Infinitesimal differential geometry: cusps and envelopes

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Abstract. We present several nonstandard definitions of cusps and we establish relations between them as well with the classical definitions of cusps. Somewhat surprisingly, differentiability does not appear to be of paramount importance. We present a new method to determine the envelope to a family of C^1 curves. In the final section we will apply the results to the well known problem of the coffee-cup caustic.

This is a first attempt of using the basic foundations in [5]; namely, we essentially use unit vectors in order to characterize particular kinds of tangency in finite dimensional Euclidian spaces; advanced treatments of Differential Geometry itself may be found in the work of Goze and Stroyan [10, 11, 16, 19].

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1 Introduction

In this section we present a brief exposition on nonstandard analysis. Basically, we will fix some terminology needed and give a nonstandard characterization of continuity and differentiability. In the references below the reader may find more on the subject. For applications on differentiable manifolds, see e.g. [2, 3, 4, 8, 10, 11, 12, 13]. For general theory on manifolds, see e.g. [1, 17].

Definition 1.1. Let x and y be two (possibly not standard) vectors in (an enlargement of) a normed space $({}^*E, |\cdot|)$. We say that

1. x is *infinitesimal* if $|x| < r$, for every $r \in \mathbb{R}^+$; the set of infinitesimals will be denoted by $inf({}^*E)$ and $x \approx 0$ will abbreviate $x \in inf({}^*E)$.

2. x is *finite* if $|x| < r$, for some $r \in \mathbb{R}^+$; the set of finite vectors is denoted by $fin({}^*E)$;

3. x is *infinite* if it is not finite; $x \approx \infty$ abbreviates $x \notin fin({}^*E)$;

4. x and y are *infinitely close* if $x - y \approx 0$ and this is also abbreviated by $x \approx y$;

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5. x is *nearstandard* if there exists $z \in E$ with $x \approx z$; z will be called the *standard part* of x and denoted $st(x)$. The set of nearstandard vectors is denoted by $ns(^*E)$.

If E is a normed space, it is clear that $ns(^*E) \subseteq fin(^*E)$. Moreover, E is finite dimensional if and only if $ns(^*E) = fin(^*E)$.

In the following, E and F are two normed spaces.

Theorem 1.1. *[14] Let $f : E \to F$ be a function. Then*

1. *f is continuous iff for all $x \in E$ and $y \in {}^*E$, if $x \approx y$ then $f(x) \approx f(y)$;*
2. *f is uniformly continuous iff for all $x, y \in {}^*E$, if $x \approx y$ then $f(x) \approx f(y)$.*

Let A be a subset of E. In the following we will denote

$$ns(^*A) := \{x \in {}^*A \,|\, x \in ns(^*E) \wedge st(x) \in A\}$$

Given a linear operator D from $^*E$ to $^*F$, we say that D is finite if $D(fin(^*E)) \subseteq fin(^*F)$.

Theorem 1.2. *[19] Let U be an open subset of E and $f : U \to F$ a function with $f(ns(^*U)) \subseteq ns(^*F)$. Then*

1. *f is differentiable iff for all $x \in U$, there exists an internal finite linear operator from $^*E$ into $^*F$, $D_x \in L(E, F)$, such that whenever $y \approx x$, there exists an infinitesimal vector η satisfying*

$$f(x) - f(y) = D_x(x - y) + |x - y|\eta;$$

2. *f is of class C^1 iff for all $x \in ns(^*U)$, there exists an internal finite linear operator $D_x \in {}^*L(E, F)$ such that, whenever $y \approx x$, there exists an infinitesimal vector η satisfying*

$$f(x) - f(y) = D_x(x - y) + |x - y|\eta.$$

When such a D_x exists we shall denote it by Df_x. More generally, let $SLin^h(E, F)$ denote the set of symmetric h-linear operators from $E \times \ldots \times E = E^h$ into F.

Theorem 1.3. Taylor's Theorem *[19] Let $f : U \to F$ be a function. Then f is of class C^k iff there exist a unique map $D^h_{(.)} : U \to SLin^h(E, F)$, $h \in \{1, \ldots, k\}$ such that, whenever $a \in ns(^*U)$ and $x \approx a$, there exists an infinitesimal $\eta \in {}^*F$ satisfying*

$$f(x) = \sum_{h=0}^{k} \frac{1}{h!} D^h_a (x - a)^{(h)} + |x - a|^k \eta.$$

Moreover, the map $D^h_{(.)}$ is the usual derivative of f, $D^h f_{(.)}$.

2 Cusps

Let $\alpha : I \to \mathbb{R}^n$ be a curve (possibly with side derivatives α'_+ and α'_-). In the literature we may find two distinct definitions of *cusp*:

1. $\alpha(t)$ is called a cusp if $\alpha'_+(t) = -\alpha'_-(t) \neq 0$;
2. $\alpha(t)$ is called a cusp if $\alpha'(t) = 0$ and $\alpha''(t) \neq 0$.

We begin this section by presenting four nonstandard definitions of cusp. Later we establish some relations between them and with Definitions 1 and 2 from above.

Let a be a positive real number and $\alpha :]-a, a[\to \mathbb{R}^n$, with $n \geq 2$, be a continuous curve such that

$$\alpha(0) = 0 \text{ and } \alpha \text{ is } 1-1.$$

Define

$$\rho(t) := \frac{|\alpha(t) - \alpha(-t)|}{|\alpha(t)|} \quad (t \neq 0).$$

Let $\theta(t)$ be the angle between $\alpha(t)$ and $\alpha(-t)$

$$\theta(t) := \angle\, \alpha(t)\, 0\, \alpha(-t),$$

so that

$$\cos(\theta(t)) = \frac{\alpha(t) \cdot \alpha(-t)}{|\alpha(t)| \cdot |\alpha(-t)|} \quad (t \neq 0).$$

Let $R(t)$ be the ratio

$$R(t) := \frac{|\alpha(-t)|}{|\alpha(t)|} \quad (t \neq 0).$$

Definition 2.1.

1. $\alpha(0)$ is a $Vector - Cusp$ (or short $V - Cusp$) if

$$\exists u \in \mathbb{R}^n \left[|u| = 1 \ \ \& \ \ \forall \epsilon \approx 0 \left[\epsilon \neq 0 \Rightarrow \frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} \approx u \right] \right]. \tag{2.1}$$

2. $\alpha(0)$ is a *symmetric* $V - Cusp$ if

$$\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} \approx \frac{\alpha(-\epsilon)}{|\alpha(-\epsilon)|} \text{ for all positive } \epsilon \approx 0. \tag{2.2}$$

3. $\alpha(0)$ is a $Triangle - Cusp$ (or short $T - Cusp$) if

$$\rho(\epsilon) \approx 0, \text{ for all non-zero } \epsilon \approx 0. \tag{2.3}$$

4. $\alpha(0)$ is a $T^+ - Cusp$ if

$$\rho(\epsilon) \approx 0, \text{ for all positive } \epsilon \approx 0. \tag{2.4}$$

For example, let $\alpha(t) = (t^2, t^3), t \in \mathbb{R}$. Then

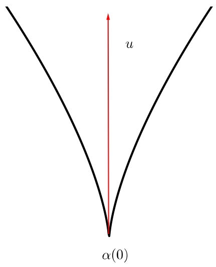

Figure 1: V-Cusp

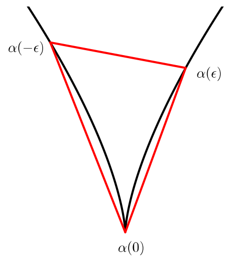

Figure 2: T-Cusp

1. $\alpha(0)$ is a V-cusp (Figure 1) since, for every non-zero ϵ,

$$\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} = \frac{(1,\epsilon)}{|(1,\epsilon)|} \approx (1,0).$$

2. $\alpha(0)$ is a T-cusp (Figure 2) since, when $0 \neq \epsilon \approx 0$,

$$\frac{|\alpha(\epsilon)-\alpha(-\epsilon)|}{|\alpha(\epsilon)|} = \frac{|(0,2\epsilon)|}{|(1,\epsilon)|} = \frac{2|\epsilon|}{\sqrt{1+\epsilon^2}} \approx 0.$$

Remark 2.2. Given a 1-1 continuous curve α, defined in a neighborhood of some real number t_0, $\alpha :]t_0-a, t_0+a[\to \mathbb{R}^n$, $\alpha(t_0)$ is, by definition, a *Cusp* of any of the previously defined types (vide Definition 2.1) if $\beta(0)$ is, where $\beta(s) = \alpha(s+t_0) - \alpha(t_0)$ ($|s| < a$).

All definitions and all results proved henceforth have then an equivalent version with adequate substitutions of t_0 for 0, and therefore of $t_0 \pm \epsilon$ for $\pm\epsilon$ respectively.

Assume that there exist

$$\alpha'_+(0) := st\left(\frac{\alpha(\epsilon)}{\epsilon}\right) \ (0 < \epsilon \approx 0) \text{ and } \alpha'_-(0) = st\left(\frac{\alpha(\epsilon)}{\epsilon}\right) \ (0 > \epsilon \approx 0).$$

Theorem 2.1. *The following relations hold:*

1. $(2.1) \Rightarrow (2.2)$*;*

2. $(2.1) \not\Leftarrow (2.2)$*,* $(2.1) \not\Leftarrow (2.3)$ *and* $(2.1) \not\Rightarrow (2.3)$*;*

3. If $\alpha'_+(0) \neq 0 \neq \alpha'_-(0)$*,*

$$(2.1) \Leftrightarrow (2.2) \Leftrightarrow \frac{\alpha'_+(0)}{|\alpha'_+(0)|} = -\frac{\alpha'_-(0)}{|\alpha'_-(0)|}. \tag{2.5}$$

Proof. Relation 1 is obvious.

On what regards relation 2, we proceed to describe a continuous 1-1 curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ which verifies (2.2) and (2.3) but not (2.1); simply define

$$\alpha(t) = \begin{cases} (t, t\sin\left(\frac{1}{t}\right), t^2) & t > 0 \\ (0,0,0) & t = 0 \\ (-t, t\sin\left(\frac{1}{t}\right), -t^2) & t < 0 \end{cases}$$

To see that (2.1) $\not\Rightarrow$ (2.3), let β be given by

$$\beta(t) = \begin{cases} (t^6, t^4) & t \geq 0 \\ (t^3, t^2) & t < 0 \end{cases}$$

This curve verifies (2.1) but not (2.3).

On what regards relation 3, consider the following. Let $\mathrm{sgn}(\epsilon) := \frac{\epsilon}{|\epsilon|}$. When $0 \neq \epsilon \approx 0$, whatever ϵ, we have

$$\begin{aligned} \frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} &= \frac{\epsilon}{|\alpha(\epsilon)|} \cdot \frac{\alpha(\epsilon)}{\epsilon} = \mathrm{sgn}(\epsilon) \frac{\frac{\alpha(\epsilon)}{\epsilon}}{\left|\frac{\alpha(\epsilon)}{\epsilon}\right|} \\ &\approx \begin{cases} \frac{\alpha'_+(0)}{|\alpha'_+(0)|} & \epsilon > 0 \\ -\frac{\alpha'_-(0)}{|\alpha'_-(0)|} & \epsilon < 0 \end{cases} \end{aligned}$$

Therefore the two standard vectors $\dfrac{\alpha'_+(0)}{|\alpha'_+(0)|}, \ -\dfrac{\alpha'_-(0)}{|\alpha'_-(0)|}$ are equal if and only if they are infinitely close if and only if for all positive $\epsilon \approx 0$, $\dfrac{\alpha(\epsilon)}{|\alpha(\epsilon)|} \approx \dfrac{\alpha(-\epsilon)}{|\alpha(-\epsilon)|}$ if and only if all the $\dfrac{\alpha(\epsilon)}{|\alpha(\epsilon)|}$ have the same standard part. □

Theorem 2.2.

1. *Conditions* (2.3)*,* (2.4) *and the following are equivalent.*

$$\theta(\epsilon) \approx 0 \ \textit{and} \ R(\epsilon) \approx 1, \ \textit{whenever} \ \ 0 \neq \epsilon \approx 0; \tag{2.6}$$

2. (2.3) $\Rightarrow$ (2.2) *but* (2.2) $\not\Rightarrow$ (2.3).

Proof.

1. According to the law of cosines, applied to the triangle with vertices $0,\ \alpha(\epsilon),\ \alpha(-\epsilon)$, whatever the sign of ϵ might be,

$$|\alpha(\epsilon) - \alpha(-\epsilon)|^2 = |\alpha(\epsilon)|^2 + |\alpha(-\epsilon)|^2 - 2|\alpha(\epsilon)| \cdot |\alpha(-\epsilon)| \cdot \cos(\theta(\epsilon))$$

so that

$$\begin{aligned} \rho(\epsilon)^2 &= 1 + R(\epsilon)^2 - 2R(\epsilon)\cos(\theta(\epsilon)) \\ &= (1 - R(\epsilon))^2 \ + \ 2R(\epsilon)(1 - \cos(\theta(\epsilon))). \end{aligned}$$

Since $(1 - R(\epsilon))^2,\ 2R(\epsilon),\ (1 - \cos(\theta(\epsilon)))$ are all nonnegative and $\rho(-\epsilon) = \rho(\epsilon)\dfrac{1}{R(\epsilon)}$, the equivalences

$$(2.3) \Leftrightarrow (2.4) \Leftrightarrow (2.6)$$

follow.

2. Observe that

$$\begin{aligned}\left|\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|}-\frac{\alpha(-\epsilon)}{|\alpha(-\epsilon)|}\right| &= \left|\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|}-\frac{|\alpha(\epsilon)|}{|\alpha(-\epsilon)|}\cdot\frac{\alpha(-\epsilon)}{|\alpha(\epsilon)|}\right| \\ &= \left|\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|}-\frac{1}{R(\epsilon)}\cdot\frac{\alpha(-\epsilon)}{|\alpha(\epsilon)|}\right|.\end{aligned}$$

If $\rho(\epsilon)\approx 0$, then $\dfrac{1}{R(\epsilon)}\approx 1$ and $\dfrac{\alpha(-\epsilon)}{|\alpha(\epsilon)|}$ is also finite (actually its norm is precisely $R(\epsilon)$) so that

$$\left|\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|}-\frac{1}{R(\epsilon)}\cdot\frac{\alpha(-\epsilon)}{|\alpha(\epsilon)|}\right|\approx\left|\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|}-\frac{\alpha(-\epsilon)}{|\alpha(\epsilon)|}\right|=\rho(\epsilon)\approx 0.$$

and (2.3) $\Rightarrow$ (2.2) is proven. Finally, the curve β defined in the proof of Theorem 2.1 verifies (2.2) but not (2.3). □

Theorem 2.3. *If $\alpha'_+(0)\neq 0$ (or $\alpha'_-(0)\neq 0$) then condition (2.3) is equivalent to the following.*

$$\alpha'_+(0)=-\alpha'_-(0). \tag{2.7}$$

Proof. For $0<\epsilon\approx 0$, there exist $\eta,\iota\approx 0$ such that

$$\begin{aligned}\alpha(\epsilon)-\alpha(-\epsilon) &= \alpha'_+(0)\epsilon+\epsilon\eta \;+\; \alpha'_-(0)\epsilon+\epsilon\iota \\ &= (\alpha'_+(0)+\alpha'_-(0))\epsilon \;+\; (\eta+\iota)\epsilon\end{aligned}$$

and

$$\rho(\epsilon)=\frac{|(\alpha'_+(0)+\alpha'_-(0))+(\eta+\iota)|}{|\alpha'_+(0)+\eta|}.$$

As $\alpha'_+(0)$ is standard and non-zero,

$$\rho(\epsilon)\approx 0 \text{ if and only if } \alpha'_+(0)+\alpha'_-(0)\approx 0.$$

Since $\alpha'_-(0)$ is also standard,

$$\rho(\epsilon)\approx 0 \text{ if and only if } \alpha'_+(0)+\alpha'_-(0)=0.$$

as required. □

Theorem 2.4. *If α is of class $C^2, \alpha'(0)=0$ and $\alpha''(0)\neq 0$, then $\alpha(0)$ is a $T-cusp$ and a $V-cusp$.*

Proof. For each positive infinitesimal ϵ, there exist $\eta,\iota\approx 0$ such that

$$\alpha(\epsilon)-\alpha(-\epsilon)=\epsilon^2(\eta-\iota);$$

and therefore

$$\rho(\epsilon)=\frac{|\eta-\iota|}{\left|\frac{\alpha''(0)}{2}+\eta\right|}\approx 0.$$

Furthermore, we have

$$\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|}\approx\frac{\alpha''(0)}{|\alpha''(0)|}$$

for all non-zero infinitesimal ϵ. □

More generally, the following is a simple application of Taylor's Theorem.

Theorem 2.5. *Let α be a curve of class C^{2k+1}.*

1. *If $\alpha(0)$ is a $T-Cusp$ (or a $V-Cusp$) and, for all $i \in \{1,\cdots,k\}$, $\alpha^{(2i)}(0)=0$, then for all $i \in \{1,\cdots,2k+1\}$ $\alpha^{(i)}(0)=0$.*
2. *If for all $i \in \{1,\cdots,k\}$, $\alpha^{(2i-1)}(0)=0$ and $\alpha^{(2k)}(0)\neq 0$, then*
   - *(a) $\alpha(0)$ is a $T-Cusp$,*
   - *(b) $\alpha(0)$ is a $V-Cusp$ and, for all infinitesimal ϵ,*

$$\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} \approx \frac{\alpha^{(2j)}(0)}{|\alpha^{(2j)}(0)|},$$

where $j := \min\{i \in \{1,\ldots,k\} \mid \alpha^{(2i)}(0)\neq 0\}$.

Proof. 1. We know that $\alpha'(0)=0$; assume that α is C^{2k+1} and

$$\alpha^{(j)}(0)=0 \quad (0\leq j\leq 2k).$$

Then, for some infinitesimals η_1,η_2,

$$0\approx\rho(\epsilon)=\frac{|\alpha(\epsilon)-\alpha(-\epsilon)|}{|\alpha(\epsilon)|}$$

$$=\frac{\left|\frac{2}{(2k+1)!}\alpha^{(2k+1)}(0)+(\eta_1+\eta_2)\right|}{\left|\frac{1}{(2k+1)!}\alpha^{(2k+1)}(0)+\eta_1\right|},$$

for all non-zero infinitesimal ϵ.
This can happen only if the standard vector $\alpha^{(2k+1)}(0)$ is 0.
Assume now that $\alpha(0)$ is a $V-cusp$. Similarly, we have

$$\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|}=\operatorname{sgn}(\epsilon)\frac{\frac{1}{(2k+1)!}\alpha^{(2k+1)}(0)+\eta}{\left|\frac{1}{(2k+1)!}\alpha^{(2k+1)}(0)+\eta\right|}\approx u$$

for some $u\in\mathbb{R}^n$ and all ϵ only if $\alpha^{(2k+1)}(0)=0$.

2. *(a)* is a straightforward adaptation of the proof above or of the proof of Theorem 2.4;

(b) is a very obvious application of Taylor's formula for there exists $\iota\approx 0$ such that $\alpha(\epsilon)=\frac{\epsilon^{2j}}{(2j)!}\left(\alpha^{(2j)}(0)+\iota\right)$. □

Actually the conditions in Definition 2.1 make sense even if one of the lateral derivatives does not exist:

Theorem 2.6. *If α is the graph of a function $f:]-a,a[\to\mathbb{R}$, then the following are equivalent.*

1. *$(0,0)=(0,f(0))$ is a $T-Cusp$.*

2. The right and left derivatives $f'_+(0)$, $f'_-(0)$ are both infinite with opposite signs.

Proof. In this case, for each non-zero infinitesimal ϵ,

$$\rho(\epsilon) = \sqrt{\frac{4+\left(\frac{f(\epsilon)}{\epsilon}-\frac{f(-\epsilon)}{\epsilon}\right)^2}{1+\left(\frac{f(\epsilon)}{\epsilon}\right)^2}}. \tag{2.8}$$

Define $\Omega = \frac{f(\epsilon)}{\epsilon}$ and $\Phi = \frac{f(-\epsilon)}{-\epsilon}$. The numerator of (2.8) is not infinitesimal, therefore $\rho(\epsilon) \approx 0$ if and only if the two following conditions are verified:

$$\Omega \quad \text{is infinite, and} \quad \rho(\epsilon)^2 = \frac{\frac{4}{\Omega^2}+\left(1+\frac{\Phi}{\Omega}\right)^2}{\frac{1}{\Omega^2}+1} \approx 0,$$

which , in turn, are equivalent to

$$\Omega \quad \text{is infinite, and} \quad \frac{\Phi}{\Omega} \approx -1,$$

which is even stronger than required. □

3 Regular cusps

Suppose further that α is obtained from two regular C^1 curves with at least C^1 contact at 0, that is, there exist two regular C^1 curves $\beta, \delta :]-a, a[\to \mathbb{R}^n$ such that

$$\begin{array}{rcl} \alpha(t) & = & \beta(t) \quad (-a < t \le 0) \\ \alpha(t) & = & \delta(t) \quad (0 \le t < a) \\ \beta'(0) & \text{and} & \delta'(0) \text{ are collinear.} \end{array}$$

If this is the case, α is *rectifiable*, in the sense that there exists an arc-length function

$$s(t) := \int_0^t |\alpha'(\tau)|\, d\tau \quad (t \in]-a, a[). \tag{3.1}$$

If we also assume that α is parameterized by this arc-length, $\alpha'(0)$ might not exist, but

$$\begin{array}{rcl} |\alpha'_+(0)| & = & 1 \\ |\alpha'_-(0)| & = & 1 \\ \alpha'_+(0) & = & \pm\alpha'_-(0). \end{array}$$

Therefore we have

Theorem 3.1. *If α is parameterized by the arc-length defined in equation* (3.1)*, then all conditions in Definition 2.1 are equivalent and equivalent to* (2.6) *and* (2.7)*.*

Proof. By theorem 2.3, we only need to prove that (2.2) ⇒ (2.3). Let $\epsilon \approx 0$; then $\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} \approx \frac{\alpha(-\epsilon)}{|\alpha(-\epsilon)|} \Rightarrow \alpha'_+(0) = -\alpha'_-(0) \Rightarrow (2.3)$. □

4 Envelopes of families of curves

Consider a family of curves $\{\alpha_\lambda \mid \lambda \in I\}$. As usual, an *envelope* of this family is a curve which at each of its points is tangent to a curve of the family. Suppose that the family of curves $\{\alpha_\lambda :]a, b[\to \mathbb{R}^2 \mid \lambda \in I\}$ is given by the equation $F(x, y, \lambda) = 0$, where F is a C^1 real valued function. The envelope is the result of eliminating λ between the two equations

$$F(x, y, \lambda) = 0 \quad \text{and} \quad \frac{\partial F}{\partial \lambda}(x, y, \lambda) = 0.$$

For example, if $F(x, y, \lambda) = (x - \lambda)^2 + y^2 - 1$, then the solution is $y = \pm 1$ (see Figure 3).

Let α_λ and $\alpha_{\lambda'}$ be two curves of this family. If they are near enough, they will meet in two distinct points; these points will be infinitely close to the lines $y = \pm 1$, if α_λ is infinitely close to $\alpha_{\lambda'}$. More precisely:

Let I and J be open intervals of $\mathbb{R}$ and Λ be a family of C^1 curves $\alpha_\lambda : J \to \mathbb{R}^n$,

$$\Lambda := \{\alpha_\lambda \mid \lambda \in I\},$$

and define $F : J \times I \to \mathbb{R}^n$ by

$$F(t, \lambda) := \alpha_\lambda(t).$$

Theorem 4.1. *Suppose that*

1. *F is of class C^1.*
2. *There exists a C^1 function $f : I \to \mathbb{R}$ which satisfies the following: for all $\lambda \in I$ and $\delta \approx 0$, there exists a pair $(t, t') \in {}^*J^2$, such that*

$$\begin{array}{rcl} t & \approx & t' \approx f(\lambda) \\ \alpha_\lambda(t) & = & \alpha_{\lambda+\delta}(t'). \end{array} \tag{4.1}$$

Define

$$\beta(\lambda) := \alpha_\lambda(f(\lambda)) = F(f(\lambda), \lambda) \quad (\lambda \in I).$$

Then $\beta'(\lambda)$ and $\alpha'_\lambda(f(\lambda))$ are collinear and thus β is an envelope of Λ.

Proof. Consider that

$$\beta'(\lambda) = f'(\lambda)\frac{\partial F}{\partial t}(f(\lambda), \lambda) + \frac{\partial F}{\partial \lambda}(f(\lambda), \lambda). \tag{4.2}$$

Next take $\lambda \in I$, a non-zero $\delta \approx 0$, the t, t' given by condition (4.1) and observe that

$$F(t', \lambda + \delta) - F(t, \lambda) = 0$$

so that, there exists $\eta \approx 0$, such that

$$DF_{(t,\lambda)}(t' - t, \delta) + |(t' - t, \delta)|\eta = 0.$$

Therefore

$$DF_{(t,\lambda)}\left(\frac{(t'-t,\delta)}{|(t'-t,\delta)|}\right) = -\eta \;\approx\; 0.$$

Since F is C^1, it follows

$$DF_{(f(\lambda),\lambda)}\left(st\left(\frac{(t'-t,\delta)}{|(t'-t,\delta)|}\right)\right) = 0,$$

hence

$$\frac{\partial F}{\partial t}(f(\lambda),\lambda) \text{ and } \frac{\partial F}{\partial \lambda}(f(\lambda),\lambda) \text{ are collinear.} \tag{4.3}$$

Finally, note that

$$\alpha'_\lambda(f(\lambda)) = \frac{\partial F}{\partial t}(f(\lambda),\lambda). \tag{4.4}$$

Conditions (4.2), (4.3) and (4.4) imply that $\beta'(\lambda)$ and $\alpha'_\lambda(f(\lambda))$ are collinear. □

Example 4.1. Let

$$\alpha_\lambda(t) := \left(t, \lambda t + \lambda^2, \frac{\lambda}{2}t + \frac{\lambda^2}{2}\right) \; (\lambda, t \in \mathbb{R}).$$

Then

$$\alpha_\lambda(t) = \alpha_{\lambda+\delta}(t') \Leftrightarrow t = t' = -2\lambda - \delta.$$

If $\delta \approx 0$ then $t = t' \approx -2\lambda$. It follows that $f(\lambda) := -2\lambda$ and the envelope is the curve $\lambda \overset{\beta}{\mapsto} (-2\lambda, -\lambda^2, -\frac{\lambda^2}{2})$ (Figure 4).

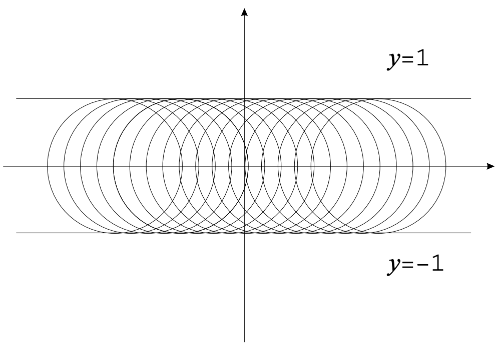

Figure 3

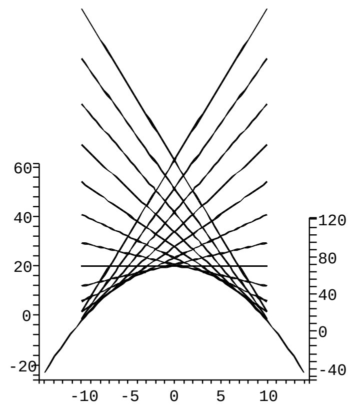

Figure 4

5 The coffee-cup caustic

The Coffee-cup Caustic is the (planar) envelope of the family of (co-planar) light rays reflected on a concave semi-cylindrical mirror, from a light source located (on the plane of the rays) at an infinite distance from the mirror, so that the produced light rays are parallel. For more detailed information, we refer [6] and [7].

We shall assume the mirror is the upper half-circle

$$(\cos(\lambda), \sin(\lambda)) \quad (0 < \lambda < \pi)$$

and that the incident rays are parallel to the y axis, with light source at $-\infty$ (Figure 5).

The reflected rays are the half-lines

$$\alpha_\lambda(t) = (\cos\lambda, \sin\lambda) + t(-\sin(2\lambda), \cos(2\lambda)), \quad t > 0, \lambda \in]0, \pi[.$$

The equation

$$\alpha_\lambda(t) = \alpha_{\lambda+\delta}(t')$$

has the solution

$$\begin{aligned} t &= \frac{\cos(\lambda+\delta) - \cos(\lambda+2\delta)}{\sin(2\delta)} \\ t' &= \frac{\cos(\lambda-\delta) - \cos(\lambda)}{\sin(2\delta)}. \end{aligned}$$

Therefore, for some $\theta, \tau \in {}^*]0,1[$ and $\delta \approx 0$, we infer

$$\begin{aligned} t &= \frac{\sin(\lambda + (2-\theta)\delta)\delta}{\sin(2\delta)} \approx \frac{\sin(\lambda)}{2} \\ t' &= \frac{\sin(\lambda - \tau\delta)\delta}{\sin(2\delta)} \approx \frac{\sin(\lambda)}{2}. \end{aligned}$$

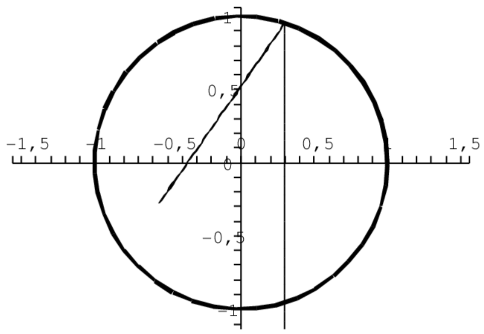

Figure 5

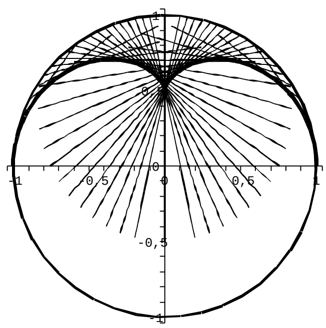

Figure 6

Taking $f(\lambda) := \frac{\sin(\lambda)}{2}$, Theorem 4.1 says that

$$\lambda \mapsto \beta(\lambda) := \left(\cos^3(\lambda), \sin(\lambda) + \frac{\sin(\lambda)}{2}\cos(2\lambda)\right)$$

is an envelope of the α_λ (Figure 6).

Also note that $\beta(\frac{\pi}{2}) = (0, \frac{1}{2})$ is a $T - cusp$ because, for $0 < \epsilon \approx 0$,

$$\begin{aligned}
\beta\left(\frac{\pi}{2}+\epsilon\right) - \beta\left(\frac{\pi}{2}-\epsilon\right) &= (-2\sin^3(\epsilon), 0) \\
\beta\left(\frac{\pi}{2}+\epsilon\right) - \beta\left(\frac{\pi}{2}\right) &= \left(-\sin^3(\epsilon), \cos(\epsilon) - \frac{\cos(\epsilon)}{2}\cos(2\epsilon) - \frac{1}{2}\right) \\
&= \left(-\sin^3(\epsilon), \sin^2(\epsilon)\left(\cos(\epsilon) - \frac{1}{2(\cos(\epsilon)+1)}\right)\right).
\end{aligned}$$

Therefore

$$\left(\frac{|\beta\left(\frac{\pi}{2}+\epsilon\right) - \beta\left(\frac{\pi}{2}-\epsilon\right)|}{|\beta\left(\frac{\pi}{2}+\epsilon\right) - \beta\left(\frac{\pi}{2}\right)|}\right)^2 = \frac{4\sin^2(\epsilon)}{\sin^2(\epsilon) + \left(\cos(\epsilon) - \frac{1}{2(\cos(\epsilon)+1)}\right)^2} \approx 0.$$

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References

[1] M. Aghasi and A. Suri, *Splitting theorems for the double tangent bundles of Frechet manifolds*, Balkan J. Geom. Appl. 15, 2 (2010), 1–13.

[2] R. Almeida, *A nonstandard characterization of regular surfaces*, Balkan J. Geom. Appl. 12, 2 (2007), 1–7.

[3] R. Almeida, *Nonstandard analysis and differentiable manifolds-foundations*, Differ. Geom. Dyn. Syst. 11 (2009), 1-19.

[4] R. Almeida, V. Neves and A. Plakhov, *Retroreflecting curves in nonstandard analysis*, Zh. Mat. Fiz. Anal. Geom. 5, 1 (2009), 12-24.

[5] G. Bouligand, *Introduction a la Gometrie Infinitesimale Directe*, Librairie Vuibert Paris 1932.

[6] J. W. Bruce, P. J. Giblin and C. G. Gibson, *On caustics of plane curves*, Amer. Math. Monthly 88 (1981), 651–667.

[7] J. W. Bruce, P. J. Giblin and C. G. Gibson, *Caustics through the looking glass*, Math. Intelligencer 6 (1984), 18–25.

[8] C. Costinescu, *Elements of Infinitesimal Riemannian Geometry* (in French), Balkan J. Geom. Appl. 6, 2 (2001), 17–26.

[9] M. Davis, *Applied Nonstandard Analysis*, John Wiley & Sons, 1977.

[10] M. Goze, *The study of the infinitely small point and applications*, (in French), Le labyrinthe du continu (Cerisy-la-Salle, 1990), 402-413, Springer, Paris, 1992.

[11] M. Goze, *Infinitesimal algebra and geometry*, Nonstandard Analysis in Practice, 91-108, Universitext, Springer, Berlin, 1995.

[12] U. Hertrich-Jeromin, *The surfaces capable of division into infinitesimal squares by their curves of curvature: a nonstandard–analysis approach to classical differential geometry*, Math. Intelligencer 22, 2 (2000), 54-61.

[13] U. Hertrich-Jeromin, *A nonstandard analysis characterization of submanifolds in Euclidean space*, Balkan J. Geom. Appl. 6, 1 (2001), 15–22.

[14] A. E. Hurd and P. A. Loeb, *An Introduction to Nonstandard Real Analysis*, Pure Appl. Math. 118, Academic Press, Inc. 1995.

[15] V. Kanovei and M. Reeken, *A nonstandard proof of the Jordan curve theorem*, Real Anal. Exchange 24, 1 (1998), 161–169.

[16] R. Lutz and M. Goze, *Nonstandard Analysis - A Practical Guide with Applications*, Springer LNM, 1981.

[17] M. Popescu and P. Popescu, *Lagrangians and higher order tangent spaces*, Balkan J. Geom. Appl. 15, 1 (2010), 142–148.

[18] A. Robinson, *Non-Standard Analysis*, North-Holland Publishing Company, 1974.

[19] K. D. Stroyan and W. A. J. Luxemburg, *Introduction to the Theory of Infinitesimals*, Pure Appl. Math. 72, Academic Press 1976.

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