A Geometry on Ricci solitons in $(LCS)_n$ manifolds

S. R. Ashoka, C. S. Bagewadi and Gurupadavva Ingalahalli

Abstract. In this paper we study and obtain results on Ricci solitons in $(LCS)_n$ manifolds satisfying $R(\xi, X)\cdot\tilde{P} = 0$, $R(\xi, X)\cdot\tilde{M} = 0$, $R(\xi, X)\cdot H = 0$, $R(\xi, X) \cdot B = 0$, and $R(\xi, X) \cdot \tilde{C} = 0$, and $R(\xi, X) \cdot C = 0$ where $\tilde{P}$, $\tilde{M}$, H, B, $\tilde{C}$, and C, are Pseudo-projective, $M-$projective, conharmonic, C-Bochner, quasi-conformal and concircular curvature tensors.

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1 Introduction

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g). A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector fileld and λ a real scalar such that

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is a Ricci tensor of M and $\mathcal{L}_V$ denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. In this paper, we prove conditions for Ricci solitons in $(LCS)_n$ manifolds to be shrinking, steady and expanding.

The authors [19], [4] and [12] have studied Ricci solitons in contact and Lorentzian manifolds. The authors [14], [5] and [6] have obtained some results on Ricci solitons for Kenmotsu manifolds.

Motivated by the above work we obtained some interesting results on Ricci soliton in $(LCS)_n$-manifolds satisfying

$$\begin{cases} R(\xi, X) \cdot \tilde{P} = 0, & R(\xi, X) \cdot B = 0, & R(\xi, X) \cdot \tilde{M} = 0, \\ R(\xi, X) \cdot H = 0, & R(\xi, X) \cdot \tilde{C} = 0 & R(\xi, X) \cdot C = 0. \end{cases}$$

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2 Preliminaries

An n-dimensional Lorentzian manifold M is said to be a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that, for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a nondegenerate inner product of signature $(-,+,....,+)$, where T_pM denotes the tangent vector space of M at P and $\mathbb{R}$ is the real number space. A nonzero vector $v \in T_pM$ is said to be timelike (resp,. non spacelike, null, and spacelike) if it satisfies $g_p(v,v) < 0$ $(rep., \leq 0, \equiv 0$ and $> 0)$.

Definition 2.1. In a Lorentzian manifold (M,g), a vector field p defined by

$$g(X,P) = A(X) \tag{2.1}$$

for any $X \in \Gamma(TM)$,is a said to be a concircular vector field if

$$(\nabla_X A)(Y) = \alpha[g(X,Y) + w(X)A(Y)], \tag{2.2}$$

where α is a non zero scalar and w is a closed $1-$from and ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let M be $n-$dimensional manifold admitting a unit time-like concircular vector field ξ, called the characteristic vector field of the manifold. Then, we have

$$g(\xi,\xi) = -1.$$

Since ξ is a unit concircular vector field, it follows that there exists a nonzero $1-$form η such that, for

$$g(X,\xi) = \eta(X),$$

the equation of the following form holds:

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0) \tag{2.3}$$

for all vector field X, Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α a non zero scalar function satisfying

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X),$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.4}$$

then from (2.3) and (2.4) we have

$$\phi X = X + \nabla_X \xi, \tag{2.5}$$

from which, it follows that ϕ is a symmetric $(1,1)$ tensor and called the structure tensor of the manifold. Thus, the Lorentzian manifold M together with the unit time like concircular vector field ξ, its associated $1-$from η, and an $(1,1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold),

[22], [1], [2], [20], [21] and [23]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto in $(LCS)_n$-manifolds for $n > 2$.

Further, on any $(LCS)_n$ manifold M, the following relations hold:

$$\phi^2 = X + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{2.6}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{2.7}$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \rho)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)), \tag{2.8}$$

$$R(X, Y)\xi = (\alpha^2 - \rho)(\eta(Y)X - \eta(X)Y), \tag{2.9}$$

$$R(\xi, X)Y = (\alpha^2 - \rho)(g(X, Y)\xi - \eta(Y)X), \tag{2.10}$$

$$R(\xi, X)\xi = (\alpha^2 - \rho)(\eta(X)\xi + X), \tag{2.11}$$

where R is the Riemannian curvature tensor.

Let (g, V, λ) be a Ricci soliton in an n-dimensional $(LCS)_n$ manifold M. From (2.4) and (2.5) we have

$$(\mathcal{L}_\xi g)(X, Y) = 2\alpha[g(X, Y) - \eta(X)\eta(Y)]. \tag{2.12}$$

From (1.1) and (2.12) we get

$$S(X, Y) = -[\alpha\eta(X)\eta(Y) + (\alpha + \lambda)g(X, Y)]. \tag{2.13}$$

The above equation yields that

$$QX = -[\alpha\eta(X)\xi + (\alpha + \lambda)X], \tag{2.14}$$

$$S(X, \xi) = -\lambda\eta(X), \tag{2.15}$$

$$r = -\lambda n - \alpha(n - 1), \tag{2.16}$$

where S is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature on M.

We have the following well known established Theorems:

Theorem 2.1. *[7] Let g be a differential real valued function on $\mathbb{R}^n$ and c is a number. Then the subset $S : g(x, y, z) = c$ of E^3 is a surface if $dg \neq 0$ at any point of S.*

Theorem 2.2. *[7] If $S : g(x, y, z) = c$ is a surface in $\mathbb{R}^3$ then the gradient vector field ∇g (connected only at a point of S) is a non vanishing normal vector field on the entire surface S.*

We can rewrite the above Theorems for higher dimensions as follows:

Corollary 2.3. *Let g be a differential real valued function on n-dimensional Euclidean space $\mathbb{R}^n$ and c a number then $S : g(x_1, \ldots, x_n) = c$ in $\mathbb{R}^n$ is a surface (abstract surface or manifold) if $dg \neq 0$ at any point of S.*

Corollary 2.4. *If $S : g(x, y, z) = c$ is a surface (abstract surface or manifold) in $\mathbb{R}^n$ then the gradient vector field ∇g (connected only at points of S) is a non vanishing normal vector field on the entire surface (abstract surface or manifold) S.*

Thus we have following results from the above:

Remark 2.2. According to Corollary 2.4 in our paper taking a real valued scalar function α associated with an $(LCS)_n$ manifold with $M = \mathbb{R}^3$ and $g = \alpha$ we have, $\nabla\alpha$ as a non vanishing normal vector field on $S \subset M$ and directional derivative of α with respect to ξ, $\xi\alpha = \xi.\nabla\alpha = |\xi||\nabla\alpha|cos(\hat{\xi}, \nabla\alpha)$

1. If ξ is tangent to S then $\xi\alpha = 0$.
2. If ξ is tangent to M but not to S then $\xi\alpha \neq 0$.
3. If the angle between ξ and $\nabla\alpha$ is acute then $0 < cos(\hat{\xi}, \nabla\alpha) < 1$, then $\xi\alpha = k|\nabla\alpha|$, $0 < k < 1$ and $\xi\alpha > 0$.
4. If the angle between ξ and $\nabla\alpha$ is obtuse then $-1 < cos(\hat{\xi}, \nabla\alpha) < 0$, then $\xi\alpha = k|\nabla\alpha|$, $-1 < k < 0$ and $\xi\alpha < 0$.

We see that Ricci solitons for $(LCS)_n$ manifolds depend on $\xi\alpha$ in Sections (3) to (8).

3 Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi, X) \cdot \tilde{P} = 0$

The Pseudo-projective curvature tensor $\tilde{P}$ is defined by

$$\begin{aligned} \tilde{P}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.1}$$

where $a, b \neq 0$ are constants. Taking $Z = \xi$ in (3.1) and using (2.11), (2.13), (2.14), we get

$$\tilde{P}(X,Y)\xi = \left[a(\alpha^2 - \rho) - b\lambda - \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right][\eta(Y)X - \eta(X)Y]. \tag{3.2}$$

Similarly using (2.8), (2.13), (2.14), (2.15) in (3.1), we get

$$\eta(\tilde{P}(X,Y)Z) = \left[a(\alpha^2 - \rho) - b\lambda - \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{3.3}$$

We assume that the condition $R(\xi, X) \cdot \bar{P} = 0$, then we have

$$\begin{aligned} &R(\xi,X)\tilde{P}(U,V)W - \tilde{P}(R(\xi,X)U,V)W \\ &-\tilde{P}(U,R(\xi,X)V)W - \tilde{P}(U,V)R(\xi,X)W = 0. \end{aligned} \tag{3.4}$$

Using (2.10) in (3.4), we find

$$\begin{aligned} &(\alpha^2 - \rho)[\eta(\tilde{P}(U,V)W)X - g(X,\tilde{P}(U,V)W)\xi - \eta(U)\tilde{P}(X,V)W + g(X,U)\tilde{P}(\xi,V)W \\ &-\eta(V)\tilde{P}(U,X)W + g(X,V)\tilde{P}(U,\xi)W - \eta(W)\tilde{P}(U,V)X + g(X,W)\tilde{P}(U,V)\xi] = 0. \end{aligned}$$

By taking an inner product with ξ then we get

$$\begin{aligned} &\eta(\tilde{P}(U,V)W)\eta(X) + g(X,\tilde{P}(U,V)W) - \eta(U)\eta(\tilde{P}(X,V)W) \\ &+g(X,U)\eta(\tilde{P}(\xi,V)W) - \eta(V)\eta(\tilde{P}(U,X)W) + g(X,V)\eta(\tilde{P}(U,\xi)W) \\ &-\eta(W)\eta(\tilde{P}(U,V)X) + g(X,W)\eta(\tilde{P}(U,V)\xi) = 0. \end{aligned} \tag{3.5}$$

By using (3.2), (3.3) in (3.5), we have

(3.6)
$$g(X,\tilde{P}(U,V)W)+\left[a(\alpha^2-\rho)-b\lambda-\tfrac{r}{n}\left(\tfrac{a}{n-1}+b\right)\right][g(X,V)g(U,W)-g(X,U)g(V,W)]=0.$$

In view of (3.1) in (3.6), we have

$$\begin{aligned}(3.7)\qquad & ag(X,R(U,V)W)+b[(\alpha+\lambda)\{g(V,X)g(U,W)-g(V,W)g(U,X)\}\\ &+\alpha[g(V,X)\eta(U)\eta(W)-\eta(V)\eta(W)g(U,X)]\\ &+[a(\alpha^2-\rho)-b(\alpha+\lambda)][g(X,V)g(U,W)-g(X,U)g(V,W)]=0.\end{aligned}$$

Taking $X=U=e_i$ in (3.7) and summing over $i=1,2,\ldots,n$, and on simplification, we get

$$(3.8)\qquad aS(V,W) = -a(\alpha^2-\rho)(1-n)g(V,W)-b\alpha(1-n)\eta(V)\eta(W).$$

Putting $V=W=\xi$ in (3.8) and by virtue of (2.15), (2.16), we get the following equation

$$(3.9)\qquad \lambda=\frac{(a(\alpha^2-\rho)-b\alpha))(1-n)}{a}.$$

Hence, Ricci solitons in $(LCS)_n$ manifold satisfy $R(\xi,X)\cdot\tilde{P}=0$. We can consequently state the following theorem:

Theorem 3.1. *A Ricci soliton in $(LCS)_n$ manifold satisfying $R(\xi,X)\cdot\bar{P}=0$ and α is a positive function is*

1. *Shrinking if $\alpha>\frac{b}{a}$, Expanding if $\alpha<\frac{b}{a}$ and steady if $\alpha=\frac{b}{a}$, provided that ξ is orthogonal to $\nabla\alpha$.*
2. *Shrinking if $\alpha^2+k|\nabla\alpha|>\frac{b}{a}\alpha$, Expanding if $\alpha^2+k|\nabla\alpha|<\frac{b}{a}\alpha$ and steady if $\alpha^2+k|\nabla\alpha|=\frac{b}{a}\alpha$, provided that angle between ξ and $\nabla\alpha$ is acute.*
3. *Shrinking if $\alpha^2>k|\nabla\alpha|+\frac{b}{a}\alpha$, Expanding if $\alpha^2<k|\nabla\alpha|+\frac{b}{a}\alpha$ and steady if $\alpha^2=k|\nabla\alpha|+\frac{b}{a}\alpha$, provided that angle between ξ and $\nabla\alpha$ is obtuse.*

Proof. 1. From the Remark 2.2 (1) and (3.9)

$$\lambda=\frac{-(n-1)(a\alpha^2-b\alpha)}{a}.$$

Hence $\lambda<0$ if $\alpha>\frac{b}{a}$, $\lambda>0$ if $\alpha<\frac{b}{a}$ and $\lambda=0$ if $\alpha=\frac{b}{a}$.

2. From Remark 2.2 (2) and (3.9),

$$\lambda=\frac{-(n-1)[a(\alpha^2+k|\nabla\alpha|)-b\alpha]}{a}.$$

Hence $\lambda<0$ if $\alpha^2+k|\nabla\alpha|>\frac{b}{a}\alpha$, $\lambda>0$ if $\alpha^2+k|\nabla\alpha|<\frac{b}{a}\alpha$ and $\lambda=0$ if $\alpha^2+k|\nabla\alpha|=\frac{b}{a}\alpha$.

3. From the Remark 2.2 (3) and (3.9)
$$\lambda = \frac{-(n-1)[a(\alpha^2 - k|\nabla\alpha|) - b\alpha]}{a}.$$
Hence $\lambda < 0$ if $\alpha^2 > k|\nabla\alpha| + \frac{b}{a}\alpha$, $\lambda > 0$ if $\alpha^2 < k|\nabla\alpha| + \frac{b}{a}\alpha$ and $\lambda = 0$ if $\alpha^2 = k|\nabla\alpha| + \frac{b}{a}\alpha$.

□

Corollary 3.2. *An LP−Sasakian manifold is shrinking if $a > b$, expending if $a < b$ and steady if $a = b$.*

This follows by putting $\alpha = 1$ in item (1) of Theorem 3.1.

4 Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi, X)B = 0$

The C-Bochner curvature tensor B is defined by

$$\begin{aligned} B(X,Y)Z &= R(X,Y)Z + \frac{1}{n+3}[g(X,Z)QY - S(Y,Z)X - g(Y,Z)QX + S(X,Z)Y \\ &+ g(\phi X,Z)Q\phi Y - S(\phi Y,Z)\phi X - g(\phi Y,Z)Q\phi X + S(\phi X,Z)\phi Y + 2S(\phi X,Y)\phi Z \\ &+ 2g(\phi X,Y)Q\phi Z + \eta(Y)\eta(Z)QX - \eta(Y)S(X,Z)\xi + \eta(X)S(Y,Z)\xi - \eta(X)\eta(Z)QY] \\ &- \frac{D+n-1}{n+3}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z] \\ &+ \frac{D}{n+3}[\eta(Y)g(X,Z)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y - \eta(X)g(Y,Z)\xi] \\ &- \frac{D-4}{n+3}[g(X,Z)Y - g(Y,Z)X], \end{aligned} \tag{4.1}$$

where $D = \frac{r+n-1}{n+1}$.

Taking $Z = \xi$ in (4.1) and using (2.11), (2.13), (2.14), we get

$$B(X,Y)\xi = \left[(\alpha^2 - \rho) + \frac{2\alpha + 3\lambda}{(n+3)} + \frac{2r + 2n - 2 - 4n - 4}{(n+1)(n+3)}\right][\eta(Y)X - \eta(X)Y]. \tag{4.2}$$

Similarly, using (2.8), (2.13), (2.14), (2.15) in (4.2), we get

$$\eta(B(X,Y)Z) = \left[(\alpha^2 - \rho) + \frac{\lambda}{(n+3)} - \frac{4-2D}{(n+3)} + \frac{2(\alpha+\lambda)}{n+3}\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{4.3}$$

We assume that the condition $R(\xi, X)B = 0$, then we have

$$\begin{aligned} &R(\xi,X)B(Y,Z)W - B(R(\xi,X)Y,Z)W \\ &-B(Y,R(\xi,X)Z)W - B(Y,Z)R(\xi,X)W = 0. \end{aligned} \tag{4.4}$$

By using (2.10) and taking an inner product with ξ then, by using (4.2) and (4.3) in (4.4), then we have

$$\begin{aligned} 4S(Z,W) &= [(\alpha^2 - \rho)(n-1)(n+3) + \lambda(n-1) - (4-2D)(n-1) \\ &+ 2(\alpha+\lambda)(n-1) + r + 4(\alpha+\lambda) - D + 3(D+n-1) - (D-4)(n-1)]g(Z,W) \\ &- [-r - 2\lambda + 4(\alpha+\lambda) + D(n-2) + 3(D+n-1)]\eta(Z)\eta(W). \end{aligned} \tag{4.5}$$

Taking $Z = W = \xi$, and by virtue of (2.15), (2.16), we get the following equation

$$\lambda = \left[\frac{-(\alpha^2 - \rho)(n-2) - 4\alpha}{5}\right] \tag{4.6}$$

Hence we can state the following result

Theorem 4.1. *A Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi, X) \cdot B = 0$ is Shrinking if α is a positive function whose gradient vector field is orthogonal to ξ.*

Proof. By the Remark 2.2 (1) and (4.6),

$$\lambda = -[\frac{\alpha^2(n-2) + 4\alpha}{5}] \implies \lambda < 0.$$

However, results for cases of acute and obtuse angles between ξ and $\nabla\alpha$ are little complex. □

5 Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi, X) \cdot \tilde{M} = 0$

The M-projective curvature tensor $\tilde{M}$ is defined by

$$\tilde{M}(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \tag{5.1}$$

Taking $Z = \xi$ in (5.1) and using (2.11), (2.13), (2.14), we get

$$\tilde{M}(X,Y)\xi = \left[(\alpha^2 - \rho) + \frac{(\alpha+\lambda)}{2(n-1)} - \frac{\alpha}{2(n-1)}\right][\eta(Y)X - \eta(X)Y]. \tag{5.2}$$

Similarly using (2.8), (2.13), (2.14), (2.15) in (5.2), we get

$$\eta(\tilde{M}(X,Y)Z) = \left[(\alpha^2 - \rho) + \frac{(\alpha+\lambda)}{2(n-1)} - \frac{\alpha}{2(n-1)}\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{5.3}$$

We assume that the condition $R(\xi, X) \cdot \tilde{M} = 0$, then we have

$$\begin{aligned} &R(\xi,X)\tilde{M}(Y,Z)W - \tilde{M}(R(\xi,X)Y,Z)W \\ &-\tilde{M}(Y,R(\xi,X)Z)W - \tilde{M}(Y,Z)R(\xi,X)W = 0. \end{aligned} \tag{5.4}$$

By using (2.10) in (5.4), we have

$$\begin{aligned} &[g(X,\tilde{M}(Y,Z)W)\xi - \eta(\tilde{M}(Y,Z)W)X) - g(X,Y)M(\xi,Z)W + \eta(Y)\tilde{M}(X,Z)W \\ &-\tilde{M}(Y,\xi)Wg(X,Z) + \eta(Z)\tilde{M}(Y,X)W - g(X,W)\tilde{M}(Y,Z)\xi + \eta(W)\tilde{M}(Y,Z)X] = 0, \end{aligned}$$

By taking an inner product with ξ then we get,

$$\begin{aligned} &[g(X,\tilde{M}(Y,Z)W) + \eta(\tilde{M}(Y,Z)W)\eta(X) + g(X,Y)\eta(\tilde{M}(\xi,Z)W) \\ &-\eta(Y)\eta(\tilde{M}(X,Z)W) + \eta(\tilde{M}(Y,\xi)W)g(X,Z) - \eta(Z)\eta(\tilde{M}(Y,X)W) \\ &+g(X,W)\eta(\tilde{M}(Y,Z)\xi) - \eta(W)\eta(\tilde{M}(Y,Z)X)] = 0. \end{aligned} \tag{5.5}$$

By using (5.2) and (5.3) in (5.5), we have
(5.6)
$$\left[(\alpha^2-\rho)+\frac{(\alpha+\lambda)}{2(n-1)}-\frac{\alpha}{2(n-1)}\right][g(Y,W)g(X,Z)-g(X,Y)g(Z,W)]+g(X,M(Y,Z)W)=0.$$

In view of (5.1) in (5.6) and taking $X=Y=e_i$, we have

$$S(Z,W)=\left[\frac{r+2(n-1)(\alpha+\lambda)+2(n-1)^2(\alpha^2-\rho)-\alpha(n-1)}{n}\right]g(Z,W). \tag{5.7}$$

Taking $Z=W=\xi$, we get

$$\lambda=-\left[(n-1)(\alpha^2-\rho)\right]. \tag{5.8}$$

Hence we can state the following theorem:

Theorem 5.1. *A Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi,X)\cdot\tilde{M}=0$ is*

1. *Shrinking if characteristic vector field ξ is orthogonal to $\nabla\alpha$.*
2. *Shrinking of the angle between characteristic vector field ξ and the gradient vector field $\nabla\alpha$ is acute.*
3. *If the angle between characteristic vector field ξ and the gradient vector field $\nabla\alpha$ is obtuse then it is Shrinking if $\alpha^2>k|\nabla\alpha|$, Expanding if $\alpha^2<k|\nabla\alpha|$ and steady $\alpha^2=k|\nabla\alpha|$.*

Proof. According to (5.8) and Remark 2.2, items (1), (3) and (4), we respectively have:

1. $\lambda=-(n-1)\alpha^2,\lambda<0;$
2. $\lambda=-(n-1)(\alpha^2+k|\nabla\alpha|),\lambda<0;$
3. $\lambda=-(n-1)(\alpha^2-k|\nabla\alpha|),\lambda<0,$

which concludes the proof. □

6 Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi,X)\cdot H=0$

The conharmonic curvature tensor H is defined by
(6.1)
$$H(X,Y)Z=R(X,Y)Z-\frac{1}{(n-2)}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY],$$

Taking $Z=\xi$ in (6.1) and using (2.11), (2.13), (2.14), we get

$$H(X,Y)\xi=\left[(\alpha^2-\rho)+\frac{2(\alpha+\lambda)}{(n-2)}-\frac{\alpha}{(n-2)}\right][\eta(Y)X-\eta(X)Y]. \tag{6.2}$$

Similarly using (2.8), (2.13), (2.14), (2.15) in (6.2), we get
(6.3)
$$\eta(H(X,Y)Z)=\left[(\alpha^2-\rho)+\frac{2(\alpha+\lambda)}{(n-2)}-\frac{\alpha}{(n-2)}\right][g(Y,Z)\eta(X)-g(X,Z)\eta(Y)].$$

We assume that the condition $R(\xi, X) \cdot H = 0$, then we have

(6.4)
$$R(\xi, X)H(Y, Z)W - H(R(\xi, X)Y, Z)W - H(Y, R(\xi, X)Z)W - H(Y, Z)R(\xi, X)W = 0.$$

By using (2.10) in (6.4), we have

$$\begin{aligned}&[g(X, H(Y, Z)W)\xi - \eta(H(Y, Z)W)X) - g(X, Y)H(\xi, Z)W + \eta(Y)H(X, Z)W\\ &-H(Y, \xi)Wg(X, Z) + \eta(Z)H(Y, X)W - g(X, W)H(Y, Z)\xi + \eta(W)H(Y, Z)X] = 0,\end{aligned}$$

By taking an inner product with ξ then we get,

$$\begin{aligned}&[g(X, H(Y, Z)W) + \eta(H(Y, Z)W)\eta(X) + g(X, Y)\eta(H(\xi, Z)W)\\ &\quad -\eta(Y)\eta(H(X, Z)W) + \eta(H(Y, \xi)W)g(X, Z) - \eta(Z)\eta(H(Y, X)W)\\ &\quad +g(X, W)\eta(H(Y, Z)\xi) - \eta(W)\eta(H(Y, Z)X)] = 0.\end{aligned} \tag{6.5}$$

By using (6.2) and (6.3) in (6.5), we have

(6.6)
$$\left[(\alpha^2 - \rho) + \frac{(\alpha + \lambda)}{2(n-1)} - \frac{\alpha}{2(n-1)}\right][g(Y, W)g(X, Z) - g(X, Y)g(Z, W)] + g(X, H(Y, Z)W) = 0.$$

In view of (6.1) in (5.6) and taking $X = Y = e_i$, we have

$$\left[\frac{(\alpha^2 - \rho)(1-n)(n-2) + 2(\alpha + \lambda)(1-n) - \alpha(1-n) - r}{n-2}\right] g(Z, W) = 0, \tag{6.7}$$

where $g(Z, W) \neq 0$; therefore we get

$$\lambda = [(\alpha^2 - \rho)(1-n)]. \tag{6.8}$$

Hence we can state the following

Theorem 6.1. *A Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi, X) \cdot H = 0$ is (6.8).*

7 Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi, X) \cdot \tilde{C} = 0$

The Quasi-conformal curvature tensor $\tilde{C}$ is defined by

$$\begin{aligned}\tilde{C}(X, Y)Z &= aR(X, Y)Z + b(S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY)\\ &\quad -\frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y, Z)X - g(X, Z)Y],\end{aligned} \tag{7.1}$$

where $a, b \neq 0$ are constants. Taking $Z = \xi$ in (7.1) and using (2.11), (2.13), (2.14), we get

$$\begin{aligned}&\tilde{C}(X, Y)\xi =\\ &\quad \left[a(\alpha^2 - \rho) - b(2\lambda + \alpha) - \tfrac{r}{n}\left(\tfrac{a}{n-1} + 2b\right)\right][\eta(Y)X - \eta(X)Y].\end{aligned} \tag{7.2}$$

Similarly using (2.8), (2.13), (2.14), (2.15) in (7.1), we get

$$\eta(\tilde{C}(X,Y)Z) = \left[a(\alpha^2-\rho) - b(2\lambda+\alpha) - \frac{r}{n}\left(\frac{a}{n-1}+2b\right)\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{7.3}$$

We assume that the condition $R(\xi,X)\cdot\tilde{C}=0$, then we have

$$\begin{aligned} &R(\xi,X)\tilde{C}(U,V)W - \tilde{C}(R(\xi,X)U,V)W \\ &-\tilde{C}(U,R(\xi,X)V)W - \tilde{C}(U,V)R(\xi,X)W = 0. \end{aligned} \tag{7.4}$$

Using (2.10) in (7.4), we find

$$\begin{aligned} &(\alpha^2-\rho)[\eta(\tilde{C}(U,V)W)X - g(X,\tilde{C}(U,V)W)\xi - \eta(U)\tilde{C}(X,V)W \\ &+g(X,U)\tilde{C}(\xi,V)W - \eta(V)\tilde{C}(U,X)W + g(X,V)\tilde{C}(U,\xi)W \\ &-\eta(W)\tilde{C}(U,V)X + g(X,W)\tilde{C}(U,V)\xi] = 0. \end{aligned} \tag{7.5}$$

By taking an inner product with ξ then we get

$$\begin{aligned} &\eta(\tilde{C}(U,V)W)\eta(X) + g(X,\tilde{C}(U,V)W) - \eta(U)\eta(\tilde{C}(X,V)W) \\ &+g(X,U)\eta(\tilde{C}(\xi,V)W) - \eta(V)\eta(\tilde{C}(U,X)W) + g(X,V)\eta(\tilde{C}(U,\xi)W) \\ &-\eta(W)\eta(\tilde{C}(U,V)X) + g(X,W)\eta(\tilde{C}(U,V)\xi) = 0. \end{aligned} \tag{7.6}$$

By using (7.2), (7.3) in (7.6), we have

$$\begin{aligned} g(X,\tilde{C}(U,V)W) + &\left[a(\alpha^2-\rho) - b(2\lambda+\alpha) - \frac{r}{n}(\frac{a}{n-1}+2b)\right] \\ &[g(X,V)g(U,W) - g(X,U)g(V,W)] = 0. \end{aligned} \tag{7.7}$$

In view of (7.1) in (7.7) and Taking $X=U=e_i$, summing over $i=1,2,\ldots,n$, and on simplification, we get

$$a + b(n-2)S(V,W) = -[a(\alpha^2-\rho)(1-n)g(V,W) - \alpha b(n-2)]g(V,W). \tag{7.8}$$

Putting $V=W=\xi$ in (7.8) and by virtue of (2.15), (2.16), we get the following equation

$$\lambda = (\alpha^2-\rho)(1-n). \tag{7.9}$$

Since, that is, the Ricci soliton in $(LCS)_n$ manifold satisfying $R(\xi,X)\cdot\tilde{C}=0$. Hence we state the following theorem:

Theorem 7.1. *A Ricci soliton in $(LCS)_n$ manifold satisfying $R(\xi,X)\cdot\tilde{C}=0$ is (7.9).*

8 Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi,X)\cdot C=0$

The concircular curvature tensor C is defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{8.1}$$

Taking $Z = \xi$ in (8.1) and using (2.11), (2.13), (2.14), we get

$$C(X,Y)\xi = \left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right][\eta(Y)X - \eta(X)Y]. \tag{8.2}$$

Similarly using (2.8), (2.13), (2.14), (2.15) in (8.2), we get

$$\eta(C(X,Y)Z) = \left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{8.3}$$

We assume that the condition $R(\xi, X) \cdot C = 0$, then we have

$$\begin{aligned} &R(\xi,X)C(Y,Z)W - C(R(\xi,X)Y,Z)W \\ &-C(Y,R(\xi,X)Z)W - C(Y,Z)R(\xi,X)W = 0. \end{aligned} \tag{8.4}$$

By using (2.10) in (8.4), we have

$$\begin{aligned} &[g(X,C(Y,Z)W)\xi - \eta(C(Y,Z)W)X - g(X,Y)C(\xi,Z)W + \eta(Y)C(X,Z)W \\ &-C(Y,\xi)Wg(X,Z) + \eta(Z)C(Y,X)W - g(X,W)C(Y,Z)\xi + \eta(W)C(Y,Z)X] = 0, \end{aligned} \tag{8.5}$$

By taking an inner product with ξ then we get,

$$\begin{aligned} &[g(X,C(Y,Z)W) + \eta(C(Y,Z)W)\eta(X) + g(X,Y)\eta(C(\xi,Z)W) \\ &\quad -\eta(Y)\eta(C(X,Z)W) + \eta(C(Y,\xi)W)g(X,Z) - \eta(Z)\eta(C(Y,X)W) \\ &\quad +g(X,W)\eta(C(Y,Z)\xi) - \eta(W)\eta(C(Y,Z)X)] = 0. \end{aligned} \tag{8.6}$$

By using (8.2) and (8.3) in (8.6), we have

$$\left[(\alpha^2 - \rho) - \frac{(r)}{n(n-1)}\right][g(U,W)g(X,V) - g(X,U)g(V,W)] + g(X,C(U,V)W) = 0. \tag{8.7}$$

In view of (5.1) in (8.7) and taking $X = Y = e_i$, we have

$$S(V,W) = -(\alpha^2 - \rho)(1-n)g(V,W). \tag{8.8}$$

Taking $V = W = \xi$, then we get

$$\lambda = (\alpha^2 - \rho)(1-n). \tag{8.9}$$

Hence we can state the following theorem:

Theorem 8.1. *A Ricci soliton in $(LCS)_n$ manifolds satisfying $R(\xi, X) \cdot C = 0$ is (8.9).*

Based on the above all results we conclude that the value of λ in (6.8), (7.9) and (8.9) is same as (5.8). Hence the results for Theorems 6.1, 7.1 and 8.1 may be explained as in Theorem 5.1.

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Authors' address:

Sailapura Ramanjaneya Ashoka, Channabasappa Shantappa Bagewadi, and Gurupadavva Ingalahalli

Department of Mathematics, Kuvempu University,
Shankaraghatta - 577 451, Shimoga, Karnataka, INDIA.

E-mail: prof_bagewadi@yahoo.co.in , srashoka@gmail.com , gurupadavva@gmail.com