Finsler-type structures and det-based classification of Mueller-type submanifolds

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Abstract. By means of the spinorial representation of matrices, two constructed determinant-induced metrics of conformal-Euclidean Riemannian and of Finsler types, respectively, are shown to produce a (h, v)-structure, whose properties are investigated from the point of view of naturally developed Einstein and Maxwell-like equations.

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Introduction

The theory of the Lorentz group provides clues for approaching problems of light polarization optics in the frames of the vector Mueller and spinor Jones formalisms. Differences in describing completely polarized vs. partly polarized light correlate with the properties of isotropic and time-like vectors in Special Relativity. The enveloping framework for the involved geometric objects is the $GL(4, \mathbb{R})$ matrix ansatz. In our work, the parametrization of 4×4-matrices M of the real linear group $GL(4, \mathbb{R})$ involves the Dirac matrices, four real 4-vectors of the form (k, m, n, l) appear as parameters for possible Mueller matrices. In this realization, assuming that the determinant is non-identically vanishing, this represents a 4-th order homogeneous polynomial in the parameters, hence naturally providing a locally-Minkowski $m-$th root metric of Finsler type. While subclasses of Mueller matrices belonging to specific Lie groups are considered, their pre-existent Lie group metric induces on the Mueller intersection a Riemannian metric, which canonically further provides jointly with the Finsler tensor field a geometric (h, v)-structure on the tangent space of the manifold. We determine the specific geometric objects of the mixed structure. Moreover, the extended Einstein and Maxwell equations of the (h, v)-geometric approach and of the associated geometric objects (curvatures, torsions, Ricci tensors and scalars of curvature, extended Einstein tensors, KCC invariants, deflections and extended electromagnetic tensors) are constructed and discussed from physical point of view. The

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last section provides information about the properties of the two det-induced metrics, of conformal-Euclidean Riemannian and of Finsler types, respectively, providing a primary classification of the Mueller-type class of matrices.

1 Mueller matrices in spinorial representation

We firstly consider the spinorial representation of $\mathbb{R}^4$ inside $\mathcal{M}_{2\times 2}(\mathbb{C})$,

$$\begin{aligned} \mathbf{a} \equiv (a_0, a_1, a_2, a_3) \in \mathbb{R}^4 \quad &\longleftrightarrow \hat{\mathbf{a}} \equiv (a_0, a_1, ia_2, a_3) \in \mathbb{R} \times \mathbb{R} \times i\mathbb{R} \times \mathbb{R} \subset \mathbb{C} \\ &\longleftrightarrow A = a_0 I_2 + a_1\sigma_1 + a_2 \cdot i\sigma_2 + a_3\sigma_3 \\ &\quad = \begin{pmatrix} a_0+a_3 & a_1+a_2 \\ a_1-a_2 & a_0-a_3 \end{pmatrix} \equiv a_0 + \hat{\mathbf{a}}\vec{\sigma} \in \mathcal{M}_{2\times 2}(\mathbb{R}), \end{aligned} \tag{1.1}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices. This allows the rewriting of the matrices in $\mathcal{M}_{4\times 4}(\mathbb{R}) \equiv \mathbb{R}^{16}$ in the form:

$$S \equiv S_x = \begin{pmatrix} k_0+k_3 & k_1+k_2 & n_0+n_3 & n_1+n_2 \\ k_1-k_2 & k_0-k_3 & n_1-n_2 & n_0-n_3 \\ \ell_0+\ell_3 & \ell_1+\ell_2 & m_0+m_3 & m_1+m_2 \\ \ell_1-\ell_2 & \ell_0-\ell_3 & m_1-m_2 & m_0-m_3 \end{pmatrix} = \begin{pmatrix} K & N \\ L & M \end{pmatrix}, \tag{1.2}$$

which is in one-to-one linear correspondence to the vector

$$x = (k_0, k_1, k_2, k_3, m_0, m_1, m_2, m_3, n_0, n_1, n_2, n_3, \ell_0, \ell_1, \ell_2, \ell_3) \in \mathbb{R}^{16}.$$

Definition 1.1. We call *Mueller matrices*, the following subset of $\mathcal{M}_{4\times 4}(\mathbb{R})$ written in the form (1.2), [1]

$$\mathcal{M}_* = \{S \in \mathcal{M}_4(\mathbb{R}) \mid (A), (B)\}, \tag{1.3}$$

given by the following two properties:

$$\forall v = (v^0, \underbrace{v^1, v^2, v^3}_{\mathbf{v}}) \in \mathbb{R}^4, \qquad v^0 > 0 \qquad \Rightarrow \qquad (Sv^t)^0 > 0, \tag{A}$$

$$\forall v = (v^0, v^1, v^2, v^3) \in \mathbb{R}^4, \qquad Q_{3,1}(v) > 0 \quad \Rightarrow \qquad Q_{3,1}(Sv^t) > 0, \tag{B}$$

where $Q_{3,1}(v) = v_0^2 - \mathbf{v}^2 \equiv v_0^2 - (v_1^2 + v_2^2 + v_3^2)$.

One can easily notice that, under the assumption of the representation (1.2), the determinant of S_x is a homogeneous of degree four polynomial in the coefficients of

[1] We further denote $||v||_{3,1} = \sqrt{(v^0)^2 - \mathbf{v}^2}$ and $\mathbf{v}^2 = (v^1)^2 + (v^2)^2 + (v^3)^2$, where a for 4-vector $w = (w^0, \mathbf{v}) \in \mathbb{R}^4$, we denote by w^0 its first component.

x, having the form

$$
\begin{aligned}
\det(S_x) = \;& (+k_0^2m_2^2 - k_0^2m_1^2 - k_0^2m_3^2 + k_0^2m_0^2) + (-k_1^2m_2^2 + k_1^2m_1^2 + k_1^2m_3^2 - k_1^2m_0^2) \\
& +(+k_2^2m_0^2 - k_2^2m_1^2 + k_2^2m_2^2 - k_2^2m_3^2) + (-k_3^2m_2^2 + k_3^2m_1^2 + k_3^2m_3^2 - k_3^2m_0^2) \\
& +(+\ell_0^2n_0^2 - \ell_0^2n_1^2 - \ell_0^2n_3^2 + \ell_0^2n_2^2) + (-\ell_1^2n_2^2 + \ell_1^2n_1^2 - \ell_1^2n_0^2 + \ell_1^2n_3^2) \\
& +(+\ell_2^2n_0^2 - \ell_2^2n_3^2 - \ell_2^2n_1^2 + \ell_2^2n_2^2) \\
& +(-\ell_3^2n_0^2 + \ell_3^2n_1^2 - \ell_3^2n_2^2 + \ell_3^2n_3^2) \\
& +2(-k_3\ell_3n_1m_1 - k_3\ell_2n_2m_3 + k_3\ell_3m_0n_0 + k_3\ell_2m_2n_3 \\
& -k_3\ell_2m_1n_0 - k_2\ell_2n_2m_2 - k_0\ell_3m_0n_3 + k_0\ell_2m_1n_3 \\
& +k_0\ell_3n_1m_2 - k_0\ell_2m_2n_0 + k_0\ell_0n_1m_1 - k_0\ell_3n_2m_1 \\
& +k_0\ell_1m_1n_0 + k_0\ell_2n_2m_0 + k_0\ell_3m_3n_0 - k_3\ell_3m_3n_3 \\
& +k_3\ell_1n_1m_3 - k_3\ell_0n_1m_2 + k_1\ell_0n_1m_0 + k_1\ell_0m_2n_3 \\
& -k_1\ell_2m_0n_3 + k_1\ell_2m_3n_0 + k_1\ell_2n_1m_2 - k_1\ell_2n_2m_1 \\
& -k_1\ell_1m_3n_3 + k_1\ell_1m_0n_0 - k_0\ell_0m_0n_0 - k_1\ell_0m_1n_0 \\
& +k_1\ell_1n_2m_2 - k_1\ell_1n_1m_1 + k_0\ell_1n_2m_3 - k_0\ell_1m_2n_3 \\
& +k_2\ell_3n_2m_3 - k_2\ell_3n_1m_0 - k_2\ell_3m_2n_3 - k_3\ell_0m_3n_0 \\
& +k_3\ell_1m_2n_0 + k_3\ell_0n_2m_1 - k_0\ell_2n_1m_3 + k_2\ell_0m_2n_0 \\
& +k_2\ell_2m_3n_3 - k_2\ell_2m_0n_0 + k_2\ell_0n_1m_3 - k_2\ell_0n_2m_0 \\
& -k_2\ell_0m_1n_3 - k_0\ell_1n_1m_0 - k_0\ell_0n_2m_2 - k_3\ell_1m_1n_3 \\
& -k_3\ell_1n_2m_0 + k_3\ell_2n_1m_0 + k_0\ell_0m_3n_3 + k_2\ell_2n_1m_1 \\
& +k_2\ell_1m_0n_3 - k_2\ell_1n_1m_2 + k_2\ell_1n_2m_1 + k_2\ell_3m_1n_0 \\
& -k_2\ell_1m_3n_0 + k_3\ell_3n_2m_2 + k_3\ell_0m_0n_3 + k_1\ell_3n_2m_0 \\
& +k_1\ell_3m_1n_3 - k_1\ell_3m_2n_0 - k_1\ell_0n_2m_3 - k_1\ell_3n_1m_3).
\end{aligned}
\tag{1.4}
$$

In [23], this large expression was shown to reduce to a considerably abbreviated form, by involving the formal cross product in $\mathbb{C}^3$ and the formal Minkowski quadratic form in $\mathbb{C}^4$. We shall assume throughout Sections 1-6 that the determinant is non-vanishing.

2 Finsler metrics and KCC stability

2.1 The Finsler structure

Definition 2.1. A real Finsler structure is a couple (M, F), where M is a real n-dimensional C^∞ manifold and $F : TM \to [0, \infty)$ is a mapping (called *Finsler fundamental function*), which satisfies the following properties:

1. F is smooth on the slit tangent space $TM \setminus \{0\} = \{(x, y) | x \in M, y \in T_xM, y \neq 0\}$ and is continuous on the image of the null section;
2. F is 1-homogeneous in the second argument, i.e., $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$;
3. F defines the smooth maps $g_{ij} : TM \setminus \{0\} \to \mathbb{R}$, $i, j \in \overline{1, n}$ which are the components of the metric Finsler tensor field $g = g_{ij} dx^i \otimes dx^j$, and form a symmetric positive definite matrix, $(g_{ij})_{i,j \in \overline{1,n}}$, where

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}. \tag{2.1}$$

The fundamental metric tensor field (2.1) and its dual will be further used for respectively lowering and raising the indices of tensors. Alternatively, under weaker conditions, i.e., for g non-degenerate and having constant signature, (M, F) is called *pseudo-Finsler structure*.

Remark. One may consider as well generalized Finsler structures whose fundamental function is not defined on the whole tangent space, but only on certain distributions of TM or have their smoothness domain strictly included in the slit tangent space. Some classic illustrative examples in this respect are the Kropina metric, the m-th root pseudo-Finsler metrics, including the Berwald-Moor metric [22].

The vertical sub-bundle $VTM = Ker(d\pi)$ of the vector bundle $(TTM, d\pi, TM)$ provided by the kernel of the linear mapping π, and a supplementary sub-bundle $HTM = Ker(N)$ is provided by a Barthel connection $N : TTM \to VTM$, $N = (N_i^j(x,y))_{i,j\in\overline{1,n}}$ which satisfies $N \circ i = id|_{VTM}$ (where $i : VTM \to TTM$ is the canonic inclusion), leads to the Whitney decomposition

$$TTM = HTM \oplus VTM, \tag{2.2}$$

This induces local adapted bases for the appropriate sections of these sub-bundles,

$$\left\{\delta_i = \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}\right\} \subset \Gamma(HTM), \quad \left\{\dot{\partial}_i = \frac{\partial}{\partial y^i}\right\} \subset \Gamma(VTM),$$

The dual splitting $T^*TM = H^*TM \oplus V^*TM$ leads to a similar dual basis,

$$\{dx^i\} \subset \Gamma(H^*TM), \quad \{\delta y^i = dy^i + N_j^i dx^j\} \subset \Gamma(V^*TM).$$

One might use then the horizontal and the vertical projectors of the decomposition (2.2),

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \qquad v = \frac{\partial}{\partial y^i} \otimes \delta y^i.$$

2.2 Finsler-Mueller structures

In the following we shall examine two particular Finsler structures, tightly related to the representation (1.2) of real 4×4 matrices. If considering on M the action of the group $H = SL(4)$, this canonically allows to extend the quartic form $Q_0(y) = \det S_y$, by left shifts

$$Q(x,y) = \det S_{xy} = \det(S_x)\det(S_y) = \det(S_y) = Q_0(y), \tag{2.3}$$

for all $x \in SL(4)$ and $y \in M \equiv T_xM$. We note that at all x, $Q(x,y)$ depends on y only, hence producing a pseudo-Finsler structure of locally Minkowski type $(M = \mathcal{M}_{4\times4}(\mathbb{R}), F)$ endowed with the $4-$th root pseudo-Finsler function

$$F(x,y) = \sqrt[4]{\det S_y}, \ \ \forall(x,y) \in TM. \tag{2.4}$$

We note that here $y \in \mathcal{M}_{4\times4}(\mathbb{R}) = T_xM, \ M \equiv \mathbb{R}^{16}$ and that the structure is of locally Minkowski type. One may apply the theory from the previous section, in order to point out the relevance of the main geometric objects of the structure, insisting on the special subcase when the directional arguments are of Mueller type ($y \in \mathcal{M}_* \cap GL_4(\mathbb{R}) \subset T_xM, \forall x \in M$).

Having in view the special role played by the Lorentz group, we additionally note that, if considering on M the action of the group $SO(3,1)$, this canonically allows to extend to $\mathcal{M}_* \cap SO(3,1)$ by left shifts the Finsler pseudo-norm, via (2.3), for all

$x \in SO(3,1)$ and $y \in T_xM$. We note that Q depends on y only, and its expression produces the pseudo-Finsler structure of locally Minkowski type (2.4),

$$F_v(y) = \sqrt[4]{\det S_y}, \ \ \forall y \in T_xM, \ x \in M. \tag{2.5}$$

As well, in each fiber T_xM there lives the flat Frobenius Euclidean norm

$$||y||_h = \sqrt{Tr(y^t \cdot y)}, \ \ \forall y \in T_xM,$$

and further, one may consider its conformal deformation:

$$F_h(x,y) = e^{Q(x)} \cdot ||y||_h, \ \ \forall (x,y) \in TM, \tag{2.6}$$

which provides a genuine Riemannian metric on M.

The metric function F_h is also Finslerian, but - unlike the vertical one F_v, it is of Riemannian type (the halved $y-$Hessian of its square provides a Remannian metric tensor field over M).

2.3 KCC invariants

The second order vector field expressed in the natural basis via:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \quad G^j = \frac{1}{2} g^{ji} \left(\frac{\partial^2 F^2}{\partial y^i \partial x^k} y^k + \frac{\partial F^2}{\partial x^i} \right), \tag{2.7}$$

is called *semispray.* The canonical Cartan nonlinear connection induced by the semispray G^j has the coefficients

$$N_i^a = \frac{\partial G^a}{\partial y^i},$$

which further leads to a horizontal distribution in (2.2), and the appropriate adapted basis.

The spray G^i and the induced nonlinear connection define the *dynamical covariant derivative*,

$$\nabla X^i = S(X^i) + N_j^i X^j,$$

which plays an essential role in defining the KCC invariants of the Finsler structure [1, 6]. The relations

$$F_{jk}^i = \frac{\partial N_j^i}{\partial y^k} = \frac{\partial^2 G^i}{\partial y^j \partial y^k}, \qquad C_{jk}^i = 0,$$

define the components of the Berwald linear connection D and further the associated horizontal and vertical Berwald covariant derivatives which act on a vector field $X \in \chi(TM)$ via

$$X^i_{|k} = \frac{\delta X^i}{\delta x^k} + F^i_{jk} X^j, \qquad X^i|_k = \frac{\partial X^i}{\partial y^k} + C^i_{jk} X^j.$$

Any Finsler connection $(N_j^i, F_{jk}^i, C_{jk}^i)$ defines a *Finsler covariant differential*,

$$\nabla X^i = DX^i - y^i_{|j} X^j, \quad DX^i = S(X^i) + F^i_{jk} X^j.$$

The considered geodesic system of second order differential equations has the form,

$$\frac{d^2x^i}{dt^2} - 2G^i\left(x, \frac{dx}{dt}\right) = 0, \tag{2.8}$$

which further produces an alternative derived geometric setup. The paths are also described by the system of differential equations

$$\begin{cases} \dot{x}^i = y^i \\ \dot{y}^i = -2G^i(x, y). \end{cases} \tag{2.9}$$

The extended trajectories do not exhibit a chaotical behavior [2], and hence their associated KCC invariants and structural stability can be investigated. The related stability theory of a second order differential (the KCC-theory), essentially deals with the five invariants which are tightly related to the behavior of the solutions of the system and to their stability [6]. It was shown in [1] that the KCC invariants have a special meaning in the Finsler framework, which is particularly convenient to describe phenomena where the dynamics depends on the present state and on the direction of change. The Finsler function produces the spray (2.7) and the associated differential geodesic system (2.8). In the Berwald basis, with emphasized horizontal and vertical part, the spray has the form

$$S = y^i \frac{\delta}{\delta x^i} + \varepsilon^i \frac{\partial}{\partial y^i},$$

where

$$\varepsilon^i(x, y) = N^i_j(x, y) y^j - 2G^i(x, y) \tag{2.10}$$

is a d-tensor field called *the first KCC invariant* or *deviation tensor*, since it exhibits the deviation of the horizontality of the spray.

The second KCC invariant S^i_j is significant for the variational equations of the system (2.8), being defined in terms of the semispray,

$$S^i_j = R^i_{jk} y^k - \varepsilon^i_{|j} = 2\frac{\partial G^i}{\partial x^j} + 2G^s F^i_{js} - \frac{\partial N^i_j}{\partial x^s} y^s - N^i_s N^s_j - \frac{\partial N^i_j}{\partial x^s} y^s, \tag{2.11}$$

where R^i_{jk} is the only nonzero component of the torsion of the Berwald connection, and the third KCC invariant,

$$B^i_{jk} = R^i_{jk} = \frac{\delta N^i_k}{\delta x^j} - \frac{\delta N^i_j}{\delta x^k}. \tag{2.12}$$

The fourth KCC invariant is the partial derivative of the latter one with respect to y, and it coincides with the Riemann-Christoffel curvature tensor the first nonzero component of the curvature of the Berwald connection,

$$B^i_{jkl} = \frac{\delta F^i_{jk}}{\delta x^l} - \frac{\delta F^i_{jl}}{\delta x^k} + F^i_{ml} F^m_{jk} - F^i_{mk} F^m_{jl}. \tag{2.13}$$

The fifth KCC invariant is defined just as the Douglas tensor of the semispray, the second (and the last) nonzero component of the curvature of the Berwald connection,

$$D^i_{jkl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}. \tag{2.14}$$

2.4 The KCC-Mueller case

In the framework of KCC geometry developed in §2.3, one can examine the geodesic differential system and the associated KCC invariants of the Finsler structures determined by F_h and F_v. We first note that the first invariant (2.10) identically vanishes, since both our investigated structures are of Finsler type.

For the locally Minkowski Finsler $m-th$ root Finsler fundamental function F_v, the independence of the metric h_{ij} on x leads to $G^i = 0$, $N^i_j = 0$ and $F^i_{jk} = 0$. Hence all the KCC invariants given by (2.11)-(2.14) are identically vanishing, and all the equations of geodesics (2.8) are trivial, with solutions consisting of segments of line.

For the Finsler particular Riemann conformally-deformed Euclidean structure (M, F_h), one has the spray G (2.7) given by $G^i = \frac{1}{2}\gamma^i_{jk}y^j y^k$ with γ^i_{jk} the Christoffel coefficients of second kind of the Riemannian metric g,

$$\gamma^i_{jk} = \frac{1}{2}g^{is}(g_{js,k} + g_{ks,j} - g_{ij,s}), \tag{2.15}$$

where the " $,i$ " indicates the partial derivative relative to x^i.

In this case, the KCC invariants are given by:

$$S^i_j = r^i_{sjt}y^s y^t, \quad B^i_{jk} = r^i_{jkt}y^t, \quad B^i_{jkl} = r^i_{jkl}, \quad D^i_{jkl} = 0,$$

and the geodesic paths are generally highly non-trivial. Several open problems are currently under research, including the relation to the physics background of the spectral data of the second KCC invariant, of the Berwald curvature KCC invariants and of the curvature of the nonlinear Cartan connection, and the related flatness case of these invariants.

3 The (h, v)-model. Einstein equations

It is known that in the framework of vector bundles, numerous models rely on $(h, v)-$metrics of the form

$$\mathcal{G} = g_{ij}(x,y)dx^i \otimes dx^j + h_{ab}(x,y)\delta y^a \otimes \delta y^b. \tag{3.1}$$

We shall briefly describe the basic geometric objects which are ingredients to the extended Maxwell and Einstein field equations, related to such structures [15]. Let M be endowed with a fixed non-linear connection $N \equiv \{N^a_i\}$. If D is a linear d-connection on TM, then it is described by its adapted coefficients $D\Gamma(N) = \{L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\}$, where:

$$D_{\delta_k}\delta_j = L^i{}_{jk}\delta_i, \quad D_{\delta_k}\dot{\partial}_b = L^a{}_{bk}\dot{\partial}_a,$$
$$D_{\dot{\partial}_c}\delta_j = C^i{}_{jc}\delta_i, \quad D_{\dot{\partial}_c}\dot{\partial}_b = C^a{}_{bc}\dot{\partial}_a.$$

We shall further denote by $_{|}$ and $|$ the $h-$ and $v-$ covariant derivatives induced by D, respectively.

As well, the torsion T of the linear connection D has the adapted components

$$\begin{array}{ll} hT(\delta_k,\delta_j)=T^i{}_{jk}\delta_i, & vT(\delta_k,\delta_j)=R^a{}_{jk}\dot\partial_a,\\ hT(\dot\partial_c,\delta_j)=C^i{}_{jc}\delta_i, & vT(\dot\partial_c,\delta_j)=P^a{}_{jc}\dot\partial_a,\\ hT(\dot\partial_c,\dot\partial_b)=0, & vT(\dot\partial_c,\dot\partial_b)=S^a{}_{bc}\dot\partial_a, \end{array}$$

while the adapted components of the curvature R are

$$\begin{array}{ll} R(\delta_l,\delta_k)\delta_j=R^i_{jkl}\delta_i, & R(\delta_l,\delta_k)\dot\partial_b=R^a_{bkl}\dot\partial_a,\\ R(\dot\partial_c,\delta_k)\delta_j=P^i_{jkc}\delta_i, & R(\dot\partial_c,\delta_k)\dot\partial_b=P^a_{bkc}\dot\partial_a,\\ R(\dot\partial_c,\dot\partial_b)\delta_j=S^i_{jbc}\delta_i, & R(\dot\partial_d,\dot\partial_c)\dot\partial_b=S^a_{bcd}\dot\partial_a. \end{array}$$

The explicit components of the torsion are explicitly given by

$$T^i_{jk}=L^i_{[jk]},\;\; R^a_{jk}=\delta_{[j}N^a_{k]},\;\; P^i_{ja}=C^i_{ja},\;\; P^a_{jb}=\frac{\partial N^a_j}{\partial y^b}-L^a_{bi},\;\; S^a_{bc}=C^a_{[bc]}, \tag{3.2}$$

while the curvature coefficients are

$$\left\{\begin{array}{l} R_j{}^i{}_{kl}=\delta_{[l}L^i_{jk]}+L^s_{j[k}L^i_{sl]}+C^i_{je}R^e_{kl},\\ R_b{}^a{}_{kl}=\delta_{[l}L^a_{bk]}+L^s_{b[l}L^a_{sl]}+C^a_{be}R^e_{kl},\\ P_j{}^i{}_{kc}=\dot\partial_c L^i_{jk}-C^i_{jc|k}+C^i_{jb}P^b_{kc},\\ P_b{}^a{}_{kc}=\dot\partial_c L^a_{bk}-C^a_{bc|k}+C^a_{be}P^e_{kc},\\ S_j{}^i{}_{cd}=\dot\partial_{[d}L^i_{jc]}+C^k_{j[c}C^i_{kd]},\\ S_b{}^a{}_{cd}=\dot\partial_{[d}L^a_{bc]}+C^e_{j[c}C^a_{ed]}, \end{array}\right.$$

where we denoted by $\tau_{\dots[i\dots j]\dots}=\tau_{\dots i\dots j\dots}-\tau_{\dots j\dots i\dots}$. As well, the generalized Ricci tensor has the components:

$$R_{jk}=R^s_{jks},\;\; \overset{1}{P}{}_{bj}=P^e_{bje},\;\; \overset{2}{P}{}_{jb}=P^k_{jkb},\;\; S_{ab}=S^e_{abe},$$

and the scalars of curvature

$$R=g^{ij}R_{ij},\;\; S=h^{ab}S_{ab}.$$

Then the Einstein equations of the (h,v)-structure are [15]

$$\left\{\begin{array}{l} R_{ij}-(R+S)g_{ij}=\kappa T_{ij},\\ S_{ab}-(R+S)h_{ab}=\kappa T_{ab},\\ \overset{1}{P}{}_{bj}=\kappa T_{bj},\;\; \overset{2}{P}{}_{jb}=-\kappa T_{jb}, \end{array}\right. \tag{3.3}$$

where κ is a constant and $T_{\alpha\beta}$ are the components of the energy-momentum tensor field. The conservation laws for these equations have the form

$$\left\{\begin{array}{l} \left(R^i_j-\dfrac{1}{2}(R+S)\delta^i_j\right)_{|i}+\left.\overset{1}{P}{}^a_j\right|_a=0,\\ \left.\left(S^a_b-\dfrac{1}{2}(R+S)\delta^a_b\right)\right|_a+\overset{2}{P}{}^i_{b|i}=0. \end{array}\right.$$

Jointly with N, the metric (3.1) produces the *canonical* metrical $d-$connection $C\Gamma(N)$ [15],

$$(3.4)\qquad \begin{cases} L^i{}_{jk} = \dfrac{1}{2}g^{ih}\left(\dfrac{\delta g_{hj}}{\delta x^k} + \dfrac{\delta g_{hk}}{\delta x^j} - \dfrac{\delta g_{jk}}{\delta x^h}\right), \\ L^a{}_{bk} = \dfrac{\partial N^a_k}{\partial y^b} + \dfrac{1}{2}h^{ac}\left(\dfrac{\delta h_{bc}}{\delta x^k} - \dfrac{\partial N^d_k}{\partial y^b}h_{dc} - \dfrac{\partial N^d_k}{\partial y^c}h_{bd}\right), \\ C^i{}_{jc} = \dfrac{1}{2}g^{ih}\dfrac{\partial g_{jh}}{\partial y^c}, \\ C^a{}_{bc} = \dfrac{1}{2}h^{ad}\left(\dfrac{\partial h_{db}}{\partial y^c} + \dfrac{\partial h_{dc}}{\partial y^b} - \dfrac{\partial h_{bc}}{\partial y^d}\right). \end{cases}$$

3.1 The (h, v)-Riemann-locally Minkowski structure

We shall specify the previous considerations to the Finsler structures (M, F_h) and (M, F_v) given respectively by (2.6) and (2.5). The pair (F_h, F_v) provides the Finsler metric tensors

$$(3.5)\qquad g_{ij} = \frac{1}{2}\frac{\partial^2 F_h^2}{\partial y^i \partial y^j}, \qquad h_{ab} = \frac{1}{2}\frac{\partial^2 F_v^2}{\partial y^a \partial y^b},$$

of which the first is positively definite and its coordinate-free expression is given by

$$(3.6)\qquad g_x(A, B) = e^{2\det(S_x)}\sqrt{Tr(A^t B)}, \quad \forall A, B \in T_x M.$$

The two metrics become ingredients for building the particular (h, v) Riemann-locally Minkowski structure (briefly, $R\ell M$) type

$$(3.7)\qquad \mathcal{G} = g_{ij}(x)dx^i \otimes dx^j + h_{ab}(y)\delta y^a \otimes \delta y^b,$$

which we shall use in further considerations.

For a given nonlinear connection $\{N^a_i\}$, we shall further denote the fields of the adapted (horizontal and respectively vertical) bases by

$$\left\{\delta_i = \frac{\delta}{\delta x^i}, \dot{\partial}_a = \frac{\partial}{\partial y^a} \;\middle|\; i, a = \overline{1,4}\right\},$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^b{}_i \frac{\partial}{\partial y^b}, \; i = \overline{1,4}.$$

We also denote the dual basis by $\{dx^i, \delta y^a \mid i, a = \overline{1,4}\}$, with $\delta y^a = dy^a + N^a{}_j dx^j$. This allows us to naturally introduce the Riemann-locally Minkowski (h, v)-metric (3.7) on TM, whose main geometric objects and extended field equations will be further described.

4 The vertical Kern $R\ell M$ model

Definition 4.1. If the (h,v)-metric

$$\mathcal{G} = g_{ij}(x,y)dx^i \otimes dx^j + h_{ab}(x,y)\delta y^a \otimes \delta y^b \tag{4.1}$$

has the property that in the neighborhood of any point $(x,y) \in TM$ there exists a local map in which $h(x,y) = h(y)$, then it is called *v-locally Minkowski.*

This is the case when the nonlinear connection is provided by the Kern spray related to the square of the vertical Finsler function F_v (2.5). For the metric h_{ab} given in (3.5), the (h,v)-metric $\mathcal{G}$ given in (3.7) is v-regular, which implies that the coefficients of its Kern canonical nonlinear connection N [8, 15] vanish,

$$N_i^a(x,y) = 0, \quad i,a = \overline{1,4}. \tag{4.2}$$

A known result provides consequences specific to this case, as follows [15]:

Theorem 4.1. *If $\mathcal{G}$ is a v-locally Minkowski metric and $h = h(y)$ is weakly regular, then the canonic linear d-connection D (3.4) given by $C\Gamma(N) = \{L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\}$ obeys the properties:*

1. *The canonical metrical linear d-connection $C\Gamma(N)$ associated to $\mathcal{G}$, is given by [15]:*

$$L^i{}_{jk} = \gamma^i{}_{jk},\ L^a{}_{bk} = 0,\ C^i{}_{jc} = 0,\ C^a{}_{bc} = \frac{1}{2}h^{ae}\left(\frac{\partial h_{eb}}{\partial y^c} + \frac{\partial h_{ec}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^e}\right),$$

   where $\gamma^i{}_{jk}$ denote the Christoffel symbols (2.15) *of g;*

2. $T^i{}_{jk} = 0,\ R^a{}_{jk} = 0,\ C^i_{ja} = 0,\ P^a_{bk} = 0,\ S^a{}_{bc} = 0;$

3. $R^i_{jkl} = r^i_{jkl},\ R^a_{b\ jk} = 0,\ P^a_{b\ kc} = 0,$ *where r^i_{jkl} are the curvature coefficients corresponding to the Levi-Civita connection γ^i_{jk}.*

Remark 4.2. In the v-Kern non-linear connection case, the following consequences hold true:

1. the equality $N^a_j = 0$ yields $\dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i}$;

2. the torsion of the canonic linear d-connection has a single non-vanishing component, namely the coefficient $P^a_{bk} = \gamma^a_{bk}$ of $hT(\dot{\partial}_a, \delta_j)$;

3. the curvature tensor fields are related to the torsion field via [15]

$$S^a_{d\ bc}y^d = S^a{}_{bc} = 0, y^d R^a_{d\ jk} = R^a{}_{jk} = 0,\ y^d P^a_{d\ kc} = P^a{}_{kc} = 0.$$

4.1 Einstein equations of the vertical Kern $R\ell M$ model

The curvatures of the canonical metrical linear d-connection associated to $\mathcal{G}$ in (3.7) with (3.5) are [15]:

$$\text{(4.3)}\qquad \begin{cases} R_{j\ kh}^{\ i} = r_{j\ kh}^{\ i}, \quad R_{b\ kh}^{\ a} = 0, \quad P_{j\ kc}^{\ i} = 0, \quad P_{b\ kc}^{\ a} = r^a_{bkc}, \quad S_{j\ bc}^{\ i} = 0, \\ S^a_{b\ cd} = \dfrac{\partial C^a_{\ bc}}{\partial y^d} - \dfrac{\partial C^a_{\ bd}}{\partial y^c} + C^f_{\ bc} C^a_{\ fd} - C^f_{\ bd} C^a_{\ fc}, \end{cases}$$

where $r^i_{j\ kh}$ are the components of the curvature tensor of the horizontal Riemannian metric. Taking into account the relations (4.3), it follows, as in [15], that the Einstein equations of the canonical metrical linear d-connection $C\Gamma(N)$ (3.4) and (4.2) can be written as

$$\text{(4.4)}\qquad \begin{cases} r_{ij} - \dfrac{1}{2}(r+S) g_{ij} = T^H_{ij}, \\ \overset{1}{T}_{bj} = 0, \quad \overset{2}{T}_{jb} = \overset{1}{P}_{jb}, \\ S_{ab} - \dfrac{1}{2}(r+S) h_{ab} = T^V_{ab}, \end{cases}$$

where r_{ij} denotes the Ricci tensor $r_{ij} = r_{i\ jh}^{\ h}$ attached to the Riemannian metric g, S_{ab} is the Ricci tensor attached to the vertical metric h_{ab}, r is the scalar curvature of r^i_{jkl}, and $T_{\alpha\beta}$ are the components of the energy-momentum tensor field. When comparing (4.4) with the classical Einstein equations of the Riemannian manifold (M, g), we have to notice in the h-part of the above equations the "perturbation" given by the term $-\dfrac{1}{2} S g_{ij}$. According to [15], the energy conservation law is identically satisfied by $C\Gamma(N)$.

4.2 Triviality of the Maxwell equations of the vertical Kern $R\ell M$ model

We note that the deflection tensors are, for the Kern trivial n-linear connection attached to the Lagrangian F_v^2,

$$D^a_{\ j} \equiv y^a_{\ |j} = \frac{\partial y^a}{\partial x^j} + y^b L^a_{\ bj} = 0, \quad d^a_{\ b} \equiv y^a|_b = \delta^a_b + y^c C^a_{\ cb}.$$

But the 0-homogeneity of the metric h_{ab} and the equality $C^a_{bc} = \frac{1}{2} h^{ae} \dfrac{\partial h_{be}}{\partial y^c}$ infer $C^a_{\ bc} y^c = 0$, whence $d^a_b = \delta^a_b$. Hence, the F_v Finslerian linear d-connection is *of Cartan type* [15], and the covariant deflection tensors are

$$D_{ij} = 0, \quad d_{ab} = h_{ab},$$

where the indices were raised/lowered using the corresponding parts of the $(h, v)-$metric. We obtain subsequently that the *electromagnetic tensors identically vanish*,

$$\begin{cases} F_{ij} = \dfrac{1}{2}(D_{ij} - D_{ji}) = 0, \\ f_{ab} = \dfrac{1}{2}(d_{ab} - d_{ba}) = 0. \end{cases}$$

5 The horizontal Kern $R\ell M$ model

Let TM be endowed with the nonlinear connection N with coefficients

$$N^a_{\ i} = \gamma^a_{ik}(x)y^k, \tag{5.1}$$

where γ^a_{jk} are the Christoffel coefficients of second kind (2.15) of the Riemannian metric g. Using (5.1) and (2.15), one can easily state the following

Proposition 5.1. *a) The Christoffel symbols of the horizontal Riemannian metric g* (3.6) *have the explicit form*

$$\gamma^i_{jk} = \frac{1}{2}(\delta^i_j Q_{,k} + \delta^i_k Q_{,j} - \delta^{is}\delta_{jk}Q_{,s}).$$

b) The Cartan nonlinear connection associated to F_h coincides with the Kern nonlinear connection provided by F^2_h, and has the components

$$N^a_i \equiv \gamma^a_{ij}y^j = \frac{1}{2}(y^a Q_{,i} + Q_0\delta^a_i - Q^a y_i),$$

where $Q_0 = Q_{,i}y^i$, $Q^a = g^{aj}Q_{,j}$, $y_i = g_{ir}y^r$ and $\{g^{ij}\}$ are the components of the metric dual to g ($g^{is}g_{sj} = \delta^i_j$).

Theorem 5.2. *The components of the canonic metric $h(hh)$- and $v(vv)$-torsionless d-linear connection D for the horizontal Kern nonlinear connection have the form:*

$$L^i_{jk} = \gamma^i_{jk}, \quad L^a_{bk} = \frac{1}{2}(\gamma^a_{bk} - 2C^a_{bd}\gamma^d_{kj}y^j - h^{ac}h_{bd}\gamma^d_{kc}),$$
$$C^i_{ja} = 0, \qquad C^a_{bc} = \tfrac{1}{4}\frac{\partial^3 F^2_v}{\partial y^b \partial y^c \partial y^e}h^{ea},$$

where r^i_{jkl} is the Riemann tensor of the Levi-Civita connection γ^i_{jk},

$$r^i_{jkl} = \delta_{[l}\gamma^i_{jk]} + \gamma^i_{s[l}\gamma^s_{jk]}.$$

While considering the non-trivial nonlinear connection $N^a_j = \gamma^a_{jk}y^k$, the results dramatically change compared to the vertical Kern non-linear connection case, as described in the following sections.

5.1 Einstein equations of the horizontal Kern $R\ell M$ model

The curvatures of the canonical metrical linear d-connection associated to $\mathcal{G}$ in (3.7) with (3.5) are

$$\begin{cases} R_j{}^i{}_{kh} = r_j{}^i{}_{kh}, \ \ P_j{}^i{}_{kc} = 0, \ \ S_j{}^i{}_{bc} = 0, \\ S^a_{b\ cd} = \dfrac{\partial C^a{}_{bc}}{\partial y^d} - \dfrac{\partial C^a{}_{bd}}{\partial y^c} + C^f{}_{bc}C^a{}_{fd} - C^f{}_{bd}C^a{}_{fc}, \end{cases} \tag{5.2}$$

where $r^i_{j\ kh}$ are the components of the curvature tensor of the horizontal Riemannian metric. We note that in this case, $P^a_{b\ kc} \neq 0$, which reveals a strong interaction between the basic and fiberwise phenomena, which further influences the shape of the mixed Einstein equations (5.3). As well, the $v-v(hv)$ curvature $R^a_{b\ kl} \neq 0$ is present, exhibiting a mixed Riemann-Christoffel endomorphism within fibers. Taking into account the relations (4.3), it follows that the Einstein equations of the canonical metrical linear d-connection $C\Gamma(N)$ (3.4) and (4.2) can be written as

$$\begin{cases} r_{ij} - \frac{1}{2}(r+S)g_{ij} = T^H_{ij}, \\ \overset{1}{P}{}_{bj} = \overset{1}{T}{}_{bj}, \quad \overset{2}{T}{}_{jb} = 0, \\ S_{ab} - \frac{1}{2}(r+S)h_{ab} = T^V_{ab}, \end{cases} \tag{5.3}$$

where r_{ij} denotes the Ricci tensor $r_{ij} = r_{i\ jh}^{\ h}$ attached to the Riemannian metric g, S_{ab} is the Ricci tensor attached to the vertical metric h_{ab}, r is the scalar curvature of r^i_{jkl} and $T_{\alpha\beta}$ are the components of the energy-momentum tensor field. Like in the vertical Kern case, the classical Einstein equations of the Riemannian manifold (M, g) are perturbed by the term $-\frac{1}{2}Sg_{ij}$. The energy conservation law is identically satisfied by $C\Gamma(N)$.

5.2 Maxwell equations of the horizontal Kern $R\ell M$ model

We note that the deflection tensors for the Kern trivial non-linear connection attached to F^2_h, are

$$D^a{}_j = y^a{}_{|j}, \quad d^a{}_b = y^a|_b = \delta^a_b + y^c C^a{}_{cb}.$$

Like in section 4.2, we infer $d^a_b = \delta^a_b$ and hence $d_{ab} = h_{ab}$. Therefore *the vertical electromagnetic tensor identically vanishes*,

$$f_{ab} = \frac{1}{2}(d_{ab} - d_{ba}) = 0,$$

whereas, regarding the horizontal deflection, we get

$$D^a{}_j = -\frac{\delta y^a}{\delta x^j} + y^b L^a{}_{bj} = -\frac{1}{2}(N^a_j + h^{ac} y_d \gamma^d_{cj}).$$

Therefore, $F_{ij} \equiv D_{[ij]} \neq 0$, whence one gets the nontrivial set of horizontal Maxwell equations

$$\mathbf{S}\, F_{ij|k} = 0 \Leftrightarrow \underset{(ijk)}{S}\ g_{\,[ie}(A^e{}_{j]})_{|k} = 0,$$

where $A^e_j = y^b \gamma^e_{bj} + y_b h^{ec} \gamma^d_{cj}$.

6 Finslerian det-classification for the Mueller set

In order to point out the shape of the considered Finsler structures induced on certain Mueller type submanifolds embedded in the space $\mathcal{M}_{4\times4}(\mathbb{R}) \equiv \mathbb{R}^{16}$, we consider the specific parametrization induced by the Dirac decomposition, given by the one-to-one correspondence

$$(6.1)\quad \begin{aligned} \mathbf{x} &= (x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}, x^{12}, x^{13}, x^{14}, x^{15}, x^{16}) \\ &\equiv (k_0, k_1, k_2, k_3, m_0, m_1, m_2, m_3, n_0, n_1, n_2, n_3, \ell_0, \ell_1, \ell_2, \ell_3) \in \mathbb{R}^{16} \\ &\leftrightarrow \underbrace{\begin{pmatrix} K & N \\ L & M \end{pmatrix}}_{\mathbf{M}} \equiv \begin{pmatrix} \begin{pmatrix} k_0+k_3 & k_1+k_2 \\ k_1-k_2 & k_0-k_3 \end{pmatrix} & \begin{pmatrix} n_0+n_3 & n_1+n_2 \\ n_1-n_2 & n_0-n_3 \end{pmatrix} \\ \begin{pmatrix} \ell_0+\ell_3 & \ell_1+\ell_2 \\ \ell_1-\ell_2 & \ell_0-\ell_3 \end{pmatrix} & \begin{pmatrix} m_0+m_3 & m_1+m_2 \\ m_1-m_2 & m_0-m_3 \end{pmatrix} \end{pmatrix} \in \mathcal{M}_{4\times 4}(\mathbb{R}). \end{aligned}$$

More explicitly, we introduce the coordinates of $\mathbb{R}^{16}$

$$(6.2)\quad \begin{array}{lll} (x^1, x^2, x^3, x^4) & \equiv (k_0, k_1, k_2, k_3) & \leftrightarrow K \\ (x^5, x^6, x^7, x^8) & \equiv (m_0, m_1, m_2, m_3) & \leftrightarrow M \\ (x^9, x^{10}, x^{11}, x^{12}) & \equiv (n_0, n_1, n_2, n_3) & \leftrightarrow N \\ (x^{13}, x^{14}, x^{15}, x^{16}) & \equiv (\ell_0, \ell_1, \ell_2, \ell_3) & \leftrightarrow L. \end{array}$$

We shall further describe the Finsler norms $\mathbb{F}_h$ and $\mathbb{F}_v$ induced by F_h and F_v on several typical Mueller classes, which were recently determined by Ovsyuk and Redkov (see [19]).

A particular class of Mueller matrices will be regarded further as a real r-dimensional submanifold N immersed in the space $\mathbb{R}^{16}M \equiv \mathcal{M}_{4\times 4}(\mathbb{R})$, considering the identification given by (6.1). We assume that the local coordinates in M are given by (6.2), and we further respectively denote the local coordinates in TN and TM by (u, v) and (x, y). Then the local equations of the $2r$-dimensional submanifold TN in TM have the form

$$(6.3)\quad \begin{cases} x^i = x^i(u^1, \ldots, u^r), \quad \text{rank} \left\| \dfrac{\partial x^i}{\partial u^\alpha} \right\| = r, \\ y^i = \dfrac{\partial x^i}{\partial u^\alpha} v^\alpha, \ i = \overline{1, 16}. \end{cases}$$

The sets of Mueller classes can be categorized according to the shape of the induced vertical and respectively horizontal Finsler norms

$$\begin{cases} \mathbb{F}_v(\mathbf{uv}) = F_v(\mathbf{x}(\mathbf{u}), \mathbf{y}(\mathbf{x}(\mathbf{u}), \mathbf{y}(\mathbf{u}, \mathbf{v})) = \det(\mathbf{M})|_{\mathbf{x}\to\mathbf{y}(\mathbf{u},\mathbf{v})} \\ \mathbb{F}_h(\mathbf{u}, \mathbf{v}) = F_h(\mathbf{x}(\mathbf{u}), \mathbf{y}(\mathbf{x}(\mathbf{u}), \mathbf{y}(\mathbf{u}, \mathbf{v})) = e^{\det(\mathbf{M})|_{\mathbf{x}\to\mathbf{x}(\mathbf{u})}} \cdot \sqrt{\sum_{i=1}^{16} y_i^2(\mathbf{u}, \mathbf{v})}. \end{cases}$$

The next table provides a rough affiliation of the Mueller of the Mueller subclasses from [19], mainly based on the decomposability of the vertical Finsler pseudo-/norm $\mathbb{F}_v$ and on the freedom of the subclass in terms of parameters[2].

[2]Here we use the notations $\bar{\mathbf{k}} = (k_1, -ik_2, k_3)$ and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_0 = I_2, \sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices described in Section 1. With these notations, we have $K = k_0 I_2 + \bar{\mathbf{k}}\vec{\sigma}$.

Det-Finsler classes on Mueller sets in spinor formalism

Mueller set	Shape	r+p	Class	(Reference in [19])
K-1	$\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$	4+0	$\mathcal{D}^*$	(3.3)/4
K-2	$\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$	4+0	$\mathcal{C}^*$	(3.4c)/4
K-3	$\begin{pmatrix} K & 0 \\ tK & 0 \end{pmatrix}$	4+1	$\mathcal{D}$	(3.5c)/4
K-4	$\begin{pmatrix} K & tK \\ 0 & 0 \end{pmatrix}$	4+1	$\mathcal{D}$	(3.6c)/5
K-5	$\begin{pmatrix} K & sK \\ tK & stK \end{pmatrix}$	4+2	$\mathcal{D}$	(3.12a)/7
K-6	$\begin{pmatrix} K & sK \\ tk_0I_2-\frac{1}{s}\vec{k}\vec{\sigma} & stk_0I_2-\vec{k}\vec{\sigma} \end{pmatrix}$	4+2	$\mathcal{D}$	(3.13a)/7
K-7	$\begin{pmatrix} K & sk_0I_2+t\vec{k}\vec{\sigma} \\ -\frac{1}{t}K & -\frac{s}{t}k_0I_2-\vec{k}\vec{\sigma} \end{pmatrix}$	4+2	$\mathcal{D}$	(3.14c)/8
M-1	$\begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}$	4+0	$\mathcal{D}^*$	(4.4)/10
M-2	$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$	4+0	$\mathcal{C}^*$	(4.7)/11
M-3	$\begin{pmatrix} 0 & 0 \\ sM & M \end{pmatrix}$	4+1	$\mathcal{D}$	(4.8c)/11
M-4	$\begin{pmatrix} 0 & tM \\ 0 & M \end{pmatrix}$	4+1	$\mathcal{D}$	(4.9c)/11
M-5	$\begin{pmatrix} stm_0I_2-\vec{m}\vec{\sigma} & sM \\ tm_0I_2-\frac{1}{s}\vec{m}\vec{\sigma} & M \end{pmatrix}$	4+2	$\mathcal{D}$	(4.14b)/13
M-6	$\begin{pmatrix} -\frac{s}{t}m_0I_2-\vec{m}\vec{\sigma} & sm_0I_2+t\vec{m}\vec{\sigma} \\ -\frac{1}{t}M & M \end{pmatrix}$	4+2	$\mathcal{D}$	(4.14c)/14
M-7	$\begin{pmatrix} stM & tM \\ sM & M \end{pmatrix}$	4+2	$\mathcal{D}$	(4.15b)/14
N-1	$\begin{pmatrix} tN & N \\ 0 & 0 \end{pmatrix}$	4+1	$\mathcal{D}$	(5.7)/16
N-2	$\begin{pmatrix} tN & N \\ t^2N & tN \end{pmatrix}$	4+1	$\mathcal{D}$	(5.8)/16
N-3	$\begin{pmatrix} sn_0I_2+t\vec{n}\vec{\sigma} & N \\ -stn_0I_2-t^2\vec{n}\vec{\sigma} & -tN \end{pmatrix}$	4+2	$\mathcal{D}$	(5.10)/17
N-4	$\begin{pmatrix} tN & N \\ stn_0I_2-t^2\vec{n}\vec{\sigma} & sn_0I_2-t\vec{n}\vec{\sigma} \end{pmatrix}$	4+2	$\mathcal{D}$	(5.11)/17
L-1	$\begin{pmatrix} tL & 0 \\ L & 0 \end{pmatrix}$	4+1	$\mathcal{D}$	(6.6)/19
L-2	$\begin{pmatrix} tL & t^2L \\ L & tL \end{pmatrix}$	4+1	$\mathcal{D}$	(6.7)/19
L-3	$\begin{pmatrix} s\ell_0I_2+t\vec{\ell}\vec{\sigma} & -st\ell_0I_2-t^2\vec{\ell}\vec{\sigma} \\ L & -tL \end{pmatrix}$	4+2	$\mathcal{D}$	(6.8)/19

Mueller set	Shape	r+p	Class	(Reference in [19])
L-4	$\begin{pmatrix} tL & st\ell_0 I_2 - t^2\vec{\ell}\vec{\sigma} \\ L & s\ell_0 I_2 - t\vec{\ell}\vec{\sigma} \end{pmatrix}$	4+2	$\mathcal{D}$	(6.9)/19
KM-1	$\begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}$	8+0	$\mathcal{B}^*$	(7.8b)/23
KM-2	$\begin{pmatrix} K & 0 \\ t(M-K) & M \end{pmatrix}$	8+1	$\mathcal{B}$	(7.14d)/25
KM-3	$\begin{pmatrix} K & tM \\ \frac{1}{t}K & M \end{pmatrix}$	8+1	$\mathcal{D}$	(7.15c)/26
KM-4	$\begin{pmatrix} K & t(K-M) \\ 0 & M \end{pmatrix}$	8+1	$\mathcal{B}$	(7.17c)/27
KM-5	$\begin{pmatrix} K & t(K-M) \\ s(K-M) & M \end{pmatrix}$	8+2	$\mathcal{A}$	(7.23b)/32
LN-1	$\begin{pmatrix} tL & N \\ L & \frac{1}{t}N \end{pmatrix}$	8+1	$\mathcal{D}$	(8.10c)/37
LN-2	$\begin{pmatrix} tN & N \\ L & \frac{1}{t}L \end{pmatrix}$	8+1	$\mathcal{D}$	(8.12c)/38
KL-1	$\begin{pmatrix} K & tK \\ L & tL \end{pmatrix}$	8+1	$\mathcal{D}$	(9.9a)/41
KL-2	$\begin{pmatrix} K & L \\ L & K+L \end{pmatrix}$	8+0	$\mathcal{A}^*$	(9.10a)/42
NM-1	$\begin{pmatrix} tN & N \\ tM & M \end{pmatrix}$	8+1	$\mathcal{D}$	(10.2)/42
NM-2	$\begin{pmatrix} M+N & N \\ N & M \end{pmatrix}$	8+0	$\mathcal{A}^*$	(10.3)/42
KN-1	$\begin{pmatrix} K & N \\ tK & tN \end{pmatrix}$	8+1	$\mathcal{D}$	(11.7a)/46
KN-2	$\begin{pmatrix} K & N \\ 0 & K \end{pmatrix}$	8+0	$\mathcal{C}^*$	(11.8b)/46
ML-1	$\begin{pmatrix} tL & tM \\ L & M \end{pmatrix}$	8+1	$\mathcal{D}$	(12.2)/47
ML-2	$\begin{pmatrix} M & 0 \\ L & M \end{pmatrix}$	8+0	$\mathcal{C}^*$	(12.3)/47
KMN-1	$\begin{pmatrix} K & N \\ 0 & M \end{pmatrix}$	12+0	$\mathcal{B}^*$	(13.5)/48
KMN-2	$\begin{pmatrix} K & N \\ -K+M+N & M \end{pmatrix}$	12+0	$\mathcal{A}^*$	(13.6a)/49
KML-1	$\begin{pmatrix} K & 0 \\ L & M \end{pmatrix}$	12+0	$\mathcal{B}^*$	(13.7a)/49
KML-2	$\begin{pmatrix} K & -M+K+L \\ L & M \end{pmatrix}$	12+0	$\mathcal{A}^*$	(13.7b)/49
NLK-1	$\begin{pmatrix} K & N \\ L & (K+tN-\frac{1}{t}L) \end{pmatrix}$	12+1	$\mathcal{A}$	(14.3b)/51
NLM-1	$\begin{pmatrix} (M+tL-\frac{1}{t}N) & N \\ L & M \end{pmatrix}$	12+1	$\mathcal{A}$	(14.5)/52

Under these circumstances, the types of Mueller solutions described in [19], regarded as submanifolds of M become Finsler spaces with the fundamental functions (denoted by $\mathbb{F}_h$ and $\mathbb{F}_v$, respectively), induced from TM to TN by means of (6.3). The local coordinates $(u^1, \ldots, u^r)$ of the submanifold N are the components of $\mathbf{x}$ which explicitly appear in the K, L, M, N blocks of the Mueller set, while the real variables s and t - in case that these explicitly appear in the description of the Mueller set - are considered as parameters which provide orbits of Finsler structures $\mathbb{F}_h$ and $\mathbb{F}_v$ on the Mueller set, respectively; we denote by $p \in \overline{0,2}$ the number of parameters. We note that, due to the specific form of the Mueller sets and the properties of the determinant, there exist four main cases based on the Finsler pseudo-norm $\mathbb{F}_v$, denoted by $\mathcal{A}, \ldots, \mathcal{D}$, as follows:

$\mathcal{A}$: $\mathbb{F}_v$ is of general pseudo-Finsler locally Minkowski pseudo-norm of $4-$th root type;

$\mathcal{B}$: $\mathbb{F}_v \in \mathcal{A}$ has the extra quality of being canonically, decomposable as the geometric mean of two pseudo-Euclidean norms $\mathbb{F}_v = \sqrt{\mathbb{F}_v{}' \cdot \mathbb{F}_v{}''}$.[3]

$\mathcal{C}$: $\mathbb{F}_v \in \mathcal{B}$ is for $r = 4$ a pseudo-Euclidean norm, whose associated metric is of signature $(++, --)$; and for $r > 4$, sub-pseudo-Euclidean, with associated metric tensor of rank 4.

$\mathcal{D}$: The vertical structure collapses: $\mathbb{F}_v$ is identically vanishing.

The descriptive table presents in detail the classes associated to the basic Mueller subsets from [19].

The starred subcases are the fixed (non-parametrized, $p = 0$) particular subcases of the non-starred cases ($p \in \overline{1,2}$).

Regarding the Finsler pseudo-norm $\mathbb{F}_h$, all the structures are conformally Euclidean. The $\mathbb{F}_h$ structures all conformally-Euclidean, and in the degenerate (zero-determinant) subcases (within the set $\mathcal{D}$), they are canonically Euclidean. We note that the $\mathbb{F}_v$ structures may be non-trivial in the $\mathcal{A}, \mathcal{B}$ sets, non-trivial within $\mathcal{C}$ and trivial within $\mathcal{D}$. The chain inclusion $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ provides a rough classification of the global set Θ of Mueller nontrivial matrices described in [19]. This is given by the disjunct union $\Theta = (\Theta \backslash \mathcal{D}) \cup \mathcal{D}$, with

$$\begin{cases} \Theta \backslash \mathcal{D} = [\mathcal{A} \backslash (\mathcal{B} \cup \mathcal{D})] \cup [\mathcal{B} \backslash (\mathcal{C} \cup \mathcal{D})] \cup \mathcal{C} \\ \mathcal{D} = [\mathcal{D} \cap (\mathcal{A} \backslash \mathcal{B})] \cup (\mathcal{D} \cap \mathcal{B}). \end{cases}$$

7 Riemannian KCC-invariants for $SO_3(\mathbb{R})$

The Lie group of Euclidean rotations $SO(3, \mathbb{R})$ of $\mathbb{R}^3$ is locally described by matrices of the form

$$\begin{pmatrix} 1-f(e_2^2+e_3^2) & -e_3 \sin b + f e_1 e_2 & e_2 \sin b + f e_1 e_3 \\ e_3 \sin b + f e_1 e_2 & 1-f(e_3^2+e_1^2) & -e_1 \sin b + f e_2 e_3 \\ -e_2 \sin b + f e_1 e_3 & e_1 \sin b + f e_2 e_3 & 1-f(e_1^2+e_2^2) \end{pmatrix} \tag{7.1}$$

[3] The associated $v-$metric is for $r = 8$ starred case, pseudo-Euclidean of signature $(++++, ----)$, while for $r > 8$, pseudo-Euclidean with the signature $(\underbrace{+, \ldots, +}_{4 \text{ times}}, \underbrace{-, \ldots, -}_{4 \text{ times}}, \underbrace{0, \ldots, 0}_{r-8 \text{ times}})$. The general conformal Riemannian norm $\mathbb{F}_h$ has in the $\mathcal{B}$ case a factorizable exponent.

where $e_1^2+e_2^2+e_3^2=1$ and $f=1-\cos b$. One may consider here, e.g.:

$$(e_1,e_2,e_3)=(\sin\varphi\cos\theta,\sin\varphi\sin\theta,\cos\varphi),\quad (b,\varphi,\theta)\in(0,\pi)\times(0,\pi)\times(0,2\pi),$$

and consider $(b,\varphi,\theta;B,\Phi,\Theta)=(\underbrace{u^1,u^2,u^3}_{\mathbf{u}};\underbrace{v^1,v^2,v^3}_{\mathbf{v}})$ as local coordinates in $TSO(3)$.

While regarding $M=SO(3,\mathbb{R})$ immersed in $\tilde{M}=\mathcal{M}(3\times 3,\mathbb{R})\ni\begin{pmatrix} x^1 & x^2 & x^3\\ x^4 & x^5 & x^6\\ x^7 & x^8 & x^9\end{pmatrix}\equiv (x^1,\ldots,x^9)\in\mathbb{R}^9$, we can apply to M the procedure of constructing the induced from $\tilde{M}$ Finsler norms F_h and F_v.

After tedious computations, after re-denoting the coordinates on TM as $(x_1,x_2,x_3;y_1,y_2,y_3)$, we get $F_v\equiv 0$ and

$$F_h=\sqrt{2y_1^2+(-4\cos x_1-4\cos^2 x_3+4+4\cos x_1\cos^2 x_3)y_2^2+(-4\cos x_1+4)y_3^2}.$$

We note that the structure F_h is of Riemannian type, and that the constructed Finsler geometric objects therefore exhibit only the horizontal part, while the mixed and vertical d-objects identically vanish. We obtain the Riemannian metric tensor field and the nonlinear connection

$$\begin{cases} g=(g_{ij})=\begin{pmatrix} 2 & 0 & 0\\ 0 & -4\cos x_1-4\cos^2 x_3+4+4\cos x_1\cos^2 x_3 & 0\\ 0 & 0 & -4\cos x_1+4\end{pmatrix},\\ N=(N^i_j)=\begin{pmatrix} 0 & -y_2\sin x_1\sin^2 x_3 & -y_3\sin x_1\\ -\frac{1}{2}y_2\frac{\sin x_1}{\cos x_1-1} & m & y_2\frac{\cos x_3}{\sin x_3}\\ -y_3\frac{1}{2(\cos x_1-1)}\sin x_1 & -y_2\cos x_3\sin x_3 & -\frac{1}{2}y_1\frac{1}{\cos x_1-1}\sin x_1\end{pmatrix},\end{cases}$$

where $m=-\frac{1}{2}\frac{(2y_3\cos x_3-2y_3\cos x_3\cos x_1+y_1\sin x_1\sin x_3)}{sinx_3\cos x_1-1}$.

The first KCC invariant vanishes (since the structure is of Riemann, and hence, of Finsler type), while the second one $P=(P^i_j)$ has the components

$$\begin{aligned}
&{P^1}_1=\tfrac{1}{2}y_2^2\cos^2 x_3\cos x_1-\tfrac{1}{2}y_2^2\cos x_1-\tfrac{1}{2}y_3^2\cos x_1-\tfrac{1}{2}y_2^2\cos^2 x_3+\tfrac{1}{2}y_2^2+\tfrac{1}{2}y_3^2\\
&{P^1}_2=y_2y_1\tfrac{1}{2}\sin^2 x_3(\cos x_1-1),\quad {P^1}_3=\tfrac{1}{2}y_1y_3(\cos x_1-1),\quad {P^2}_1=-\tfrac{1}{4}y_2y_1\\
&{P^2}_2=\tfrac{1}{4}\frac{-2y_3^2\cos x_1+2y_3^2\cos^2 x_3\cos x_1+y_1^2-2y_3^2\cos^2 x_3-y_1^2\cos^2 x_3+2y_3^2}{\sin^2 x_3}\\
&{P^2}_3=-\tfrac{1}{2}y_2y_3(-\cos x_1-\cos^2 x_3+1+\cos x_1\cos^2 x_3)\frac{1}{\sin^2 x_3}\\
&{P^3}_1=-\tfrac{1}{4}y_1y_3,\quad {P^3}_2=-\tfrac{1}{2}y_2y_3(-\cos x_1-\cos^2 x_3+1+\cos x_1\cos^2 x_3)\\
&{P^3}_3=-\tfrac{1}{2}y_2^2\cos x_1+\tfrac{1}{2}y_2^2\cos^2 x_3\cos x_1+\tfrac{1}{2}y_2^2-\tfrac{1}{2}y_1^2y_2^2\cos^2 x_3+\tfrac{1}{4}y_1^2.
\end{aligned}$$

The last three invariants $R=({R^i}_{jk})$, $B=({B^i}_{jkl})$ and $D=(D_{ijkl})$ have the nontrivial coefficients

$R^1{}_{12} = -R^1{}_{21} = \frac{1}{6}\sin^2 x_3 \sin x_1 y_2 y_1;$
$R^1{}_{13} = -R^1{}_{31} = -\frac{1}{3}y_2^2 \sin x_3 \cos x_3 \cos x_1 + \frac{1}{3}y_2^2 \sin x_3 \cos x_3 + \frac{1}{6}y_3 y_1 \sin x_1;$
$R^1{}_{23} = -R^1{}_{32} = \frac{1}{3}y_2 y_1 \sin x_3(\cos x_1 - 1)\cos x_3;$
$R^2{}_{12} = -R^2{}_{21} = -\frac{1}{6}y_3^2 \sin x_1;$
$R^2{}_{13} = -R^2{}_{31} = \frac{1}{6}y_2 y_3 \sin x_1;$
$R^3{}_{12} = -R^3{}_{21} = \frac{1}{6}y_2 y_3 \sin x_1 \sin^2 x_3;$
$R^3{}_{13} = -R^3{}_{31} = -\frac{1}{6}y_2^2 \sin x_1 \sin^2 x_3;$
$R^3{}_{23} = -R^3{}_{32} = \frac{1}{3}y_2 y_3 \sin x_3 \cos x_3(\cos x_1 - 1);$

$B^1{}_{112} = -B^1{}_{121} = -\frac{1}{6}y_2 y_1 \sin^2 x_3 \cos x_1;$
$B^1{}_{123} = -B^1{}_{132} = \frac{1}{3}y_2 y_1 \sin x_3 \sin x_1 \cos x_3;$
$B^1{}_{131} = -B^1{}_{113} = \frac{1}{3}y_2^2 \sin x_1 \sin x_3 \cos x_3 + \frac{1}{6}y_3 y_1 \cos x_1;$
$B^1{}_{312} = -B^1{}_{321} = -\frac{1}{3}y_2 y_1 \sin x_3 \sin x_1 \cos x_3;$
$B^1{}_{313} = -B^1{}_{331} = \frac{1}{3}y_2^2(2\cos x_1 \cos^2 x_3 - \cos x_1 - 2\cos^2 x_3 + 1);$
$B^1{}_{323} = -B^1{}_{332} = -\frac{1}{3}y_2 y_1(2\cos x_1 \cos^2 x_3 - \cos x_1 - 2\cos^2 x_3 + 1);$
$B^2{}_{112} = -B^2{}_{121} = \frac{1}{6}y_3^2 \cos x_1; \quad B^2{}_{113} = -\frac{1}{6}y_2 \cos x_1 y_3;$
$B^3{}_{112} = -B^3{}_{121} = -\frac{1}{6}y_2 y_3 \cos x_1 \sin^2 x_3; \quad B^3{}_{113} = \frac{1}{6}y_2^2 \sin^2 x_3 \cos x_1 :$
$B^2{}_{113} = -B^2{}_{131} - \frac{1}{6}y_2 \cos x_1 y_3;$
$B^3{}_{123} = -B^3{}_{132} = \frac{1}{3}y_2 y_3 \sin x_1 \sin x_3 \cos x_3;$
$B^3{}_{131} = -B^3{}_{313} = -\frac{1}{6}y_2^2 \sin^2 x_3 \cos x_1;$
$B^3{}_{312} = -B^3{}_{321} = -\frac{1}{3}y_2 y_3 \sin x_1 \sin x_3 \cos x_3;$
$B^3{}_{323} = -B^3{}_{332} = -\frac{1}{3}y_2 y_3(2\cos x_1 \cos^2 x_3 - \cos x_1 - 2\cos^2 x_3 + 1);$
$B^3{}_{331} = -B^3{}_{313} = -\frac{1}{3}y_2^2 \sin x_1 \sin x_3 \cos x_3;$

$D^1{}_{111} = -\frac{1}{2}y_2^2 \cos^2 x_3 \cos x_1 + \frac{1}{2}y_2^2 \cos x_1 + \frac{1}{2}y_3^2 \cos x_1;$
$D^1{}_{113} = D^1{}_{131} = D^1{}_{311} = y_2^2 \sin x_1 \sin x_3 \cos x_3;$
$D^1{}_{133} = D^1{}_{313} = D^1{}_{331} = -y_2^2 \cos x_1(2\cos^2 x_3 - 1);$
$D^1{}_{333} = 4y_2^2 \sin x_1 \sin x_3 \cos x_3; \quad D^2{}_{111} = -\frac{1}{2}y_2 y_1 \frac{\cos x_1 + 2}{\cos^2 x_1 - 2\cos x_1 + 1};$
$D^2{}_{333} = -2y_2 y_3 \frac{2\cos^2 x_3 + 1}{\sin^4 x_3}; \quad D^3{}_{111} = -\frac{1}{2}y_1 y_3 \frac{\cos x_1 + 2}{\cos^2 x_1 - 2\cos x_1 + 1};$
$D^3{}_{333} = 2y_2^2(2\cos^2 x_3 - 1).$

Then the norm of the third KCC invariant R and the scalar curvatures of the last two invariants (B and D) are

$$\begin{aligned} R_\Delta = \;& -\tfrac{1}{72}y_2^4\cos^2 x_1\cos^4 x_3+\tfrac{1}{36}y_2^2y_3^2\cos^2 x_3\cos^2 x_1+\\ & \tfrac{1}{36}y_2^4\cos^2 x_3\cos^2 x_1-\tfrac{1}{72}y_3^4\cos^2 x_1-\tfrac{1}{72}y_2^4\cos^2 x_1-\\ & \tfrac{1}{36}y_2^2y_3^2\cos^2 x_1+\tfrac{1}{36}y_2^4\cos^4 x_3\cos x_1-\tfrac{1}{36}y_2^4\cos^2 x_3\cos x_1-\\ & \tfrac{1}{144}y_2^2y_1^2\cos x_1\cos^2 x_3-\tfrac{1}{36}y_2^2y_3^2\cos^2 x_3\cos x_1+\\ & \tfrac{1}{144}y_3^2y_1^2\cos x_1+\tfrac{1}{144}y_2^2y_1^2\cos x_1-\tfrac{1}{72}y_2^4\cos^4 x_3-\tfrac{1}{144}y_2^2y_1^2\cos^2 x_3+\\ & \tfrac{1}{16}y_2^2y_3y_1\sin x_3\cos x_3\sin x_1+\tfrac{1}{72}y_2^4+\tfrac{1}{144}y_3^2y_1^2+\\ & \tfrac{1}{144}y_2^2y_1^2+\tfrac{1}{36}y_2^2y_3^2+\tfrac{1}{72}y_3^4; \end{aligned}$$

$$\begin{aligned} B_\Delta = \;& -\tfrac{1}{12}y_2^2\cos^2 x_3\cos x_1+\tfrac{1}{12}y_2^2\cos x_1+\tfrac{1}{12}y_3^2\cos x_1+\tfrac{1}{6}y_2^2\cos^2 x_3-\tfrac{1}{12}y_2^2;\\ D_\Delta = \;& -\tfrac{1}{4}(y_2^2\cos^2 x_1\cos^2 x_3-3y_2^2\cos^2 x_3\cos x_1-y_2^2\cos^2 x_1+\\ & 2y_2^2\cos x_1-y_3^2\cos^2 x_1+y_3^2\cos x_1+4y_2^2\cos^2 x_3-2y_2^2)/(\cos x_1-1); \end{aligned}$$

The horizontal nontrivial components $L=(L^i{}_{jk})$ of the Cartan d-connection (which is in the Riemannian case the Levi-Civita Christoffel symbols of second kind of the metric g), are

$$\begin{aligned} & L^1{}_{22}=-\sin x_1\sin^2 x_3;\;\;(1,3,3)=-\sin x_1;\\ & L^2{}_{12}=L^2{}_{21}=\tfrac{1}{2}\tfrac{1}{-\cos x_1-\cos^2 x_3+1+\cos x_1\cos^2 x_3}\sin x_1\sin^2 x_3;\\ & L^2{}_{23}=L^2{}_{32}=\tfrac{1}{\sin x_3}\cos x_3;\\ & L^3{}_{13}=L^3{}_{31}=-\tfrac{1}{2}\tfrac{1}{\cos x_1-1}\sin x_1;\;\; L^3{}_{22}=-\sin x_3\cos x_3, \end{aligned}$$

and the nontrivial components of the h-curvature $(R^i{}_{jkl})$ are

$$\begin{aligned} & R^1{}_{212}=-R^1{}_{221}=\sin^2 x_3\cos x_1;\\ & R^1{}_{223}=-R^1{}_{232}=-2\sin x_1\sin x_3\cos x_3;\\ & R^1{}_{313}=-R^1{}_{331}=\cos x_1;\\ & R^2{}_{112}=-R^2{}_{121}=\tfrac{1}{2}\tfrac{\sin^2 x_3}{-\cos x_1-\cos^2 x_3+1+\cos x_1\cos^2 x_3};\\ & R^2{}_{323}=-R^2{}_{332}=-\tfrac{1}{\sin^2 x_3};\\ & R^3{}_{113}=-R^3{}_{131}=-\tfrac{1}{2}\tfrac{1}{\cos x_1-1};\\ & R^3{}_{223}=-R^3{}_{232}=-2\cos^2 x_3+1. \end{aligned}$$

The h-Ricci tensor Ric has the associated matrix

$$(Ric^H_{ij})=\text{diag}\,(Ric^H_{11},Ric^H_{22},Ric^H_{33}),$$

where

$$\begin{cases} Ric^H_{11}=\frac{\sin^2 x_3}{-\cos x_1-\cos^2 x_3+1+\cos x_1\cos^2 x_3}\\ Ric^H_{22}=\cos x_1\cos^2 x_3-\cos x_1-2\cos^2 x_3+1\\ Ric^H_{33}=\frac{\cos x_1\cos^2 x_3-\cos x_1+1}{\sin^2 x_3}, \end{cases}$$

the h-scalar of curvature R_* is

$$R_* = -\frac{1}{2}\frac{\cos x_1 \cos^2 x_3 - 2\cos^2 x_3 - \cos x_1 + 2}{\sin^2 x_3(\cos x_1 - 1)},$$

while the h-Einstein tensor Ein_h has the matrix

$$(Ein_{ij}) = \text{diag}\,(Ein_{11}, Ein_{22}, Ein_{33}),$$

where

$$\begin{cases} Ein_{11} = \frac{1}{2}(-1+\cos^2 x_3)\frac{\cos x_1}{\sin^2 x_3(\cos x_1 - 1)} \\ Ein_{22} = -1; \qquad Ein_{33} = \frac{2\cos^2 x_3 - 1}{\sin^2 x_3}, \end{cases}$$

and the deflection-generated l.h.s. 3-differential form (M_{ijk}) of the horizontal Maxwell equation has the generating non-trivial components

$$M^1{}_{23} = M^2{}_{31} = M^3{}_{12} = -\tfrac{2}{3}y_2 \sin x_1 \sin x_3 \cos x_3;$$
$$M^1{}_{32} = M^2{}_{13} = M^3{}_{21} = \tfrac{2}{3}y_2 \sin x_1 \sin x_3 \cos x_3.$$

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