A Geometry on Ricci solitons in \((LCS)_n\) manifolds

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Abstract. In this paper we study and obtain results on Ricci solitons in \((LCS)_n\) manifolds satisfying
\[ R(\xi, X) \cdot \tilde{P} = 0, \quad R(\xi, X) \cdot \tilde{M} = 0, \quad R(\xi, X) \cdot \tilde{H} = 0, \quad R(\xi, X) \cdot B = 0, \quad R(\xi, X) \cdot \tilde{C} = 0, \quad R(\xi, X) \cdot C = 0 \]
where \(\tilde{P}, \tilde{M}, \tilde{H}, B, \tilde{C}\), and \(C\) are Pseudo-projective, \(M\)-projective, conharmonic, \(C\)-Bochner, quasi-conformal and concircular curvature tensors.


Key words: Ricci soliton; \((LCS)_n\) manifold; Einstein metrics; pseudo-projective curvature tensor; \(M\)-projective curvature tensor; conharmonic curvature tensor; \(C\)-Bochner curvature tensor; quasi-conformal curvature tensor; concircular curvature tensor.

1 Introduction

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold \((M, g)\). A Ricci soliton is a triple \((g, V, \lambda)\) with \(g\) a Riemannian metric, \(V\) a vector field and \(\lambda\) a real scalar such that
\[ \mathcal{L}_V g + 2S + 2\lambda g = 0, \]
where \(S\) is a Ricci tensor of \(M\) and \(\mathcal{L}_V\) denotes the Lie derivative operator along the vector field \(V\). The Ricci soliton is said to be shrinking, steady and expanding according as \(\lambda\) is negative, zero and positive respectively. In this paper, we prove conditions for Ricci solitons in \((LCS)_n\) manifolds to be shrinking, steady and expanding.

The authors [19], [4] and [12] have studied Ricci solitons in contact and Lorentzian manifolds. The authors [14], [5] and [6] have obtained some results on Ricci solitons for Kenmotsu manifolds.

Motivated by the above work we obtained some interesting results on Ricci soliton in \((LCS)_n\)-manifolds satisfying
\[ \begin{align*}
   R(\xi, X) \cdot \tilde{P} &= 0, & R(\xi, X) \cdot B &= 0, & R(\xi, X) \cdot \tilde{M} &= 0, \\
   R(\xi, X) \cdot \tilde{H} &= 0, & R(\xi, X) \cdot \tilde{C} &= 0, & R(\xi, X) \cdot C &= 0.
\end{align*} \]
2 Preliminaries

An $n$-dimensional Lorentzian manifold $M$ is said to be a smooth connected paracom- pact Hausdorff manifold with a Lorentzian metric $g$, that is, $M$ admits a smooth symmetric tensor field $g$ of type $(0, 2)$ such that, for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a nondegenerate inner product of signature $(-, +, \ldots, +)$, where $T_pM$ denotes the tangent vector space of $M$ at $P$ and $\mathbb{R}$ is the real number space. A nonzero vector $v \in T_pM$ is said to be timelike (resp., non spacelike, null, and spacelike) if it satisfies $g_p(v, v) < 0$ (resp., $\leq 0$, $\equiv 0$ and $> 0$).

**Definition 2.1.** In a Lorentzian manifold $(M, g)$, a vector field $p$ defined by
\begin{equation}
(2.1) \quad g(X, P) = A(X)
\end{equation}
for any $X \in \Gamma(TM)$, is said to be a concircular vector field if
\begin{equation}
(2.2) \quad (\nabla_X A)(Y) = \alpha[g(X, Y) + w(X)A(Y)],
\end{equation}
where $\alpha$ is a non zero scalar and $w$ is a closed 1–from and $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$.

Let $M$ be $n$–dimensional manifold admitting a unit time-like concircular vector field $\xi$, called the characteristic vector field of the manifold. Then, we have
\begin{equation}
(\xi, \xi) = -1.
\end{equation}
Since $\xi$ is a unit concircular vector field, it follows that there exists a nonzero 1–form $\eta$ such that, for
\begin{equation}
(2.3) \quad (\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0)
\end{equation}
for all vector field $X, Y$ where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$ and $\alpha$ a non zero scalar function satisfying
\begin{equation}
\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X),
\end{equation}
$\rho$ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put
\begin{equation}
(2.4) \quad \phi X = \frac{1}{\alpha}\nabla_X \xi,
\end{equation}
then from (2.3) and (2.4) we have
\begin{equation}
(2.5) \quad \phi X = X + \nabla_X \xi,
\end{equation}
from which, it follows that $\phi$ is a symmetric $(1, 1)$ tensor and called the structure tensor of the manifold. Thus, the Lorentzian manifold $M$ together with the unit time like concircular vector field $\xi$, its associated 1–from $\eta$, and an $(1, 1)$ tensor field $\phi$ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold),
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Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto in $(LCS)_n$-manifolds for $n > 2$.

Further, on any $(LCS)_n$ manifold $M$, the following relations hold:

$$\phi^2 = X + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \eta \circ \phi = 0, \quad \phi \xi = 0,$$

$$(2.6)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X),$$

$$(2.7)$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \rho)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)),$$

$$(2.8)$$

$$R(X, Y)\xi = (\alpha^2 - \rho)g(X, Y)\xi - \eta(Y)X,$$

$$(2.9)$$

$$R(\xi, X)Y = (\alpha^2 - \rho)(\eta(X)\xi + X),$$

$$(2.10)$$

$$R(\xi, X)\xi = (\alpha^2 - \rho)(\eta(X)\xi + X),$$

$$(2.11)$$

where $R$ is the Riemannian curvature tensor.

Let $(g, V, \lambda)$ be a Ricci soliton in an $n$-dimensional $(LCS)_n$ manifold $M$. From (2.4) and (2.5) we have

$$(2.12)$$

$$(L_\xi g)(X, Y) = 2\alpha[g(X, Y) - \eta(X)\eta(Y)].$$

From (1.1) and (2.12) we get

$$(2.13)$$

$$S(X, Y) = -[\alpha \eta(X)\eta(Y) + (\alpha + \lambda)g(X, Y)].$$

The above equation yields that

$$(2.14)$$

$$QX = -[\alpha \eta(X)\xi + (\alpha + \lambda)X],$$

$$(2.15)$$

$$S(X, \xi) = -\lambda \eta(X),$$

$$(2.16)$$

$$r = -\lambda n - \alpha(n - 1),$$

where $S$ is the Ricci tensor, $Q$ is the Ricci operator and $r$ is the scalar curvature on $M$.

We have the following well known established Theorems:

**Theorem 2.1.** [7] Let $g$ be a differential real valued function on $\mathbb{R}^n$ and $c$ is a number. Then the subset $S : g(x, y, z) = c$ of $E^3$ is a surface if $dg \neq 0$ at any point of $S$.

**Theorem 2.2.** [7] If $S : g(x, y, z) = c$ is a surface in $\mathbb{R}^3$ then the gradient vector field $\nabla g$ (connected only at a point of $S$) is a non vanishing normal vector field on the entire surface $S$.

We can rewrite the above Theorems for higher dimensions as follows:

**Corollary 2.3.** Let $g$ be a differential real valued function on $n$-dimensional Euclidean space $\mathbb{R}^n$ and $c$ a number then $S : g(x_1, \ldots, x_n) = c$ in $\mathbb{R}^n$ is a surface (abstract surface or manifold) if $dg \neq 0$ at any point of $S$.

**Corollary 2.4.** If $S : g(x, y, z) = c$ is a surface (abstract surface or manifold) in $\mathbb{R}^n$ then the gradient vector field $\nabla g$ (connected only at points of $S$) is a non vanishing normal vector field on the entire surface (abstract surface or manifold) $S$.

Thus we have following results from the above:
Remark 2.2. According to Corollary 2.4 in our paper taking a real valued scalar function \( \alpha \) associated with an \((LCS)_n\) manifold with \( M = \mathbb{R}^3 \) and \( g = \alpha \) we have, \( \nabla \alpha \) as a non vanishing normal vector field on \( S \subset M \) and directional derivative of \( \alpha \) with respect to \( \xi, \xi \nabla \alpha = |\xi||\nabla \alpha|\cos(\xi, \nabla \alpha)\)

1. If \( \xi \) is tangent to \( S \) then \( \xi \alpha = 0 \).
2. If \( \xi \) is tangent to \( M \) but not to \( S \) then \( \xi \alpha \neq 0 \).
3. If the angle between \( \xi \) and \( \nabla \alpha \) is acute then \( 0 < \cos(\hat{\xi}, \nabla \alpha) < 1 \), then \( \xi \alpha = k|\nabla \alpha|, 0 < k < 1 \) and \( \xi \alpha > 0 \).
4. If the angle between \( \xi \) and \( \nabla \alpha \) is obtuse then \( -1 < \cos(\hat{\xi}, \nabla \alpha) < 0 \), then \( \xi \alpha = k|\nabla \alpha|, -1 < k < 0 \) and \( \xi \alpha < 0 \).

We see that Ricci solitons for \((LCS)_n\) manifolds depend on \( \xi \alpha \) in Sections (3) to (8).

3 Ricci soliton in \((LCS)_n\) manifolds satisfying
\( R(\xi, X) \cdot \vec{P} = 0 \)

The Pseudo-projective curvature tensor \( \vec{P} \) is defined by
\[
\vec{P}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] - \frac{r}{n} \left( \frac{a}{n-1} + b \right) [g(Y, Z)X - g(X, Z)Y],
\]
(3.1)
where \( a, b \neq 0 \) are constants. Taking \( Z = \xi \) in (3.1) and using (2.11), (2.13), (2.14), we get
\[
\vec{P}(X, Y)\xi = \left[ a(\alpha^2 - \rho) - b\lambda - \frac{r}{n} \left( \frac{a}{n-1} + b \right) \right] [\eta(Y)X - \eta(X)Y].
\]
(3.2)

Similarly using (2.8), (2.13), (2.14), (2.15) in (3.1), we get
\[
\eta(\vec{P}(X, Y)Z) = \left[ a(\alpha^2 - \rho) - b\lambda - \frac{r}{n} \left( \frac{a}{n-1} + b \right) \right] [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].
\]
(3.3)

We assume that the condition \( R(\xi, X) \cdot \vec{P} = 0 \), then we have
\[
R(\xi, X)\vec{P}(U, V)W - \vec{P}(R(\xi, X)U, V)W = \vec{P}(U, R(\xi, X)V)W - \vec{P}(U, V)R(\xi, X)W = 0.
\]
(3.4)

Using (2.10) in (3.4), we find
\[
(\alpha^2 - \rho)\eta(\vec{P}(U, V)W)X - g(X, \vec{P}(U, V)W)\xi - \eta(\xi)\vec{P}(U, V)X + g(X, U)\vec{P}(\xi, V)W - \eta(V)\vec{P}(U, X)W + g(X, V)\vec{P}(U, \xi)W - \eta(W)\vec{P}(U, V)X + g(X, W)\vec{P}(U, V)\xi = 0.
\]

By taking an inner product with \( \xi \) then we get
\[
\eta(\vec{P}(U, V)W)\eta(\xi) + g(X, \vec{P}(U, V)W)\xi - \eta(\xi)\eta(\vec{P}(U, V)W) + g(X, U)\vec{P}(\xi, V)W - \eta(V)\eta(\vec{P}(U, X)W) + g(X, V)\eta(\vec{P}(U, \xi)W) - \eta(W)\eta(\vec{P}(U, V)X) + g(X, W)\eta(\vec{P}(U, V)\xi) = 0.
\]
(3.5)
By using (3.2), (3.3) in (3.5), we have

\[(3.6)\]
\[g(X, P(U, V)W) + [a(\alpha^2 - \rho) - b\lambda - \frac{\alpha}{\eta} \left(\frac{a}{n-1} + b\right)] [g(X, V)g(U, W) - g(X, U)g(V, W)] = 0.\]

In view of (3.1) in (3.6), we have

\[(3.7)\]
\[ag(X, R(U, V)W) + b[(\alpha + \lambda)\{g(V, X)g(U, W) - g(V, W)g(U, X)\} + a\{g(V, X)\eta(W) - \eta(V)\eta(W)g(U, X)\} + [a(\alpha^2 - \rho) - b(\alpha + \lambda)] [g(X, V)g(U, W) - g(X, U)g(V, W)] = 0.\]

Taking \(X = U = e_i\) in (3.7) and summing over \(i = 1, 2, \ldots, n\), and on simplification, we get

\[(3.8)\]
\[aS(V, W) = -a(\alpha^2 - \rho)(1 - n)g(V, W) - b\alpha(1 - n)\eta(V)\eta(W).\]

Putting \(V = W = \xi\) in (3.8) and by virtue of (2.15), (2.16), we get the following equation

\[(3.9)\]
\[\lambda = \frac{(a(\alpha^2 - \rho) - b\alpha)(1 - n)}{a}.\]

Hence, Ricci solitons in \((LCS)_n\) manifold satisfy \(R(\xi, X) \cdot \bar{P} = 0\). We can consequently state the following theorem:

**Theorem 3.1.** A Ricci soliton in \((LCS)_n\) manifold satisfying \(R(\xi, X) \cdot \bar{P} = 0\) and \(\alpha\) is a positive function is

1. Shrinking if \(\alpha > \frac{b}{\alpha}\), Expanding if \(\alpha < \frac{b}{\alpha}\) and steady if \(\alpha = \frac{b}{\alpha}\), provided that \(\xi\) is orthogonal to \(\nabla \alpha\).

2. Shrinking if \(\alpha^2 + k|\nabla \alpha| > \frac{b}{\alpha}\), Expanding if \(\alpha^2 + k|\nabla \alpha| < \frac{b}{\alpha}\) and steady if \(\alpha^2 + k|\nabla \alpha| = \frac{b}{\alpha}\), provided that angle between \(\xi\) and \(\nabla \alpha\) is acute.

3. Shrinking if \(\alpha^2 > k|\nabla \alpha| + \frac{b}{\alpha}\), Expanding if \(\alpha^2 < k|\nabla \alpha| + \frac{b}{\alpha}\) and steady if \(\alpha^2 = k|\nabla \alpha| + \frac{b}{\alpha}\), provided that angle between \(\xi\) and \(\nabla \alpha\) is obtuse.

**Proof.**

1. From the Remark 2.2 (1) and (3.9)

\[\lambda = \frac{-(n - 1)(aa^2 - b\alpha)}{a}.\]

Hence \(\lambda < 0\) if \(\alpha > \frac{b}{\alpha}\), \(\lambda > 0\) if \(\alpha < \frac{b}{\alpha}\) and \(\lambda = 0\) if \(\alpha = \frac{b}{\alpha}\).

2. From Remark 2.2 (2) and (3.9),

\[\lambda = \frac{-(n - 1)[a(\alpha^2 + k|\nabla \alpha|) - b\alpha]}{a}.\]

Hence \(\lambda < 0\) if \(\alpha^2 + k|\nabla \alpha| > \frac{b}{\alpha}\), \(\lambda > 0\) if \(\alpha^2 + k|\nabla \alpha| < \frac{b}{\alpha}\) and \(\lambda = 0\) if \(\alpha^2 + k|\nabla \alpha| = \frac{b}{\alpha}\).
3. From the Remark 2.2 (3) and (3.9)

\[ \lambda = -\frac{(n-1)[a(\alpha^2 - k|\nabla \alpha|) - b\alpha]}{a}. \]

Hence \( \lambda < 0 \) if \( \alpha^2 > k|\nabla \alpha| + \frac{b}{a}\alpha \), \( \lambda > 0 \) if \( \alpha^2 < k|\nabla \alpha| + \frac{b}{a}\alpha \) and \( \lambda = 0 \) if \( \alpha^2 = k|\nabla \alpha| + \frac{b}{a}\alpha \).

\[ \square \]

**Corollary 3.2.** An LP–Sasakian manifold is shrinking if \( a > b \), expanding if \( a < b \) and steady if \( a = b \).

This follows by putting \( \alpha = 1 \) in item (1) of Theorem 3.1.

## 4 Ricci soliton in \((LCS)_n\) manifolds satisfying \( R(\xi, X)B = 0 \)

The C-Bochner curvature tensor \( B \) is defined by

\[
B(X, Y)Z = R(X, Y)Z + \frac{1}{n+3}[g(X, Z)QY - S(Y, Z)X - g(Y, Z)QX + S(X, Z)Y
+ g(\phi X, Z)Q\phi Y - S(\phi Y, Z)\phi X - g(\phi Y, Z)\phi QX + S(\phi X, Z)\phi Y + 2S(\phi X, Z)\phi Z
+ 2g(\phi X, Y)Q\phi Z + \eta(Y)\eta(Z)QX - \eta(Y)S(X, Z)\xi + \eta(X)S(Y, Z)\xi - \eta(X)\eta(Z)QY]
- \frac{D + n - 1}{n+3}[g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X + 2g(\phi X, Y)\phi Z]
+ \frac{D}{n+3}[\eta(Y)g(X, Z)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y - \eta(X)g(Y, Z)\xi]
- \frac{D - 4}{n+3}[g(X, Z)Y - g(Y, Z)X],
\]

where \( D = \frac{r+n-1}{n+1} \).

Taking \( Z = \xi \) in (4.1) and using (2.11), (2.13), (2.14), we get

\[ (4.2) \]

\[
\eta(B(X, Y)\xi) = \left[ (\alpha^2 - \rho) + \frac{2\alpha + 3\lambda}{n+3} + \frac{2r + 2n - 2 - 4n - 4}{(n+1)(n+3)} \right] [\eta(Y)X - \eta(X)Y].
\]

Similarly, using (2.8), (2.13), (2.14), (2.15) in (4.2), we get

\[ \eta(B(X, Y)Z) = \left[ (\alpha^2 - \rho) + \frac{\lambda}{n+3} - \frac{4 - 2D}{n+3} + \frac{2(\alpha + \lambda)}{n+3} \right] [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].
\]

We assume that the condition \( R(\xi, X)B = 0 \), then we have

\[ R(\xi, X)B(Y, Z)W - B(R(\xi, X)Y, Z)W
- B(Y, R(\xi, X)Z)W - B(Y, Z)R(\xi, X)W = 0.\]

By using (2.10) and taking an inner product with \( \xi \) then, by using (4.2) and (4.3) in (4.4), then we have

\[ 4S(Z, W) = \left[ (\alpha^2 - \rho)(n-1)(n+3) + \lambda(n-1) - (4 - 2D)(n-1)
+ 2(\alpha + \lambda)(n-1) + r + 4(\alpha + \lambda) - D + 3(D + n - 1) - (D - 4)(n-1) \right] g(Z, W)
- (4.5) \]

\[
- r - 2\lambda + 4(\alpha + \lambda) + D(n-2) + 3(D + n - 1)\eta(Z)\eta(W).
\]
Taking $Z = W = \xi$, and by virtue of (2.15), (2.16), we get the following equation

$$\lambda = \left[-\frac{(\alpha^2 - \rho)(n - 2) - 4\alpha}{5}\right]$$

(4.6)

Hence we can state the following result

**Theorem 4.1.** A Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi, X) \cdot B = 0$ is Shrinking if $\alpha$ is a positive function whose gradient vector field is orthogonal to $\xi$.

**Proof.** By the Remark 2.2 (1) and (4.6),

$$\lambda = -\frac{\alpha^2(n - 2) + 4\alpha}{5} \implies \lambda < 0.$$

However, results for cases of acute and obtuse angles between $\xi$ and $\nabla \alpha$ are little complex. 

\[\Box\]

## 5 Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi, X) \cdot \tilde{M} = 0$

The $M$-projective curvature tensor $\tilde{M}$ is defined by

$$\tilde{M}(X, Y) \xi = R(X, Y)Z - \frac{1}{2(n-1)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY],$$

Taking $Z = \xi$ in (5.1) and using (2.11), (2.13), (2.14), we get

(5.2) $$\tilde{M}(X, Y) \xi = \left[(\alpha^2 - \rho) + \frac{(\alpha + \lambda)}{2(n-1)} - \frac{\alpha}{2(n-1)}\right]\eta(Y)X - \eta(X)Y].$$

Similarly using (2.8), (2.13), (2.14), (2.15) in (5.2), we get

(5.3) $$\eta(\tilde{M}(X, Y)Z) = \left[(\alpha^2 - \rho) + \frac{(\alpha + \lambda)}{2(n-1)} - \frac{\alpha}{2(n-1)}\right][g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].$$

We assume that the condition $R(\xi, X) \cdot \tilde{M} = 0$, then we have

$$R(\xi, X)\tilde{M}(Y, Z)W - \tilde{M}(R(\xi, X)Y, Z)W$$

(5.4) $$\tilde{M}(Y, R(\xi, X)Z)W - \tilde{M}(Y, Z)R(\xi, X)W = 0.$$ By using (2.10) in (5.4), we have

$$[g(X, \tilde{M}(Y, Z)W)\xi - \eta(\tilde{M}(Y, Z)W)X - g(X, Y)\tilde{M}(\xi, Z)W + \eta(Y)\tilde{M}(X, Z)W - \tilde{M}(Y, \xi)Wg(X, Z) + \eta(Z)\tilde{M}(Y, X)W - g(X, W)\tilde{M}(Y, Z)\xi + \eta(W)\tilde{M}(Y, Z)X] = 0,$$

By taking an inner product with $\xi$ then we get,

(5.5) $$[g(X, \tilde{M}(Y, Z)W) + \eta(\tilde{M}(Y, Z)W)\eta(X) + g(X, Y)\eta(\tilde{M}(\xi, Z)W) - \eta(Y)\eta(\tilde{M}(X, Z)W) + \eta(M(Y, \xi)W)g(X, Z) - \eta(Z)\eta(\tilde{M}(Y, X)W) + g(X, W)\eta(\tilde{M}(Y, Z)\xi) - \eta(W)\eta(\tilde{M}(Y, Z)X)] = 0.$$
By using (5.2) and (5.3) in (5.5), we have

\[(\alpha^2 - \rho) + \frac{(\alpha + \lambda)}{2(n-1)} - \frac{\alpha}{2(n-1)} \right] [g(Y, W)g(X, Z) - g(X, Y)g(Z, W)] + g(X, M(Y, Z)W) = 0. \]

In view of (5.1) in (5.6) and taking \(X = Y = e_i\), we have

\[S(Z, W) = \left[ r + 2(n - 1)(\alpha + \lambda) + 2(n - 1)^2(\alpha^2 - \rho) - \alpha(n - 1) \right] g(Z, W). \]

Taking \(Z = W = \xi\), we get

\[\lambda = - \left[ (n - 1)(\alpha^2 - \rho) \right]. \]

Hence we can state the following theorem:

**Theorem 5.1.** A Ricci soliton in \((LCS)_n\) manifolds satisfying \(R(\xi, X) \cdot \tilde{M} = 0\) is

1. Shrinking if characteristic vector field \(\xi\) is orthogonal to \(\nabla \alpha\).
2. Shrinking of the angle between characteristic vector field \(\xi\) and the gradient vector field \(\nabla \alpha\) is acute.
3. If the angle between characteristic vector field \(\xi\) and the gradient vector field \(\nabla \alpha\) is obtuse then it is Shrinking if \(\alpha^2 > k|\nabla \alpha|\), Expanding if \(\alpha^2 < k|\nabla \alpha|\) and steady \(\alpha^2 = k|\nabla \alpha|\).

**Proof.** According to (5.8) and Remark 2.2, items (1), (3) and (4), we respectively have:

1. \(\lambda = -(n - 1)\alpha^2, \lambda < 0;\)
2. \(\lambda = -(n - 1)(\alpha^2 + k|\nabla \alpha|), \lambda < 0;\)
3. \(\lambda = -(n - 1)(\alpha^2 - k|\nabla \alpha|), \lambda < 0,\)

which concludes the proof.

\[\square\]

**6 Ricci soliton in \((LCS)_n\) manifolds satisfying \(R(\xi, X) \cdot H = 0\)**

The conharmonic curvature tensor \(H\) is defined by

\[H(X, Y)Z = R(X, Y)Z - \frac{1}{(n-2)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]. \]

Taking \(Z = \xi\) in (6.1) and using (2.11), (2.13), (2.14), we get

\[H(X, Y)\xi = \left[ (\alpha^2 - \rho) + \frac{2(\alpha + \lambda)}{(n-2)} - \frac{\alpha}{(n-2)} \right] [\eta(Y)X - \eta(X)Y]. \]

Similarly using (2.8), (2.13), (2.14), (2.15) in (6.2), we get

\[\eta(H(X, Y)Z) = \left[ (\alpha^2 - \rho) + \frac{2(\alpha + \lambda)}{(n-2)} - \frac{\alpha}{(n-2)} \right] [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \]
We assume that the condition $R(\xi, X) \cdot H = 0$, then we have

By using (2.10) in (6.4), we have
\[ [g(X, H(Y, Z)W)\xi - \eta(H(Y, Z)W)X] - g(X, Y)H(\xi, Z)W + \eta(Y)H(X, Z)W - H(Y, \xi)W g(X, Z) + \eta(Z)H(Y, X)W - g(X, W)H(Y, Z)\xi + \eta(W)H(Y, Z)X = 0, \]

By taking an inner product with $\xi$ then we get,
\[ [g(X, H(Y, Z)W) + \eta(H(Y, Z)W)\eta(X) + g(X, Y)\eta(H(\xi, Z)W) - \eta(Y)\eta(H(X, Z)W) + \eta(H(Y, \xi)W)g(X, Z) - \eta(Z)\eta(H(Y, X)W) + g(X, W)\eta(H(Y, Z)\xi) - \eta(W)\eta(H(Y, Z)X)] = 0. \]

By using (6.2) and (6.3) in (6.5), we have
\[ (\alpha^2 - \rho) \left( \frac{1}{n - 1} \right) g(Y, W)g(X, Z) - g(X, Y)g(Z, W) + g(X, H(Y, Z)W) = 0. \]

In view of (6.1) in (5.6) and taking $X = Y = e_i$, we have
\[ \left( (\alpha^2 - \rho)(1 - n)(n - 2) + 2(\alpha + \lambda)(1 - n) - \alpha(1 - n) - r \right) \frac{g(Z, W)}{n - 2} = 0, \]

where $g(Z, W) \neq 0$; therefore we get
\[ \lambda = [(\alpha^2 - \rho)(1 - n)]. \]

Hence we can state the following

**Theorem 6.1.** A Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi, X) \cdot H = 0$ is (6.8).

### 7 Ricci soliton in $(LCS)_n$ manifolds satisfying
\[ R(\xi, X) \cdot \tilde{C} = 0 \]

The Quasi-conformal curvature tensor $\tilde{C}$ is defined by
\[ \tilde{C}(X, Y)Z = aR(X, Y)Z + b(S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY) \]
\[ - \frac{r}{n} \left( \frac{a}{n - 1} + 2b \right) [g(Y, Z)X - g(X, Z)Y], \]

where $a, b \neq 0$ are constants. Taking $Z = \xi$ in (7.1) and using (2.11), (2.13), (2.14), we get
\[ \tilde{C}(X, Y)\xi = \]
\[ \left[ a(\alpha^2 - \rho) - b(2\lambda + \alpha) - \frac{r}{n} \left( \frac{a}{n - 1} + 2b \right) \right] [\eta(Y)X - \eta(X)Y]. \]
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Similarly using (2.8), (2.13), (2.14), (2.15) in (7.1), we get
\[
\eta(\tilde{C}(X,Y)Z) = \left[ a(\alpha^2 - \rho) - b(2\lambda + \alpha) - \frac{r}{n} \left( \frac{\alpha}{n-1} + 2b \right) \right] [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].
\]

We assume that the condition \(R(\xi,X) \cdot \tilde{C} = 0\), then we have
\[
R(\xi,X)\tilde{C}(U,V)W - \tilde{C}(R(\xi,X)U,V)W
\]
\[
- \tilde{C}(U,R(\xi,X)V)W - \tilde{C}(U,V)R(\xi,X)W = 0.
\]

Using (2.10) in (7.4), we find
\[
\eta(\tilde{C}(U,V)W)X - g(X,\tilde{C}(U,V)W)\xi - \eta(U)\tilde{C}(X,V)W
\]
\[
+ g(X,U)\tilde{C}(\xi,V)W - \eta(V)\tilde{C}(U,X)W + g(X,V)\tilde{C}(U,\xi)W
\]
\[
- \eta(W)\tilde{C}(U,V)X + g(X,W)\tilde{C}(U,V)\xi = 0.
\]

By taking an inner product with \(\xi\) then we get
\[
\eta(\tilde{C}(U,V)W)\eta(X) + g(X,\tilde{C}(U,V)W)\eta(U)\eta(\tilde{C}(X,V)W)
\]
\[
+ g(X,U)\eta(\tilde{C}(\xi,V)W) - \eta(V)\eta(\tilde{C}(U,X)W) + g(X,V)\eta(\tilde{C}(U,\xi)W)
\]
\[
- \eta(W)\eta(\tilde{C}(U,V)X) + g(X,W)\eta(\tilde{C}(U,V)\xi) = 0.
\]

By using (7.2), (7.3) in (7.6), we have
\[
g(X,\tilde{C}(U,V)W) + \left[ a(\alpha^2 - \rho) - b(2\lambda + \alpha) - \frac{r}{n} \left( \frac{\alpha}{n-1} + 2b \right) \right]
\]
\[
\left[ g(X,V)g(U,W) - g(X,U)g(V,W) \right] = 0.
\]

In view of (7.1) in (7.7) and Taking \(X = U = e_i\), summing over \(i = 1,2,\ldots,n\), and on simplification, we get
\[
(7.8) a + b(n-2)S(V,W) = -[a(\alpha^2 - \rho)(1-n)g(V,W) - ab(n-2)]g(V,W).
\]

Putting \(V = W = \xi\) in (7.8) and by virtue of (2.15), (2.16), we get the following equation
\[
(7.9) \quad \lambda = (\alpha^2 - \rho)(1-n).
\]

Since, that is, the Ricci soliton in \((LCS)\) manifold satisfying \(R(\xi,X) \cdot \tilde{C} = 0\). Hence we state the following theorem:

**Theorem 7.1.** A Ricci soliton in \((LCS)\) manifold satisfying \(R(\xi,X) \cdot \tilde{C} = 0\) is (7.9).

8 **Ricci soliton in \((LCS)\) manifolds satisfying \(R(\xi,X) \cdot C = 0\)**

The concircular curvature tensor \(C\) is defined by
\[
C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y],
\]
Taking $Z = \xi$ in (8.1) and using (2.11), (2.13), (2.14), we get

\[(8.2)\]
\[C(X, Y)\xi = \left[ (\alpha^2 - \rho) - \frac{r}{n(n-1)} \right] [\eta(Y)X - \eta(X)Y].\]

Similarly using (2.8), (2.13), (2.14), (2.15) in (8.2), we get

\[(8.3)\]
\[\eta(C(X, Y)Z) = \left[ (\alpha^2 - \rho) - \frac{r}{n(n-1)} \right] [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].\]

We assume that the condition $R(\xi, X) \cdot C = 0$, then we have

\[(8.4)\]
\[-C(Y, R(\xi, X)Z)W - C(Y, Z)R(\xi, X)W = 0.\]

By using (2.10) in (8.4), we have

\[(8.5)\]
\[[g(X, C(Y, Z)W)\xi - \eta(C(Y, Z)W)X - g(X, Y)C(\xi, Z)W + \eta(Y)C(X, Z)W - C(Y, Z)\eta(X)\]
\[- C(Y, W)\eta(C(Y, Z)\xi) - \eta(W)\eta(C(Y, Z)X)] = 0.

By taking an inner product with $\xi$ then we get,

\[(8.6)\]
\[\left[ (\alpha^2 - \rho) - \frac{(r)}{n(n-1)} \right] [g(U, W)g(X, V) - g(X, U)g(V, W)] + g(X, C(U, V)W) = 0.\]

In view of (5.1) in (8.7) and taking $X = Y = e_i$, we have

\[(8.8)\]
\[S(V, W) = -(\alpha^2 - \rho)(1-n)g(V, W).\]

Taking $V = W = \xi$, then we get

\[(8.9)\]
\[\lambda = (\alpha^2 - \rho)(1-n).\]

Hence we can state the following theorem:

**Theorem 8.1.** A Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi, X) \cdot C = 0$ is (8.9).

Based on the above all results we conclude that the value of $\lambda$ in (6.8), (7.9) and (8.9) is same as (5.8). Hence the results for Theorems 6.1, 7.1 and 8.1 may be explained as in Theorem 5.1.
References


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