Notes on Finsler manifolds with a compact submanifold

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Abstract. In this paper, we study the relationship between a Finsler manifold and its submanifolds, prove some rigidity theorems, and then obtain some results which have the form of the well-known Bonnet-Myers theorem.


Key words: Chern connection; Finsler geometry; totally geodesic; compact.

1 Introduction

Recently, there has been a surge of interest in Finsler geometry, especially in its global and analytic aspects (see [1,8,10]). One of the fundamental problems is to study the relationship between a Finsler manifold and its compact submanifolds. This study started in [3,4,5,6,9]. The purpose of this paper is to discuss some properties on a Finsler manifold with a compact submanifold.

At the beginning, using the Chern connection and the first variation formula of arc-length, we obtain the result that \( γ'(l) \) is perpendicular to \( T_{x_1}N \), where \( N \) is a submanifold of \( M \) and \( γ : [0, l] \rightarrow M \) is a shortest geodesic curve such that \( γ(0) \notin N, γ(l) = x_1 \in N \). It is noteworthy that this conclusion is the basis of the ensuing discussion.

As is known to all, the Bonnet-Myers theorem states that every geodesic of length \( \frac{l}{\sqrt{c}} \) has conjugate points and a manifold \( M \) is compact under some particular curvature conditions. Also, because totally geodesic submanifolds are the higher dimension generalizations of geodesic curves, we expect to obtain some rigidity results for Finsler manifolds with a totally geodesic submanifold. Therefore we define \( \text{Ric}_r M \) and \( K(X, H) \) by analyzing the characteristic of the flag curvature, then compute the second variation formula of arc-length. Finally we get some results similar to the Bonnet-Myers theorem.

In the end of this article we point out that to conclude our results, ” totally geodesic ” cannot be weakened to ” minimal ” submanifolds.
2 Preliminaries

Let \((M, F)\) be a \(m\)-dimensional complete connected Finsler manifold with Finsler metric \(F : TM \rightarrow [0, +\infty)\). Let \((x, v) = (x^i, v^i)\) be local coordinates on \(TM\), and \(\pi : TM \setminus 0 \rightarrow M\) be the natural projection. Then we present some fundamental quantities:

\[
g_{ij} := \frac{1}{2} \frac{\partial^2 F^2(x, v)}{\partial v^i \partial v^j}, \quad \text{(fundamental tensor)}
\]

\[
C_{ijk} := \frac{1}{4} \frac{\partial^3 F^2(x, v)}{\partial v^i \partial v^j \partial v^k}. \quad \text{(Cartan tensor)}
\]

According to [1], the pulled-back bundle \(\pi^*TM\) admits a unique linear connection, named Chern connection. Its connection forms are characterized by the following structural equations:

\[
dx^j \wedge \omega^j_i = 0, \quad \text{(Torsion freeness)}
\]

\[
dg_{ij} - g_{kj} \omega^k_i - g_{ik} \omega^k_j = 2C_{ijk} \omega^k_n + C_{ijk}, \quad \text{(Almost \(g\)-compatibility)}
\]

Let \(V = v^i \frac{\partial}{\partial x^i}\) be a non-vanishing vector field on an open subset \(U \subset M\). One can introduce a Riemannian metric \(g_V\) and a linear connection \(\nabla^V\) on the tangent bundle over \(U\) as follows:

\[
g_V(X, Y) = X^i Y^j g_{ij}(x, V), \quad \forall X = X^i \frac{\partial}{\partial x^i}, Y = Y^j \frac{\partial}{\partial x^j}
\]

\[
\nabla^V_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \Gamma^k_{ij}(x, V) \frac{\partial}{\partial x^k}.
\]

From the torsion freeness and \(g\)-compatibility of Chern connection, we have (see [1][12])

\[
\nabla^V_X Y - \nabla^V_Y X = [X, Y]
\]

\[
X g_V(Y, Z) = g_V(\nabla^V_X Y, Z) + g_V(Y, \nabla^V_X Z) + 2C_V(\nabla^V_X V, Y, Z),
\]

where \(C_V\) is defined by \(C_V(X, Y, Z) = X^i Y^j Z^k C_{ijk}(x, v)\).

The Chern curvature \(R^V(X, Y)Z\) for vector fields \(X, Y, Z\) on \(U\) is defined by

\[
R^V(X, Y)Z := \nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X,Y]} Z.
\]

For a flag \((V; \sigma)\) (or \((V; W)\)), the flag curvature \(K(V; \sigma)\) is defined as follows:

\[
K(V; \sigma) = K(V; W) := \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g^2_V(V, W)}.
\]

where \(W\) is a tangent vector such that \(V, W\) span the two-plane \(\sigma\).
Let \( H^r \subset T_x M \) be an \( r \)-plane spanned by \( r \)-mutually orthogonal unit tangent vectors \( e_1, e_2, \ldots, e_r \in T_x M \) and \( V \in T_x M \) be a tangent vector orthogonal to \( H^r \).

Then the \( r \)-th Ricci curvature and the \( V \)-\( H^r \) flag curvature of \( M \) are defined by

\[
\begin{align*}
\text{Ric}_r M & := \sup_{1 \leq i \leq r} K(V, e_i), \\
K(V, H^r) & := \sum_{i=1}^r K(V, e_i).
\end{align*}
\]

It is obvious that \( V \)-\( H^r \) flag curvature is independent of choice of the vectors \( e_1, e_2, \ldots, e_r \). In fact, let \( b_1, b_2, \ldots, b_r \) be another orthogonal unit tangent vectors which span \( H^r \), then there exists \( \lambda_{ij} \in \mathfrak{N} \), such that \( b_i = \sum_{j=1}^r \lambda_{ij} e_j \), where \( (\lambda_{ij}) \) is an orthogonal matrix. Thus

\[
\sum_{i=1}^r K(V, b_i) = \sum_{i=1}^r K(V, \sum_{j=1}^r \lambda_{ij} e_j) = \frac{1}{g_V(V, V)} \sum_{i,j,k=1}^r \lambda_{ij} \lambda_{ik} g_V(R^V(V, e_j)e_k, V)
\]

\[
= \frac{1}{g_V(V, V)} \sum_{j,k=1}^r \delta_{jk} g_V(R^V(V, e_j)e_k, V) = \sum_{j=1}^r K(V, e_j) = K(V, H^r).
\]

\section{Main theorems}

Now let us first fix \( x \in M \) and let \( N \) be a \( n \)-dimensional compact submanifold of \( M \). Then there is a point \( x_1 \in N \), such that \( d := d(x, N) = d(x, x_1) \). Let \( \gamma(t), t \in [0, d] \) be a minimizing geodesic in \( M \) parametrized by arc-length from \( x \) to \( x_1 \) such that \( \gamma \) realizes the distance from \( x \) to \( N \), then define \( V = \gamma'(t) \). First of all, we obtain an available theorem as follows:

\textbf{Theorem 3.1.} Let \( (M, F) \) be a complete connected Finsler manifold, \( N \) be a compact submanifold of \( M \), and \( \gamma : [0, l] \rightarrow M \) be a geodesic such that \( \gamma(0) \notin N, \gamma(l) = x_1 \in N \). If \( \gamma \) is the shortest curve from \( \gamma(0) \) to \( N \), then \( \gamma'(l) \) is perpendicular to \( T_{x_1} M \).

\textbf{Proof.} If \( \gamma'(l) \) is not perpendicular to \( T_{x_1} M \), choose \( X \in T_{x_1} M \) such that \( g_V(\gamma'(l), X) > 0 \). Let \( c(u) \) be a curve starting from \( x_1 \) with initial tangent vector \( X \) in \( N \), then we can construct a variation \( b : [0, l] \times [-\varepsilon, \varepsilon] \rightarrow M \) such that \( b|[0,l] \times \{0\} = \tilde{\gamma}, b(0, u) = c(u), b(l, u) = \gamma(0) \). Denote \( \tilde{\gamma}_u = b|[0,l] \times u, \tilde{\gamma}_u(t, u) = \frac{\partial b}{\partial u}(t, u), \tilde{\gamma}(t, u) = \frac{\partial b}{\partial u}(t, u). \) From the first variation formula of arc-length (see [1], [11]), we have

\[
\frac{d}{du} L(\tilde{\gamma}_u)_{u=0} = \|\tilde{\gamma}\|^{-1} \int_0^l g_V(\nabla^V \tilde{\gamma}_u, \tilde{\gamma}_u) dt
\]

\[
= \|\tilde{\gamma}\|^{-1} [g_V(\tilde{\gamma}, \tilde{\gamma})]_0^l - \int_0^l g_V(\tilde{\gamma}_u, \nabla^V \tilde{\gamma}_u) dt],
\]

where \( \tilde{\gamma} \) is \( \gamma \) itself, but its direction is from \( \gamma(0) \) to \( \gamma(0) \). Since \( \gamma \) is a geodesic, then the terms \( \nabla^V \tilde{\gamma}_u|_{u=0} = 0 \) and \( g_V(\tilde{\gamma}, \tilde{\gamma})|_{u=1} = 0 \). As a result,

\[
\frac{d}{du} L(\tilde{\gamma}_u)_{u=0} = -\|\tilde{\gamma}\|^{-1} g_V(\tilde{\gamma}, \tilde{\gamma})|_{u=0} = -\|\tilde{\gamma}\|^{-1} g_V(\gamma'(l), X) < 0.
\]
Therefore, for a small \( u \), we have \( L(\tilde{\gamma}_u) < L(\tilde{\gamma}) = L(\gamma) \), which contradicts the assumption that \( \gamma \) is the shortest curve from \( \gamma(0) \) to \( N \).

Next let \( N \) be a totally geodesic submanifold, then the second fundamental form of \( N \) is zero and the tangent vector field along the geodesics in \( N \) are also the geodesics in \( M \). Take an orthogonal basic \( e_1, e_2, \ldots, e_n \) of \( T_x N \) and let \( \{E_i(t)\} \) be the parallel translate of \( \{e_i\} \) along \( \gamma \). Then set \( W_i(t) = (\sin \frac{\pi}{2}) E_i(t) \) \((i = 1, \ldots, n)\) is a vector field on \( \gamma \). Obviously, each \( W_i \) gives rise to a geodesic variation of the variational curves of the geodesic \( \gamma \) by keeping one end point \( x \) fixed and the other end point on submanifold \( N \). Let \( L_i \) be the arc-length of the variational curve induced by \( W_i \), it is easy to see that \( L_i(0) = 0 \). Now we compute \( L''_i(0) \) by using the second variation formula of arc-length (see [1][11]).

Noting that \( g(V, \nabla_V W_i)\big|_0^d = 0 \), we have

\[
L''_i(0) = \frac{d^2}{du^2} L(\gamma_u)|_{u=0} = g(V, \nabla_V W_i)|_0^d + \int_0^d [g(V, \nabla^2 V W_i) - g(V, R^V (V, W_i) W_i, V)] dt = \int_0^d [g(V, \nabla^2_V W_i) - g(V, R^V (V, W_i) W_i, V)] dt.
\]

(3.3)

Based on this argument, it can be concluded the following result:

**Theorem 3.2.** Let \((M, F)\) be a complete connected Finsler manifold and \( N \) be a totally geodesic compact submanifold whose dimension is \( n \). For all \( x \in M \), if \( K(\gamma'(t), H^r(t)) \geq rc > 0 \) \((r \leq n)\), along each minimizing geodesic \( \gamma \) starting from \( x \) for any \( r \)-plane \( H^r(0) \subset \gamma(0)^\perp \), then \( d(x, N) \leq \frac{\pi}{2} \sqrt{rc} \); where \( \gamma(0)^\perp \) denotes the orthogonal complement of \( \gamma(0) \) in \( T_x M \) and \( H^r(t) \) denotes the parallel translate of the plane \( H^r(0) \) \((H^r(0) \subset T_x M)\) along \( \gamma \).

**Proof.** Since \( K(\gamma'(t), H^r(t)) \geq rc \), for \( i_1, i_2, \ldots, i_r \in \{1, \ldots, n\}, i_j \neq i_k \) \((j \neq k)\), \( r \leq n \) and \( \gamma'(t) = V \), we have \( \sum_{j=1}^{r} g(V, R^V (V, E_{i_j}(t)) E_{i_j}(t), V) \geq rc \). As a result, we get

\[
\sum_{i=1}^{n} g(V, R^V (V, E_i(t)) E_i(t), V) = \frac{1}{r} (r \sum_{i=1}^{n} g(V, R^V (V, E_i(t)) E_i(t), V)) \geq \frac{1}{r} \cdot \frac{n}{r^2} (\sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r} \sum_{j=1}^{r} g(V, R^V (V, E_{i_j}(t)) E_{i_j}(t), V)) \geq \frac{n}{r c_n} \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n} rc = nc.
\]

(3.4)
Let $L_i$ be minimal, then geodesic in $Ric$ totally geodesic compact submanifold whose dimension is $n$. From (3.4) and (3.5), we have

$$\sum_{i=1}^{n} L_i''(0) = \int_0^{d} \left( \sum_{i=1}^{n} g_V(\nabla^V_i W_i, \nabla^V_i W_i) - \sum_{i=1}^{n} g_V(R^V(V, W_i)W_i, V) \right) dt$$

(3.5)

$$= \int_0^{d} \left( \sum_{i=1}^{n} \left( \frac{\pi}{\sqrt{2r}} \cos \frac{\pi}{2r} \right)^2 - \sum_{i=1}^{n} \sin^2 \frac{\pi}{2r} g_V(R^V(V, E_i)E_i, V) \right) dt$$

$$= \int_0^{d} \left( \frac{\pi^2}{4r^2} \cos^2 \frac{\pi}{2r} - \sin^2 \frac{\pi}{2r} \sum_{i=1}^{n} g_V(R^V(V, E_i)E_i, V) \right) dt.$$
References


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