac{\ell}{F} (F \ell_j) x^i - \ell \frac{N_{jk}^i}{F} - \Gamma^l_{jk} \right\} v_{kl} \right) v^{kl} \delta_y^j,$$

$$\tilde{\nabla}_{\delta_x^i} \delta_y^j = \frac{1}{2} \left( h_{kj} - \ell \bar{R}_{h}^{j}_{ki} v_i \right) h^{kl} \delta_x^j$$

$$+ \frac{1}{2} \left( v_{jk} + 2 \Gamma_{ij}^h v_h + \dot{A}^l_{ij} v_l - \dot{\Lambda}^l_{i} v_j \right) v^{kl} \delta_y^j.$$


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\[
\bar{\nabla}_{\delta_x^i} \delta_x^j = \frac{1}{2} \left( h_{jki} + h_{kij} - h_{ijk} + 2 T_{ij}^l h_{lk} \right) h^{ks} \delta_x^s + \frac{1}{2} \left( -h_{ij,k} - \ell^h R_{h}^{l} v^s_{ij} v_{lk} \right) v^{sk} \delta_y^s.
\]

We assume the following ([6])

\[
\xi := \frac{1}{\bar{g}} \bar{e}_m = \frac{\ell^i}{\bar{g}} \delta_x^i,
\]

by considering

\[
\bar{e}_m = p^i_{\bar{m}} \delta_x^i = \ell^i \delta_x^i,
\]

where

\[
g := |\bar{e}_m| = \sqrt{\bar{g}(\bar{e}_m, \bar{e}_m)} = \sqrt{\ell^i \ell^j h_{ij}},
\]

for the covector \( \eta \) of \( \xi \), it follows that \( \eta \) determines a contact structure on \( PTM \).

Consequently, there exists a contact metric structure \((\phi, \xi, \eta, \bar{g})\) such that

\[
\bar{g}(\phi X, Y) = d\eta(X, Y),
\]

that is,

\[
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta(X) = \bar{g}(X, \xi), \quad \bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y), \quad \bar{g}(\phi X, Y) = d\eta(X, Y).
\]

Here, we have the following forth lemmas ([6]).

**Lemma 2.2.** Derivatives of the \( \bar{g} \)-norm \( g \) of \( \bar{e}_m \) are given by

\[
\delta_{x^k} g = \frac{1}{2g} \ell^\ell \ell^j h_{ij,k}, \quad (2.5)
\]

\[
\delta_{y^k} g = \frac{1}{2g} \left( 2\ell^\ell h_{kj} - 2g^2 \ell_k + \ell^\ell \ell^j h_{ij,k} \right), \quad (2.6)
\]

and

\[
\ell^k (\delta_{x^k} g) = \frac{1}{2g} \ell^\ell \ell^j h_{ij,k}, \quad \ell^k (\delta_{y^k} g) = \frac{1}{2g} \ell^\ell \ell^k h_{ij,k}. \quad (2.7)
\]

**Lemma 2.3.** For \( \bar{g} \) on \( TPTM \), we have the formulas

\[
\bar{g}(\delta_{y^s}, \xi) = 0, \quad (2.8)
\]

\[
\bar{g}(\delta_{x^i}, \xi) = \frac{\ell^i}{\bar{g}} h_{ij}. \quad (2.9)
\]
Lemma 2.4. \(d\eta\) satisfies the following formulas:

\begin{align}
(2.10) & \quad d\eta(\delta y_i, \delta y_j) = 0, \\
(2.11) & \quad d\eta(\delta x_i, \delta y_j) = -\frac{1}{g} h_{ij} + \frac{1}{2g^3} \ell^k h_{iik} (2\ell^s h_{js} + \ell^s \ell^t h_{st,j}) - \frac{\ell^k}{g} h_{ikj}, \\
(2.12) & \quad d\eta(\delta x_i, \delta x_j) = -\frac{1}{2g^3} \ell^s \ell^t h_{stj} \ell^k h_{jik} + \frac{\ell^k}{g} h_{jik} + \frac{1}{2g^3} \ell^s \ell^t h_{stj} \ell^k h_{ik} - \frac{\ell^k}{g} h_{ikj}.
\end{align}

Lemma 2.5. The semicolon of \(h_{ij}\) has the following formula.

\begin{align}
(2.13) & \quad \ell^s \ell^t h_{stj} = 0.
\end{align}

Here, we prove the following lemma.

Lemma 2.6. For the \(g\)-norm \(g\) of \(\tilde{e}_m\), we have the formulas

\begin{align}
(2.14) & \quad \ell^i \delta_{y^i}(\delta_{y^i} g) = 0, \quad \ell^i \delta_{y^i}(\delta_{y^i} g) = -\delta_{y^i} g.
\end{align}

Proof. From (1.7), we get

\begin{align}
(2.15) & \quad \delta_{y^i} g - \delta_{y^i} g = \frac{1}{g} \left( \ell^k h_{iik} - \ell^i h_{ik} \right).
\end{align}

Using (2.15) and (2.17), we have

\begin{align}
(2.16) & \quad \ell^i \delta_{y^i}(\delta_{y^i} g) = \ell^i \delta_{y^i}(\delta_{y^i} g) - \delta_{y^i} g.
\end{align}

Since \(\ell^i(\delta_{y^i} g) = 0\), we obtain

\begin{align*}
0 = \delta_{y^i} \left\{ \ell^i(\delta_{y^i} g) \right\} &= (\delta_{y^i} \ell^i)(\delta_{y^i} g) + \ell^i \delta_{y^i}(\delta_{y^i} g) \\
&= \ell^i(\delta_{y^i} g) + \ell^i \delta_{y^i}(\delta_{y^i} g) - (\delta_{y^i} g) \\
&= (\delta_{y^i} - \ell^i \ell_j)(\delta_{y^i} g) + \ell^i \delta_{y^i}(\delta_{y^i} g) - (\delta_{y^i} g) \\
&= (\delta_{y^i} g) + \ell^i \delta_{y^i}(\delta_{y^i} g) - (\delta_{y^i} g) = \ell^i \delta_{y^i}(\delta_{y^i} g).
\end{align*}

Hence we have the first formula in (2.14). From (2.16) and the first formula in (2.14), we get the second one in (2.14). \(\square\)

From (2.7) and (2.12), we have

\begin{align}
(2.17) & \quad \delta_{y^i} g = \frac{1}{g} (\ell^i h_{ikj} - g^2 \ell_k).
\end{align}
Since $d\eta(X,Y) = \tilde{g}(\phi X,Y)$ and $\phi \xi = 0$, we have

$$0 = \tilde{g}(\phi \xi, X) = d\eta(\xi, X) = \frac{1}{\mathcal{g}} d\eta(\ell^k \delta_x, X)$$

or equivalently,

$$d\eta(\ell^k \delta_x, X) = 0$$

for any tangent vector field on $PTM$. From (2.12) and (2.18), it follows that

$$\ell^k (2h_{ji} - h_{iij}) = \frac{1}{\mathcal{g}} \ell^s \ell^t h_{stij} \ell^k h_{jk}.$$  

From Lemma 2.2, Lemma 2.3 and Lemma 2.4, we obtain the following proposition.

**Proposition 2.7.** Let $\tilde{g}$ be an $h$-$v$ metric on $PTM$ and $(\phi, \xi, \eta, \tilde{g})$ be a contact metric structure on $PTM$ determined by $\xi := \frac{1}{\mathcal{g}} \epsilon_m$. Then the following formulas hold:

$$h_{ji} = \mathcal{g}^2 g_{ji} - \ell^k h_{tjki} + \mathcal{g} \delta_{y^j}(\delta_{y^i} \mathcal{g}) + 2\mathcal{g}(\delta_{y^j} \mathcal{g}) \ell_j + \mathcal{g}(\delta_{y^j} \mathcal{g}) \ell_i + (\delta_{y^j} \mathcal{g})(\delta_{y^i} \mathcal{g})$$

and

$$\ell^k h_{tjki} = \ell^k h_{tij}.$$  

**Proof.** From (2.17), it follows that

$$\ell^k h_{kj} = \mathcal{g}^2 \ell_k + \mathcal{g}(\delta_{y^j} \mathcal{g}).$$

According to (2.2), (2.18) and (2.22),

$$\tilde{g}(\delta_x, \xi) = \frac{1}{\mathcal{g}} \delta_x - \ell_i \xi - \frac{1}{\mathcal{g}} (\delta_{y^j} \mathcal{g}) \xi + \frac{\ell^j}{2 \mathcal{g}} (h_{jk;i} + \ell^h R_{jki}^{\ell} v_{i}) h^{ks} \delta_x + \frac{\ell^j}{2 \mathcal{g}} (v_{kij} - \hat{A}_{jkl}^{t_l} v_{kl} - \hat{A}_{jkl}^{t_l} v_{kl}) v^{ks} \delta_x.$$  

From (2.9) and (2.22), it follows that

$$\tilde{g}(\xi, \delta_x) = \mathcal{g} \ell_j + (\delta_{y^j} \mathcal{g}).$$

Applying $\delta_{y^j}$ to the left-hand side of (2.24) and using (2.2), (2.23), (2.24) and (2.9), we have

$$\delta_{y^j}(\tilde{g}(\xi, \delta_x)) = \frac{1}{\mathcal{g}} h_{ij} - \mathcal{g} \ell_i \ell_j - (\delta_{y^j} \mathcal{g}) \ell_i - (\delta_{y^j} \mathcal{g})(\delta_{y^i} \mathcal{g}) + \frac{\ell^l}{\mathcal{g}} h_{tjki}.$$  

Also, applying $\delta_{y^j}$ to the right-hand side of (2.24), we get

$$\delta_{y^j}(\mathcal{g} \ell_j + (\delta_{y^j} \mathcal{g})) = (\delta_{y^j} \mathcal{g}) \ell_j + \delta_{y^j}(\delta_{y^j} \mathcal{g}) + \mathcal{g} g_{ij} - \mathcal{g} \ell_i \ell_j.$$  

From (2.25) and (2.26), we obtain (2.20).

By exchanging $i$ and $j$ in (2.20), we have

$$h_{ji} = \mathcal{g}^2 g_{ji} + \mathcal{g} \delta_{y^i}(\delta_{y^j} \mathcal{g}) + 2\mathcal{g}(\delta_{y^i} \mathcal{g}) \ell_i + \mathcal{g}(\delta_{y^j} \mathcal{g}) \ell_j + (\delta_{y^i} \mathcal{g})(\delta_{y^j} \mathcal{g}) - \ell^k h_{tij}.$$ 

Subtracting (2.27) from (2.20) and using (2.22), we get (2.21). 

□
3 Two conditions of \( h_{ij} \)

Using (2.10), (2.11), (2.12) and (2.13), we get

\[
\tilde{g}(\phi \delta x^i, \delta x^j) = \eta(\delta x^i, \delta x^j) = \frac{1}{2g^3} \ell^k h_{ij,kl} + \frac{1}{2g^3} \ell^k \ell^l h_{ij,kl} h_{ik} - \frac{\ell^k}{g} h_{ik,j} \tag{3.1}
\]

\[
\tilde{g}(\phi \delta y^i, \delta y^j) = \eta(\delta y^i, \delta y^j) = \frac{1}{g} \ell^k h_{ij,i} = 0 \tag{3.2}
\]

\[
\tilde{g}(\phi \delta x^i, \delta y^j) = \eta(\delta x^i, \delta y^j) = \frac{1}{g} \ell^k h_{ij,k} - \frac{\ell^k}{g} h_{ik,j} \tag{3.3}
\]

Here, we set the following conditions (\( C_1 \)) and (\( C_2 \)):

\( (C_1) \): For any \( i, j \in \{1, \ldots, m\} \),

\[
\ell^i h_{ij} = 0. \tag{3.4}
\]

\( (C_2) \): For any \( i, j \in \{1, \ldots, m\} \),

\[
\ell^k h_{ik,j} = 0. \tag{3.5}
\]

First, we have the following lemma.

**Lemma 3.1.** We assume that the condition (\( C_1 \)) or (\( C_2 \)) hold on \( PTM \). Then we have

\[
\delta x^i g = 0. \tag{3.6}
\]

*Proof.* First, we assume that the condition (\( C_1 \)) holds on \( PTM \). Making use of (3.4) and (2.19), we have

\[
\ell^i \ell^k h_{ij} = 0. \tag{3.7}
\]

According to (2.5) and (3.7), we have (3.6).

Second, we assume that the condition (\( C_2 \)) holds on \( PTM \). Plugging (3.5) into (2.5) yields (3.6). \( \Box \)

Also, we get the following lemma.

**Lemma 3.2.** We assume that (3.6) holds. Then we have

\[
\ell^i h_{ij} = g \delta x^i (\delta y^k g). \tag{3.8}
\]
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Proof. Applying $\delta_x$ to the left-hand side of (2.22) and using the definition of the slash, we have

$$
(3.9) \quad \delta_x (\ell^j h_{kj}) = (\ell^j s - \ell^j \Gamma^r_{js} h_{kj} + \ell^j (h_{kj}|s + h_{rj} \Gamma^r_{ks} + h_{kr} \Gamma^r_{js})).
$$

Similarly, applying $\delta_x$ to the right-hand side of (2.22) and making use of (3.6), we get

$$
(3.10) \quad \delta_x \left( g^2 \ell_k \right) + \delta_x \left( g(\delta_y \ell_k) \right) = g^2 \ell_k |s + g^2 \ell_r \Gamma^r_{ks} + g \delta_x (\delta_y \ell_k).
$$

Using (1.3), (3.9) and (3.10), (3.8) yields. □

Here, we put

$$
(3.11) \quad \tilde{H}_{ij} := -\frac{1}{g} h_{ij} + \frac{1}{g^2} \ell^k h_{ik} \ell^s h_{js} - \frac{\ell^k}{g} h_{ik;j}.
$$

From Lemma 3.1 and Lemma 3.2, we have the following proposition.

Proposition 3.3. We assume that the condition $(C_2)$ holds on PTM. Then, on $TPTM$ with the contact metric structure $(\phi, \xi, \eta, \tilde{g})$, we have

$$
(3.12) \quad \delta_x (\delta_y \ell_k) = 0,
$$

and

$$
(3.13) \quad \phi \delta_x = \tilde{H}_{ij} v^j \delta_y, \quad \phi \delta_y = -\tilde{H}_{ij} h^j \delta_x.
$$

Proof. We assume that the condition $(C_2)$ holds on PTM. From (3.8), it follows that

$$
(3.14) \quad \delta_x (\delta_y \ell_k) = 0.
$$

Making use of (3.1), (3.2), (3.3) and (3.11), we get

$$
\tilde{g}(\phi \delta_x, \delta_x) = d\eta(\delta_x, \delta_x) = 0,
$$

$$
\tilde{g}(\phi \delta_x, \delta_y) = -\tilde{g}(\phi \delta_y, \delta_x) = d\eta(\delta_x, \delta_y) = \tilde{H}_{ij},
$$

$$
\tilde{g}(\phi \delta_y, \delta_y) = d\eta(\delta_y, \delta_y) = 0,
$$

or equivalently,

$$
\phi \delta_x = \tilde{H}_{ij} v^j \delta_y, \quad \phi \delta_y = -\tilde{H}_{ij} h^j \delta_x.
$$

□

Making use of (2.20), (2.22) and (1.4), (3.11) can be rewritten as

$$
(3.15) \quad \tilde{H}_{ij} = -\tilde{g} h_{ij} - \delta_y (\delta_y \ell_k) - (\delta_y \ell_k) \ell_j.
$$
Here, we define the following set whose the matrix $N$ is non-singular,

$$\mathcal{U} := \{ p \in PTM \mid \det N(p) \neq 0 \},$$

where,

$$N = \begin{pmatrix} N_{1}^{1} & \ldots & N_{1}^{m} \\ \vdots & \ddots & \vdots \\ N_{m}^{1} & \ldots & N_{m}^{m} \end{pmatrix} \quad (N_{i}^{j} \text{ is the Chern-Rund connection coefficients}).$$

From Lemma 3.1 and Lemma 3.2, we obtain the following proposition.

**Proposition 3.4.** We assume that the condition $(C_2)$ holds on $PTM$ and $\mathcal{U}$ isn’t empty. Then we have (3.6) and

$$\delta_{y^{i}} g = 0,$$

i.e., the $\tilde{g}$-norm $g$ of $\hat{e}_{m}$ is constant on $\mathcal{U}$.

**Proof.** From Lemma 3.1, we get (3.6). Using Lemma A, (3.15), (3.6) and (3.14), the following formula is calculated, that is,

$$\tilde{H}_{ij}|s = -\delta_{x^{r}}(\delta_{y^{r}}(\delta_{y^{i}} g)) + \delta_{y^{r}}(\delta_{y^{r}} g)\Gamma_{js}^{k} + \delta_{y^{r}}(\delta_{y^{r}} g)\Gamma_{is}^{k} + (\delta_{y^{r}} g)\ell_{j}\Gamma_{is}^{k}.$$  

From (2.14), we get

$$\ell_{i}\delta_{x^{s}}(\delta_{y^{j}} g) = \ell_{k}\Gamma_{i}s\delta_{y^{j}} g.$$  

Similarly, using (2.22) and (3.14), we have

$$\ell_{j}\delta_{x^{s}}(\delta_{y^{i}} g) = \ell_{k}\Gamma_{js}\delta_{y^{i}} g.$$  

Contracting (3.17) with $\ell_{i}$ and using (3.18), we get

$$\ell_{i}\tilde{H}_{ij}|s = (\delta_{y^{r}} g)\ell_{j}\Gamma_{is}^{k} = \frac{1}{F}(\delta_{y^{r}} g)\ell_{j}N_{s}^{k}.$$  

Also, contracting (3.17) with $\ell_{j}$ and using (3.19), we have

$$\ell_{j}\tilde{H}_{ij}|s = 0.$$  

From (3.20) and (3.21), it follows that

$$0 = \ell_{i}\ell_{j}\tilde{H}_{ij}|s = \frac{1}{F}(\delta_{y^{r}} g)N_{s}^{k}.$$  

(3.22) yields (3.16) on $\mathcal{U}$. □

From Proposition 3.4 and Lemma 3.2, we have the following theorem.

**Theorem 3.5.** Let $\tilde{g}$ be an h-v metric on $PTM$ and $(\phi, \xi, \eta, \tilde{g})$ be a contact metric structure on $PTM$ determined by $\xi := \frac{1}{F}\hat{e}_{m}$. Let $\mathcal{U}$ be the set whose the matrix $N$ is non-singular, i.e.,

$$\mathcal{U} := \{ p \in PTM \mid \det N(p) \neq 0 \}.$$

We assume that $\mathcal{U}$ isn’t empty. Then the condition $(C_2)$ holds if and only if the $\tilde{g}$-norm $g$ of $\hat{e}_{m}$ is constant on $\mathcal{U}$.
4 The flag curvatures on PTM

We define a tensor field $\tilde{h}$ on a contact manifold with the contact metric structure $(\varphi, \xi, \eta, \tilde{g})$ by

$$\tilde{h}X := -(L_{\xi}\phi)(X) = [\phi X, \xi] - \phi[X, \xi]$$

for any vector field $X$ on $PTM$ ([7]).

In [1](p47), we have

$$\ell^t \dot{A}_{it} = \ell^s g^{sr} \dot{A}_{rli} = 0.$$  

(4.1)

From now on, we assume that the condition $(C_2)$ holds and $\mathfrak{h}$ defined in §3 is non-empty. The $\tilde{g}$-norm of $\hat{e}_m$ is constant on $\mathfrak{h}$.

From (3.16), (3.15) and (3.13), it follows that

$$\phi \delta_{x^i} = -g \tilde{h}_{ij} v^j \delta_{y^i}, \quad \phi \delta_{y^i} = \tilde{g} h_{ij} h^{jk} \delta_{x^j}.$$  

(4.2)

By directly calculating, we get

$$\xi \tilde{h}_{ij} = \frac{1}{g} \tilde{h}_{ij} \xi \delta_{x^i} + \frac{1}{g} \tilde{h}_{ij} \Gamma^r_{ik} \ell^k = \frac{1}{g} F (\tilde{h}_{ij} N_i^t + \tilde{h}_{it} N_j^t).$$

Using Lemma B, (4.1), (4.2) and (4.3), we obtain

$$[\phi \delta_{x^i}, \xi] = \frac{1}{F} (\tilde{h}_{ij} N_i^t + \tilde{h}_{it} N_j^t) v^j \delta_{y^i} + \tilde{g} h_{ij} (\xi v^j) \delta_{y^i} - \tilde{h}_{ij} v^j \delta_{x^i} + \tilde{g} h_{ij} \ell_k \xi + \tilde{h}_{ij} N^t \delta_{y^i}.$$  

(4.4)

Similarly, by directly calculating, we get

$$-\phi [\delta_{x^i}, \xi] = -\frac{N^t}{F} \tilde{h}_{st} v^j \delta_{y^i} - \ell^t \ell^j R_{h_{ji}} \tilde{h}_{st} h^{jl} \delta_{x^i}.$$  

(4.5)

From (4.4) and (4.5), it follows that

$$\tilde{h} \delta_{x^i} = \frac{1}{F} \tilde{h}_{ij} N_i^j v^j \delta_{y^i} + \tilde{g} h_{ij} (\xi v^j) \delta_{y^i} - \tilde{h}_{ij} v^j \delta_{x^i} + \tilde{g} h_{ij} v^j \ell_k \xi + \frac{1}{F} \tilde{h}_{ij} N^t \delta_{y^i} - \ell^t \ell^j R_{h_{ji}} \tilde{h}_{st} h^{jl} \delta_{x^i}.$$  

(4.6)

By similar way, we get

$$\tilde{h} \delta_{y^i} = -\frac{1}{F} \tilde{h}_{ij} N_i^j h^j \delta_{x^i} - \tilde{g} h_{ij} (\xi h^j) \delta_{x^i} - \frac{1}{F} \tilde{h}_{ij} h^j N_i^t \delta_{x^i} + \tilde{h}_{ij} h^j \ell^t R_{h_{ji}} \tilde{h}_{st} v^q \delta_{y^i}.$$  

(4.7)

We define the following notations:

$$G^s := FN^s \ell_j$$

and

$$G_s := FN^j \ell_s.$$  

(cf. [1])

First, we prove that the following proposition holds.
Proposition 4.1. Let \( \tilde{g} \) be an h-v metric on \( PTM \) and \((\phi, \xi, \eta, \tilde{g})\) be a contact metric structure on \( PTM \) determined by \( \xi := \frac{1}{\tilde{g}} \tilde{e}_m \). Let \( \mathcal{U} \) be the set whose the matrix \( N \) is non-singular, i.e.,

\[ \mathcal{U} := \{ p \in PTM | \det(N(p)) \neq 0 \}. \]

We assume that \( \mathcal{U} \) isn’t empty and the condition \((C_2)\) holds. Then the following equation holds:

\[ (1 - g^2) \tilde{h}_{ij} h^{ij} N^t_i \ell_t = 0. \]

In particularly, if \( g^2 \neq 1 \) and the zero set of \( \Lambda := G^t \ell_t \) is non-empty, then we have

\[ \ell_t = \frac{F N^t_i \ell_t}{\Lambda}. \]

Proof. By taking inner product with \( \xi \) and making use of (2.9), (4.7) yields

\[ -\eta(\tilde{h}\delta_{\gamma'}) = \frac{1}{\tilde{g}F}(1 - g^2) \tilde{h}_{ij} h^{ij} N^t_i \ell_t. \]

Since \( \tilde{h} \) is a symmetric operator and \( \tilde{h}\xi = 0 \), we get

\[ \eta(\tilde{h}\delta_{\gamma'}) = \tilde{g}(\xi, \tilde{h}\delta_{\gamma'}) = \tilde{g}(\tilde{h}\xi, \delta_{\gamma'}) = 0. \]

From (4.10) and (4.11), we get (4.8).

Here, (4.8) is written as the following equation:

\[ \tilde{h}_{ij} h^{ij} N^t_i \ell_t = (g_{ij} - \ell_t \ell_t) h^{ij} N^t_i \ell_t = g_{ij} h^{ij} N^t_i \ell_t - \ell_t \ell_t N^t_i \ell_t = g_{ij} h^{ij} N^t_i \ell_t - \frac{1}{g^2 F} \ell_t G^t \ell_t. \]

If \( g^2 \neq 1 \), then

\[ 0 = \tilde{h}_{ij} h^{ij} N^t_i \ell_t = g_{ij} h^{ij} N^t_i \ell_t - \frac{1}{g^2 F} \ell_t G^t \ell_t. \]

or equivalently,

\[ g_{ij} h^{ij} N^t_i \ell_t = \frac{1}{g^2 F} \ell_t G^t \ell_t. \]

From (4.12), we have (4.9). \[ \square \]

Making use of

\[ \nabla_{\xi} X = \frac{\ell_t}{g} \nabla_{\delta_x} X, \quad \nabla X \xi = \frac{1}{g} (X \ell_t) \delta_x + \frac{\ell_t}{g} \nabla_X \delta_x, \quad [X, \xi] = \nabla_X \xi - \nabla_{\xi} X, \]

we get the following theorem:
Theorem 4.2. Let $\tilde{g}$ be an $h$-$v$ metric on PTM and $(\phi, \xi, \eta, \tilde{g})$ be a contact metric structure on PTM determined by $\xi := \frac{1}{\tilde{e}} \xi_m$. Let $\Xi$ be the set whose the matrix $N$ is non-singular, i.e.,

$$\Xi := \{p \in \text{PTM} \mid \det(N(p)) \neq 0\}.$$ 

We assume that $\Xi$ isn’t empty and the condition $(C_2)$ holds. Then we get

$$\ell^h \ell^j R^s_{h^i j} = \left\{ h_{ik} - 2g^2 \ell_i \ell_k + g^2 h_{ik} - 2g^2 \tilde{h}_{ij} \tilde{v}^j \tilde{h}_{ik} \right\} v^k s.$$  \hfill (4.13)

Proof. Using (2.4), (2.20), (3.12) and (3.14), we get

$$\tilde{\nabla}_{\delta_x} \xi = \frac{\ell^j}{2g} h_{ki} l h^k s \delta_x s + \frac{1}{2g} \left( h_{ik} - 2g^2 \ell_i \ell_k + g^2 h_{ik} - \ell^h \ell^j R^l_{h^i j} v^l v^k \right) v^k s \delta_y s.$$  \hfill (4.14)

By calculating $\phi \delta_x s = -\phi \delta_x s + \tilde{\nabla}_{\delta_x} \xi$ and using (4.14) and (4.2), we get

$$\phi \delta_x s = \frac{\ell^j}{2g} h_{ki} l h^k s \delta_x s + \frac{1}{2g} \left( h_{ik} - 2g^2 \ell_i \ell_k + g^2 h_{ik} - \ell^h \ell^j R^l_{h^i j} v^l v^k \right) v^k s \delta_y s.$$  \hfill (4.15)

Similarly, by calculating $\phi \delta_y s = -\phi \delta_y s + \tilde{\nabla}_{\delta_y} \xi$ and using (2.24) and (4.2), we get

$$\phi \delta_y s = \frac{1}{2g} \left( h_{ki} - 2g^2 g_{ki} + \ell^h \ell^j R^l_{h^i j} v^l v^k \right) h^k s \delta_x s + \frac{\ell^j}{2g} v_{kli} v^k s \delta_y s.$$  \hfill (4.16)

Here, applying $\phi$ to (4.6), we get

$$\phi \delta_x s = \frac{1}{F^*} \tilde{h}_{ij} N^j \mu^l \phi \delta_y s + \frac{\tilde{g} h_{ij} (\xi \psi^i) \phi \delta_y s + \frac{1}{F^*} \tilde{h}_{ij} \psi^i N^j \phi \delta_y s}{-\tilde{h}_{ij} \psi^i \phi \delta_x s - \ell^h \ell^j R^l_{h^i j} \tilde{h}_{is} h^l \phi \delta_x s}.$$  \hfill (4.17)

Similarly, applying $\phi$ to (4.7), we obtain

$$\phi \delta_y s = -\frac{1}{F^*} \tilde{h}_{ij} N^j \mu^l \phi \delta_x s - \frac{\tilde{g} h_{ij} (\xi \psi^i) \phi \delta_x s - \frac{1}{F^*} \tilde{h}_{ij} \psi^i N^j \phi \delta_x s}{\tilde{h}_{ij} \psi^i \phi \delta_y s + \tilde{h}_{ij} h^l \ell \ell^l R^s_{a^i j} \phi \delta_y s}.$$  \hfill (4.18)

From (4.15) and (4.17), we get

$$\frac{1}{2g} \left( h_{ik} - 2g^2 \ell_i \ell_k + g^2 h_{ik} - \ell^h \ell^j R^l_{h^i j} v^l v^k \right) v^k s \delta_y s = g (\tilde{h}_{ij} \psi^l h_{ik} + \ell^h \ell^j R^l_{h^i j} \tilde{h}_{is} h^l \psi^i h_{ik}) v^k s \delta_y s,$$

or equivalently,

$$h_{ik} - 2g^2 \ell_i \ell_k + g^2 h_{ik} - \ell^h \ell^j R^l_{h^i j} v^l v^k = 2g^2 (\tilde{h}_{ij} \psi^l h_{ik} + \ell^h \ell^j R^l_{h^i j} \tilde{h}_{is} h^l \psi^i h_{ik}).$$  \hfill (4.19)

By directly calculating, we get

$$v_{ij} = g^2 \tilde{h}_{ij} l s \tilde{h}_{st}.$$  \hfill (4.20)

Using (4.20), (4.19) is written as follows;

$$h_{ik} - 2g^2 \ell_i \ell_k + g^2 h_{ik} = 2g^2 \tilde{h}_{ij} v^l h_{ik} + \ell^h \ell^j R^l_{h^i j} v^l v^k.$$  \hfill (4.21)

Thus (4.21) yields (4.13).
From Theorem 4.2, we obtain the following corollary.

**Corollary 4.3.** Let $\tilde{g}$ be an h-v metric on $PTM$ and $(\phi, \xi, \eta, \tilde{g})$ be a contact metric structure on $PTM$ determined by $\xi := \frac{1}{\tilde{g}} e_m$. We assume that $\tilde{g}$ coincides with a Sasaki type metric $g^s$. Then we obtain the flag curvature on $PTM$ is zero.

**Proof.** We assume that $\tilde{g} = g^s$. Then we get
\[
\tilde{h}_{ij}g^{jl} = (g_{ij} - \ell_i \ell_j)g^{jl} = (\delta_i^j - \ell^j \ell_i) = g_{ik} - \ell_i \ell_k - \ell_k \ell_i = \tilde{h}_{ik}.
\]
From these above equations and (4.13), we have
\[
(4.22) \quad \ell^h \ell^i R^k_{h \ ij} = 0.
\]
From (4.22), the flag curvature
\[
K(\ell, V, W) := \frac{V^i(\ell^j R_{ijkl})W^k}{g(V, W) - g(\ell, V)g(\ell, W)}
\]
is zero (cf. [1] p.68). \hfill \Box

**Remark 4.1.** Corollary 4.3 is the same theorem as Theorem 3.2 in [8], so that Theorem 4.2 is a generalization of Theorem 3.2 in [8].

Moreover, we get the following theorem.

**Theorem 4.4.** Let $\tilde{g}$ be an h-v metric on $PTM$ and $(\phi, \xi, \eta, \tilde{g})$ be a contact metric structure on $PTM$ determined by $\xi := \frac{1}{\tilde{g}} e_m$. Let $U$ be the set whose the matrix $N$ is non-singular, i.e.,
\[
U := \{ p \in PTM \mid \det N(p) \neq 0 \}.
\]
We assume that $U$ isn’t empty and the condition (C2) holds. Then we get
\[
(4.23) \quad \tilde{h}_{ik} = \frac{1}{g^2} \left( \ell^h \ell^i R^l_{h \ jk}v_{lk} + g^2 \ell^h \ell^i R^q_{h \ ij} \tilde{h}_{kr} h^r \tilde{h}_{ql} \right).
\]

**Proof.** Using (4.16) and (4.18), we get
\[
(4.24) \quad h_{ki} - g^2 g_{ki} + \ell^i \ell^h R^l_{h \ jk}v_{li} = 2g^2 \left( \tilde{h}_{ij} v^j \tilde{h}_{lk} - \ell^h \ell^i R^q_{h \ ij} \tilde{h}_{kr} h^r \tilde{h}_{ql} \right).
\]
Subtracting (4.24) from (4.21), we obtain
\[
(4.25) \quad 2g^2 (g_{ik} - \ell_i \ell_k) = 2\ell^h \ell^i R^l_{h \ jk}v_{lk} + 2g^2 \ell^h \ell^i R^q_{h \ ij} \tilde{h}_{kr} h^r \tilde{h}_{ql}.
\]
Hence (4.25) yields (4.23). \hfill \Box
On the flag curvatures on PTM deduced from contact metric structures

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