Real hypersurfaces in complex space forms with pseudo-parallel Ricci operator

Jun-ichi Inoguchi

Abstract. Real hypersurfaces in non-flat complex space forms with pseudo-parallel Ricci operator are classified. As a result we exhibit some examples of homogeneous Riemannian manifolds with pseudo-parallel Ricci operator.

Key words: real hypersurface; complex space form; pseudo-parallel.

1 Introduction

Construction and classification problems for Einstein hypersurfaces in Riemannian manifolds have been paid much attention of differential geometers.

As a generalisation of Einstein manifold, the notion of semi-parallel Ricci operator is introduced as follows.

A Riemannian manifold $(M, g)$ has a semi-parallel Ricci operator if its Ricci operator $S$ satisfies

$$R(X, Y) \cdot S = 0$$

for all vector fields $X$ and $Y$ on $M$. Here $R$ is the Riemannian curvature of $M$.

On the other hand, as a generalisation of locally symmetric Riemannian manifolds, (locally) semi-symmetric spaces have been studied intensively.

A Riemannian manifold $(M, g)$ is said to be (locally) semi-symmetric if

$$R(X, Y) \cdot R = 0$$

for all vector fields $X$ and $Y$ on $M$.

In 1972, P. J, Ryan [36] proposed a question:

"Euclidean hypersurfaces with semi-parallel Ricci operator ($R \cdot S = 0$) are semi-symmetric ($R \cdot R = 0$) ?"

One of the generalisation of Ricci semi-parallelity is the Ricci pseudo-parallelity:
\[
R(X, Y) \cdot S = L \{(X \wedge Y) \cdot S\},
\]
where \(L\) is a function.

In differential geometry of hypersurfaces, the Ricci pseudo-parallelism is actually an interesting property. In fact, every Cartan’s isoparametric hypersurface in spheres has pseudo-parallel Ricci operator [6, 7, [13].

On the other hand, in real hypersurface geometry of complex space forms, the Einstein condition is a too strong restriction for real hypersurfaces. In fact, there are no real hypersurfaces in non-flat complex space forms with parallel Ricci operator (cf. [15]). More generally, M. Kimura and S. Maeda [23] and U-H. Ki, H. Nakagawa and Y. J. Suh [16] showed the nonexistence of real hypersurfaces with semi-parallel Ricci operator in non-flat complex space forms.

These results motivate us to study more mild conditions on Ricci operator of real hypersurfaces.

In our previous work [9], J. T. Cho, T. Hamada and the present author classified Hopf hypersurfaces with pseudo-parallel Ricci operator in complex projective plane \(\mathbb{CP}_2\) and complex hyperbolic plane \(\mathbb{CH}_2\).

In [18], I.-B. Kim, H. J. Park and H. Song studied real hypersurfaces in complex projective \(\mathbb{CP}_n\) and complex hyperbolic \(\mathbb{CH}_n\) with pseudo-parallel Ricci tensor and \(n > 2\). They claimed that the only real hypersurfaces of dimension greater than 3 in \(\mathbb{CP}_n\) and \(\mathbb{CH}_n\) are \((A_2)\) minimal tubes over totally geodesic complex projective subspace \(\mathbb{CP}_\ell \subset \mathbb{CP}_n (1 \leq \ell \leq n - 2)\) and \((A_0)\) horospheres in \(\mathbb{CH}_n\).

However, one can easily check that every geodesic sphere in \(\mathbb{CP}_n\) and \(\mathbb{CH}_n\) have pseudo-parallel Ricci tensor. These real hypersurfaces are counterexamples to the classification result due to [18].

As a continuation, in this paper, we shall give a classification of real hypersurfaces with pseudo-parallel Ricci operator in \(\mathbb{CP}_n\) and \(\mathbb{CH}_n\) for \(n \geq 3\).

**Theorem 1.** Let \(M \subset \mathbb{CP}_n(c)\) be a real hypersurface in a complex projective space of constant holomorphic sectional curvature \(c (n > 2)\). Then \(M\) has pseudo-parallel Ricci operator if and only if it is locally holomorphically congruent to one of the following hypersurfaces:

\(\text{(A}_1\text{)}\) a geodesic sphere of radius \(r\), where \(0 < r < \frac{\pi}{\sqrt{c}}\),

\(\text{(A}_2\text{)}\) a tube of radius \(r = \frac{\pi}{2\sqrt{c}}\) over a totally geodesic \(\mathbb{CP}_{(n-1)/2}\). In this case \(M\) is minimal in \(\mathbb{CP}_n(c)\).

**Theorem 2.** Let \(M \subset \mathbb{CH}_n(c)\) be a real hypersurface in a complex hyperbolic space of constant holomorphic sectional curvature \(c (n > 2)\). Then \(M\) has pseudo-parallel Ricci operator if and only if it is locally holomorphically congruent to one of the following hypersurfaces:

\(\text{(A}_0\text{)}\) a horosphere,

\(\text{(A}_1\text{)}\) a geodesic sphere or a tube over a complex hyperbolic hyperplane \(\mathbb{CH}_{n-1}\).
In our previous paper [9], we have shown that there are no ruled real hypersurfaces in $\mathbb{C}P_2(c)$ with pseudo-parallel Ricci operator.

In the final section of this paper, we shall show the nonexistence of ruled real hypersurfaces in $\mathbb{C}H_2(c)$ with pseudo-parallel Ricci operator.

**Theorem 3.** Ruled real hypersurfaces in $\mathbb{C}H_2(c)$ do not admit pseudo-parallel Ricci operator.

## 2 Preliminaries

### 2.1

Let $(M, g)$ be a Riemannian manifold with its Levi-Civita connection $\nabla$. Denote by $\mathfrak{X}(M)$ the Lie algebra of all vector fields on $M$. A tensor field $F$ of type $(1, 3)$;

$$F : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is said to be curvature-like provided that $F$ has the symmetric properties of $R$. For example,

$$\begin{align*}
(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y, \quad &X, Y, Z \in \mathfrak{X}(M)
\end{align*}$$

defines a curvature-like tensor field on $M$. Note that the curvature $R$ of a Riemannian manifold $(M, g)$ of constant curvature $c$ satisfies the formula $R(X, Y) = c(X \wedge Y)$.

Every curvature-like tensor field $F$ acts on the algebra $T^1_s(M)$ of all tensor fields on $M$ of type $(1, s)$ as a derivation:

$$(F \cdot P)(X_1, \cdots, X_s; Y, X) = F(X, Y)\{P(X_1, \cdots, X_s)\}$$

$$- \sum_{j=1}^s P(X_1, \cdots, F(X, Y)X_j, \cdots, X_s),$$

$$X_1, \cdots, X_s \in \mathfrak{X}(M), \quad P \in T^1_s(M).$$

The derivative $F \cdot P$ of $P$ by $F$ is a tensor field of type $(1, s + 2)$.

For a tensor filed $P$ of type $(1, s)$, we denote the by $Q(g, P)$ the derivative of $P$ with respect to the curvature-like tensor defined by (2.1).

**Definition 2.1.** A tensor field $P$ is said to be

- **parallel** if $\nabla P = 0$.
- **semi-parallel** if $R \cdot P = 0$.
- **pseudo-parallel** if there exists a function $L$ such that $(R \cdot P) = LQ(g, P)$.

In particular, a pseudo-parallel tensor field $P$ is said to be **proper** if $L \neq 0$. 

2.2

A Riemannian manifold \((M, g)\) is said to be semi-symmetric if \(R \cdot R = 0\). Obviously, locally symmetric spaces \((\nabla R = 0)\) are semi-symmetric.

R. Deszcz [12] introduced the notion of pseudo-symmetric space. A Riemannian manifold \((M, g)\) is said to be pseudo-symmetric if \(R\) is pseudo-parallel, i.e.,

\[
R \cdot R = L \cdot \mathcal{Q}(g, R)
\]

for some function \(L\). A pseudo-symmetric space is said to be proper if it is not semi-symmetric.

2.3

The Ricci tensor field \(\rho\) of a Riemannian manifold \((M, g)\) is defined by

\[
\rho(X, Y) = \text{trace}(Z \mapsto R(Z, X)Y), \quad X, Y \in \mathfrak{X}(M).
\]

The endomorphism field \(S\);

\[
\rho(X, Y) = g(SX, Y), \quad X, Y \in \mathfrak{X}(M)
\]

metrically associated to \(\rho\) is called the Ricci operator of \(M\). The trace \(s\) of \(S\) is called the scalar curvature of \(M\).

A Riemannian manifold is said to be Einstein if \(\rho = cg\) for some constant \(c\). In this case, \(c = s/\dim M\).

On can see that every Einstein manifold has parallel Ricci tensor field, i.e., \(\nabla \rho = 0\) (equivalently \(\nabla S = 0\)). More generally, Einstein manifolds have semi-parallel Ricci operator \((R \cdot S = 0)\).

Remark 2.2. Riemannian manifolds with pseudo-parallel Ricci operator are also called Ricci pseudo-symmetric spaces. Some authors call Riemannian manifolds with semi-parallel Ricci operator by the name Ryan spaces.

2.4

The Riemannian curvature \(R\) of a Riemannian 3-manifold \((M, g)\) is expressed as

\[
R(X, Y)Z = \rho(Y, Z)X - \rho(Z, X)Y + g(Y, Z)SX - g(Z, X)SY - \frac{s}{2}(X \wedge Y)Z
\]

for all \(X, Y, Z \in \mathfrak{X}(M)\).

The formula (2.2) implies that a Riemannian 3-manifold is Einstein if and only if it is of constant curvature. Moreover we have

Proposition 2.1. A Riemannian 3-manifold \((M, g)\) is a pseudo-symmetric space such that \(R \cdot R = L \mathcal{Q}(g, R)\) if and only if \((M, g)\) has pseudo-parallel Ricci operator such that \(R \cdot S = L \mathcal{Q}(g, S)\). In particular \(M\) is semi-symmetric \((R \cdot R = 0)\) if and only if \(R \cdot S = 0\).
2.5

The pseudo-parallelity of tensor fields of type $(1,1)$ is characterized as follows (see [9], [11]).

**Lemma 2.2.** Let $(M, g)$ be a Riemannian 3-manifold and $B$ a tensor field on $M$ of type $(1,1)$ which is self-adjoint with respect to $g$. Take a local orthonormal frame field $\{e_1, e_2, e_3\}$ which diagonalizes $B$ so that $Be_j = b_j e_j$ ($j = 1, 2, 3$). Assume that $M$ is not of constant curvature and $B$ is not of the form $B = \mu I$ for some function $\mu$, where $I$ denotes the identity transformation. Then $B$ is pseudo-parallel such that $R \cdot B = LQ(g, B)$ for some function $L$ if and only if the eigenvalues of $B$ and the sectional curvature function $K$ locally satisfy the following relations (up to numeration):

$$b_1 = b_2 \neq b_3, \quad K_{13} = K_{23} = L.$$ 

Here $K_{ij} = K(e_i \wedge e_j)$ denotes the sectional curvature of the plane $e_i \wedge e_j$ spanned by $e_i$ and $e_j$.

Corollary 2.1 together with Lemma 2.2 imply the following criterion for pseudo-symmetry [11].

**Proposition 2.3.** A Riemannian 3-manifold $(M, g)$ of non-constant curvature is pseudo-symmetric if and only if it is quasi-Einstein. Namely there exists a one-form $\omega$ such that the Ricci tensor field $S$ has the form:

$$S = a g + b \omega \otimes \omega.$$ 

Here $a$ and $b$ are functions. In this case $M$ satisfies $R \cdot R = LQ(g, R)$ with $2L = a + b$.

The preceding proposition can be rephrased as follows ([27, Proposition 0.1]):

**Proposition 2.4.** A Riemannian 3-manifold of non-constant curvature is a pseudo-symmetric space with $R \cdot R = LQ(g, R)$ if and only if the eigenvalues of the Ricci tensor locally satisfy the following relations (up to numeration):

$$\rho_1 = \rho_2, \quad \rho_3 = 2L.$$ 

3 Real hypersurfaces

3.1

Let us denote by $\overline{M}_n(c)$ the simply connected $n$-dimensional complex space form of constant holomorphic sectional curvature $c$ with Kähler structure $(\overline{g}, J)$. As is well known, $\overline{M}_n(c)$ is holomorphically isometric to complex projective space $\mathbb{C}P_n(c)$, complex Euclidean space $\mathbb{C}^n$ or complex hyperbolic space $\mathbb{C}H_n(c)$ according as $c > 0$, $c = 0$ and $c < 0$, respectively.

Throughout this paper we assume that $\overline{M}_n(c)$ is non-flat, that is, $c \neq 0$.

Let $M \subset \overline{M}_n(c)$ be a real hypersurface with (local) unit normal vector field $N$. Let us denote by $g$ the induced Riemannian metric on $M$. Then the Levi-Civita connections $\overline{\nabla}$ of $\overline{M}_n(c)$ and $\nabla$ of $M$ are related by the Gauss formula:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad X, Y \in \mathfrak{X}(M),$$

(3.1)
where the endomorphism field $A$ on $M$ is defined by the Weingarten formula:

$$\nabla_X N = -AX, \ X \in \mathfrak{X}(M).$$  (3.2)

The endomorphism field $A$ is called the shape operator of $M$ derived from $N$. An eigenvector $X \in T_x M$ of $A$ at a point $x \in M$ is called a principal curvature vector at $x$. The corresponding eigenvalue $\lambda$ of $A$ is called a principal curvature of $M$ at $x$.

The structure vector field $\xi$ of $M$ and its dual 1-form $\eta$ are defined by

$$\eta(X) = g(\xi, X) = \tilde{g}(JX, N), \ X \in \mathfrak{X}(M).$$

We define an endomorphism $\phi$ on $M$ by

$$g(\phi X, Y) = \tilde{g}(JX, Y), \ X, Y \in \mathfrak{X}(M).$$

One can easily check that this structure satisfies the following formulae:

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi \xi = 0.$$  (3.3)

These formulae show that $(\phi, \xi, \eta, g)$ is an almost contact Riemannian structure on $M$. The shape operator $A$ is related to the almost contact Riemannian structure by

$$\nabla_X \xi = \phi AX.$$

The Ricci operator $S$ of $M$ is given by

$$SX = \frac{c}{4} \{ (2n+1)X - 3\eta(X)\xi \} + hAX - A^2 X, \ X \in \mathfrak{X}(M),$$

where $h = \text{tr} A$.

Here, we recall the following two fundamental results (See eg., [34]).

**Lemma 3.1.** If $\xi$ is a principal curvature vector, then the corresponding principal curvature $\alpha$ is locally constant.

**Lemma 3.2.** Assume that $\xi$ is a principal curvature vector and the corresponding principal curvature is $\alpha$. If $AX = \lambda X$ for $X \perp \xi$, then we have $(2\lambda - \alpha)A\phi X = (\alpha \lambda + \frac{c}{2})\phi X$.

A real hypersurface $M \subset \tilde{M}_n(c)$ is said to be Hopf if $\xi$ is a principal curvature vector field.

### 3.2

R. Takagi [38]–[39] classified the homogeneous real hypersurfaces of $\mathbb{C}P_n(c)$ into six types. T. E. Cecil and P. J. Ryan [8] extensively studied Hopf hypersurfaces, which are realized as tubes over certain submanifolds in $\mathbb{C}P_n(c)$, by using its focal map $\varphi_r$. By making use of those results and the work of Takagi, M. Kimura [20] gave a local classification theorem for Hopf hypersurfaces of $\mathbb{C}P_n(c)$ all of whose principal curvatures are constant.
Theorem 3.3. ([20]) Let $M$ be a Hopf hypersurface in complex projective $n$-space $\mathbb{C}P_n(c)$ of constant holomorphic sectional curvature $c > 0$ and $n > 2$. Then $M$ has constant principal curvatures if and only if $M$ is locally congruent to one of the following:

(A) a geodesic hypersphere of radius $r$, where $0 < r < \frac{\pi}{\sqrt{c}}$,

(B) a tube of radius $r$ over a totally geodesic $\mathbb{C}P_\ell$ ($1 \leq \ell \leq n - 2$), where $0 < r < \frac{\pi}{\sqrt{c}}$,

(C) a tube of radius $r$ over $\mathbb{C}P_1 \times \mathbb{C}P_{(n-1)/2}$, where $0 < r < \frac{\pi}{\sqrt{c}}$ and $n(\geq 5)$ is odd,

(D) a tube of radius $r$ over a complex Grassmannian $G_{2,5}(\mathbb{C})$, where $0 < r < \frac{\pi}{\sqrt{c}}$ and $n = 9$,

(E) a tube of radius $r$ over a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{\sqrt{c}}$ and $n = 15$.

For complex hyperbolic space $\mathbb{C}H_n(c)$, J. Berndt [3] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant.

Theorem 3.4. ([3]) Let $M$ be a Hopf hypersurface in complex hyperbolic $n$-space $\mathbb{C}H_n(c)$ of constant holomorphic sectional curvature $c < 0$ and $n > 2$. Then $M$ has constant principal curvatures if and only if $M$ is locally congruent to one of the following:

(A) a horosphere,

(B) a tube over a totally geodesic $\mathbb{C}H_\ell$ ($1 \leq \ell \leq n - 2$),

(c) a tube over a totally geodesic and Lagrangian imbedded real hyperbolic space $\mathbb{R}H_n$.

We call simply type (A) for real hypersurfaces of type (A), (A) in $\mathbb{C}P_n(c)$ and ones of type (A), (A) or (A) in $\mathbb{C}H_n(c)$.

3.3

Kimura and Maeda studied real hypersurfaces in complex projective whose Ricci operator are constant along the trajectory of the structure vector field, i.e., $\mathcal{L}_\xi S = 0$.

One can see that a real hypersurface $M \subset \tilde{M}_n(c)$ of a non-flat complex space form satisfies $\mathcal{L}_\xi S = 0$ if and only if

$$||S\phi - \phi S||^2 + \frac{3c}{2}||\phi A\xi||^2 = 0.$$  (3.5)

This equation implies the following fundamental result.
**Proposition 3.5** ([24]). Let $M \subset \mathbb{C}P_n(c)$ be a real hypersurface with $n > 2$. Then $M$ satisfies $\mathcal{L}_\xi S = 0$ if and only if $M$ is Hopf and satisfies $S\phi = \phi S$.

In complex projective space, the following classification is known.

**Theorem 3.6** ([22], [24]). Let $M \subset \mathbb{C}P_n(c)$ be a real hypersurface with $n > 2$. Then $M$ satisfies $\mathcal{L}_\xi S = 0$ if and only if $M$ is Hopf and satisfies $S\phi = \phi S$, in addition except for the null set on which the focal map $\varphi_r$ degenerates, $M$ lies on a tube of radius $r$ over one of the following Kähler submanifolds in $\mathbb{C}P_n(c)$:

(A1) a geodesic hypersphere of radius $r$, where $0 < r < \frac{\pi}{\sqrt{c}}$,

(A2) a tube of radius $r$ over a totally geodesic $\mathbb{C}P_\ell$ $(1 \leq \ell \leq n - 2)$, where $0 < r < \frac{\pi}{\sqrt{c}}$,

(B) a tube of radius $r$ over $\mathbb{C}P_1 \times \mathbb{C}P_{n-1}/2$, where $0 < r < \frac{\pi}{\sqrt{c}}$ and $n(\geq 5)$ is odd,

(D) a tube of radius $r$ over a complex Grassmannian $G_{2,5}(\mathbb{C})$, where $0 < r < \frac{\pi}{\sqrt{c}}$ and $n = 9$,

(E) a tube of radius $r$ over a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{\sqrt{c}}$ and $n = 15$.

On the other hand, in complex hyperbolic space, (3.5) implies that a Hopf hypersurface satisfies $S\phi = \phi S$ if and only if $\mathcal{L}_\xi S = 0$.

**Theorem 3.7** ([19]). Let $M$ be a Hopf hypersurface in complex hyperbolic $n$-space $\mathbb{C}H_n(c)$ of constant holomorphic sectional curvature $c < 0$ and $n > 2$. Then $M$ satisfies $\mathcal{L}_\xi S = 0$ if and only if $M$ is locally congruent to one of the following:

(A0) a horosphere,

(A1) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $\mathbb{C}H_{n-1}$,

(A2) a tube over a totally geodesic $\mathbb{C}H_{\ell}$ $(1 \leq \ell \leq n - 2)$,

3.4

A real hypersurface $M \subset \tilde{M}_n(c)$ is said to be pseudo-Einstein (or $\eta$-Einstein) if there exist constants $a$ and $b$ such that $S = aI + b\eta \otimes \xi$.

**Theorem 3.8** ([8], [25]). Let $M$ be a real hypersurface in $\mathbb{C}P_n(c)$ $(n > 2)$ with Ricci operator $S = aI + b\eta \otimes \xi$, where $a$ and $b$ are functions. Then $a$ and $b$ are constants. Moreover $M$ is locally holomorphically congruent to
\(A_1\) a geodesic sphere,

\(A_2\) a tube of radius \(r = 2u/\sqrt{c}\) over a complex projective subspace \(\mathbb{CP}_p\), with \(1 < p < n - 2\), \(0 < u < \pi/2\), and \(\cot^2 u = p/q\) or

\(B\) a tube of radius \(r = 2u/\sqrt{c}\) over a complex quadric \(Q_{n-1}\), where \(0 < u < \pi/4\) and \(\cot^2 2u = n - 2\).

**Theorem 3.9 ([8], [32])**. Let \(M\) be a real hypersurface in \(\mathbb{CH}_n(c)\) \((n > 2)\) with Ricci operator \(S = aI + b\eta \otimes \xi\), where \(a\) and \(b\) are functions. Then \(a\) and \(b\) are constants.

\(A_1\) a geodesic sphere,

\(A_1\) a tube of a complex hyperbolic hyperplane \(\mathbb{CH}_{n-1}\), or

\(A_0\) a horosphere.

From these classification we notice that all the pseudo-Einstein real hypersurfaces in non-flat complex space forms are homogeneous if \(n > 2\).

4 Real hypersurfaces with pseudo-parallel Ricci operator

4.1

In [18], Kim, Park and Song showed that every real hypersurface in a non-flat complex space form \(\tilde{M}_n(c)\) with pseudo-parallel Ricci operator with \(n > 2\) is a Hopf hypersurface.

In this section we refine and extend some results of [18], specifically the classification of real hypersurfaces with pseudo-parallel Ricci operator.

In [18], the authors claimed that such a hypersurface is locally holomorphically congruent to one of the following hypersurfaces:

1. If \(c > 0\), then \(M\) is a minimal hypersurface of type \((A_2)\) in \(\mathbb{CP}_n(c)\).

2. If \(c < 0\), then \(M\) is a horosphere in \(\mathbb{CH}_n(c)\).

Unfortunately, there are two classes of real hypersurfaces with pseudo-parallel Ricci dropped in their classification. In this section we give a correct classification table for real hypersurfaces with pseudo-parallel Ricci operator.

The following two results were obtained in [18].

**Proposition 4.1.** Let \(M\) be a real hypersurface in a non-flat complex space form \(\tilde{M}_n(c)\) with \(n > 2\). If \(M\) has pseudo-parallel Ricci tensor, then

1. \(M\) is a Hopf hypersurface, i.e., \(\xi\) is a principal curvature vector field with constant principal curvature \(\alpha\).

2. The structure vector field \(\xi\) satisfies \(S\xi = \sigma\xi\) with

\[
\sigma = \frac{c}{2}(n - 1) + (h - \alpha)\alpha.
\]
3. Let $\lambda$ be a principal curvature of $M$ with principal curvature vector field $X_{\lambda}$ orthogonal to $\xi$, then $\lambda \neq 0$ and

$$A\phi X_{\lambda} = \frac{L}{\lambda} \phi X_{\lambda}.$$ 

Moreover any vector field $Z$ orthogonal to $\xi$ satisfies

$$SZ = k_\lambda Z, \quad k_\lambda = \frac{c}{4}(2n + 1) + (h - \lambda)\lambda \neq \sigma.$$

4. The Ricci operator $S$ commutes with $\phi$, i.e., $\phi S = S \phi$.

5. The function $L$ is related to $\lambda$ by the formula

$$(4.1) \quad \frac{c}{4} - L + \alpha \lambda = 0.$$

**Lemma 4.2.** Let $M$ be a real hypersurface in a non-flat complex space form $\widetilde{M}_n(c)$ with $n > 2$. Denote by $g = g(x)$ the number of distinct principal curvatures at a point $x \in M$. If $M$ has pseudo-parallel Ricci tensor then

1. $g \leq 3$ on $M$.
2. If $g = 3$ then $\alpha = 0$.
3. The multiplicity $m_0$ of $\alpha$ is 1.

Now we study real hypersurfaces $M$ in $\widetilde{M}_n(c)$ with $n > 2$ and pseudo-parallel Ricci operator $S$.

**4.2 The case $g = 2$:**

First we study the case $M$ has distinct two principal curvatures. Proposition 4.1 implies that $M$ has two distinct principal curvatures $\alpha$ corresponding to $\xi$ and $\lambda \neq 0$. Hence $M$ is an $\eta$-umbilical real hypersurface.

**Proposition 4.3.** A real hypersurface $M \subset \widetilde{M}_n(c)$ with $n > 2$ and $c \neq 0$ which has pseudo-parallel Ricci operator and distinct two principal curvatures is $\eta$-umbilical.

Here we recall the classification of $\eta$-umbilical real hypersurfaces.

**Proposition 4.4 ([38]).** The only $\eta$-umbilical real hypersurfaces in $\mathbb{C}P_n(c)$ ($n \geq 2$) are geodesic spheres.

**Proposition 4.5 ([33], [34]).** The only $\eta$-umbilical real hypersurfaces in $\mathbb{C}H_n(c)$ ($n \geq 2$) are horospheres, geodesic spheres and tubes over complex hyperbolic hyperplane.

Take a principal vector field $X$ corresponding to $\lambda$. Then the principal curvature corresponding to $\phi X$ is $L/\lambda$ by Proposition 4.1. Since $\phi X$ is orthogonal to $\xi$, $\phi X$
is a principal curvature vector field corresponding to \( \lambda \). Thus we have \( L/\lambda = \lambda \), equivalently \( L = \lambda^2 > 0 \). Inserting this into (4.1), we have

\[
\lambda^2 - \alpha \lambda - c/4 = 0.
\]

Since \( \lambda \) is real, this quadratic equation has nonnegative discriminant, i.e.,

\[
\alpha^2 + c \geq 0.
\]

The principal curvature \( \lambda \) is given by

\[
\lambda = \frac{\alpha \pm \sqrt{\alpha^2 + c}}{2}.
\]

**Proposition 4.6.** Let \( M \) be a real hypersurface with pseudo-parallel Ricci operator in \( \tilde{M}_n(c) \) with \( c \neq 0 \) and \( n > 2 \). If \( g = 2 \), then \( M \) has constant principal curvatures \( \alpha \) corresponding to \( \xi \) and \( \lambda \). The latter principal curvature is a solution to \( \lambda^2 - \alpha \lambda - c/4 = 0 \).

**Remark 4.1 (cf. [8], [32]).** Real hypersurfaces in non-flat complex space form \( \tilde{M}_n(c) \) \( (n > 2) \) which satisfy \( g \leq 2 \) at each point are locally holomorphically congruent to one of the following hypersurfaces:

- (A\(_1\)) a geodesic hypersphere in \( \mathbb{C}P_n(c) \),
- (A\(_1\)) a geodesic sphere in \( \mathbb{C}H_n(c) \),
- (A\(_1\)) a tube over a complex hyperbolic hyperplane, or
- (A\(_0\)) a horosphere.

Now we investigate Hopf hypersurfaces with two distinct constant principal curvatures in detail.

**Example 4.2 (Geodesic spheres in \( \mathbb{C}P_n(c) \)).** Let \( M \) be a geodesic sphere of radius \( r \in (0, \pi/\sqrt{c}) \) in \( \mathbb{C}P_n(c) \). Put \( r = 2u/\sqrt{c} \), then the principal curvatures of \( M \) are represented as

\[
\alpha = \sqrt{c} \cot(2u), \quad \lambda = \frac{\sqrt{c}}{2} \cot u, \quad 0 < u < \pi/2
\]

with multiplicities \( m_0 = 1 \) and \( m_1 = 2n - 2 \), respectively. The eigenspace \( V_x(\lambda) \) of \( A \) at \( x \) is invariant under \( \phi_x \). One can see that

\[
\lambda = \frac{\alpha + \sqrt{\alpha^2 + c}}{2}.
\]

Thus \( \lambda \) is a solution to (4.2).

**Example 4.3 (Horospheres in \( \mathbb{C}H_n(c) \)).** Let \( M \) be a horosphere in complex hyperbolic space \( \mathbb{C}H_n(c) \) with \( c < 0 \). Then the principal curvatures of \( M \) are given by

\[
\alpha = \sqrt{-c}, \quad \lambda = \frac{\sqrt{-c}}{2} = \frac{\alpha}{2}.
\]

Their multiplicities are \( m_0 = 1 \) and \( m_1 = 2n - 2 \), respectively. One can easily check that \( \alpha^2 + c = 0 \) and \( \lambda \) satisfies the quadratic equation (4.2).
Example 4.4 (Geodesic spheres in $\mathbb{C}H_n(c)$). Let $M$ be a geodesic sphere of radius $r$ in complex hyperbolic $n$-space $\mathbb{C}H_n(c)$ with $c < 0$. Put $r = 2u/\sqrt{-c}$, then the principal curvatures of $M$ are given by

$$\alpha = \sqrt{-c}\coth(2u) = \sqrt{-c}\coth(r\sqrt{-c}), \quad \lambda = \frac{\sqrt{-c}}{2}\coth(r\sqrt{-c}/2).$$

Their multiplicities are $m_0 = 1$ and $m_1 = 2n - 2$, respectively. One can see that

$$\lambda = \frac{\alpha + \sqrt{\alpha^2 + c}}{2}.$$

Thus $\lambda$ is a solution to (4.2).

Example 4.5 (Tubes over complex hyperbolic hyperplane). Let $M$ be a tube of radius $r = 2u/\sqrt{-c}$ over the hyperplane $\mathbb{C}H_{n-1}$ in $\mathbb{C}H_n(c)$. Then the principal curvatures of $M$ are

$$\alpha = \sqrt{-c}\coth(2u) = \sqrt{-c}\coth(r\sqrt{-c}), \quad \lambda = \frac{\sqrt{-c}}{2}\tanh u = \frac{\sqrt{-c}}{2}\tanh(r\sqrt{-c}/2)$$

with multiplicities $m_0 = 1$ and $m_1 = 2n - 2$, respectively. We can see that $\lambda$ satisfies

$$\lambda = \frac{\alpha - \sqrt{\alpha^2 + c}}{2}.$$

Example 4.6 (Tubes over $\mathbb{R}H_n$). Let $M$ be a tube of radius $r$ over $\mathbb{R}H_n$. Then $M$ has three principal curvatures

$$\alpha = \sqrt{-c}\tanh(2u), \quad \lambda_1 = \frac{\sqrt{-c}}{2}\coth u, \quad \lambda_2 = \frac{\sqrt{-c}}{2}\tanh u,$$

where $u = r\sqrt{-c}/2$. The multiplicities of these principal curvatures are

$$m_0 = 1, \quad m_1 = n - 1, \quad m_2 = n - 1.$$

In case $r = \log(2 + \sqrt{3})/\sqrt{-c}$, $M$ has two distinct constant principal curvatures

$$\alpha = \lambda_1 = \frac{\sqrt{-3c}}{2}, \quad \lambda_2 = \frac{\sqrt{-c}}{2\sqrt{3}}.$$  

These two principal curvatures have multiplicities $n$ and $n - 1$, respectively. Moreover $\lambda_2$ satisfies

$$\lambda_2^2 - \alpha\lambda_2 - c/4 \neq 0.$$

Hence the tube of radius $\log(2 + \sqrt{3})/\sqrt{-c}$ does not satisfy $\lambda_2^2 - \alpha\lambda_2 - c/4 = 0$. Thus the tube $M$ of radius $\log(2 + \sqrt{3})/\sqrt{-c}$ has the properties:

- $M$ has two distinct principal curvatures, but
- $M$ is not $\eta$-umbilical and
- the Ricci operator $S$ of $M$ is not pseudo-parallel.
Now we arrive at the following classification result.

**Theorem 4.7.** Real hypersurfaces with pseudo-parallel Ricci operator in non-flat complex space form $\tilde{M}_n(c)$ with $n > 2$ and two distinct principal curvatures are locally holomorphically congruent to

(A) a horosphere in $\mathbb{CH}_n(c)$,

(A1) a geodesic sphere in $\mathbb{CP}_n(c)$, a geodesic sphere $\mathbb{CH}_n(c)$ or a tube over a totally geodesic complex hyperbolic hyperplane.

4.3 The case $g = 3$:

Next we study real hypersurfaces with pseudo-parallel Ricci operator and 3 distinct principal curvatures.

Let $M$ be a real hypersurface with pseudo-parallel Ricci operator $S$ and $g = 3$. Then by Proposition 4.1, $M$ is a Hopf hypersurface with $A\xi = 0$. The formula (4.1) implies that $L = c/4$.

Here we recall the following useful fact due to Bönig (see also Kimura [20] and Wang [40], Berndt[3], [4]).

**Proposition 4.8 ([5]).** Let $M \subset \tilde{M}_n(c)$ be a Hopf hypersurface with $c \neq 0$ and $n > 2$. Assume that $\alpha^2 + c \neq 0$ and $g = 3$ on $M$. Then $M$ is locally holomorphically congruent to

(A) a tube of radius $r$ over a totally geodesic $\mathbb{CP}_\ell (1 \leq \ell \leq n - 2)$, where $0 < r < \frac{\pi}{\sqrt{c}}$,

(A) a tube over a totally geodesic $\mathbb{CH}_\ell (1 \leq \ell \leq n - 2)$,

(B) a tube of radius $r$ over a complex quadric $Q^{n-1}$ and totally geodesic and Lagrangian imbedded real projective space $\mathbb{RP}_n$, where $0 < r < \frac{\pi}{2\sqrt{c}}$, or

(B) a tube over a totally geodesic and Lagrangian imbedded real hyperbolic space $\mathbb{RH}_n \subset \mathbb{CH}_n(c)$ of radius $r \neq \log(2 + \sqrt{3})/\sqrt{-c}$.

By virtue of Bönig’s result, $M$ is either of type $A_2$ or type $B$ in $\mathbb{CP}_n(c)$ or $\mathbb{CH}_n(c)$. This claim can be deduced alternatively as follows. Since $M$ is Hopf and satisfies $S\phi = \phi S$ and $g = 3$, by Theorem 3.6 and Theorem 3.7, $M$ is of type $A_2$ or type $B$ in $\mathbb{CP}_n(c)$ or $\mathbb{CH}_n(c)$. However $\alpha \neq 0$ for type $A_2$ and type $B$ hypersurfaces in $\mathbb{CH}_n(c)$.

Now we check those Hopf hypersurfaces in detail.

Let $M \subset \tilde{M}_n(c)$ be a real hypersurface with pseudo-parallel Ricci operator and $g = 3$. Then we have $A\xi = 0$. Denote by $\lambda$ and $\mu$, the nonzero principal curvatures ($\lambda \neq \mu$). Take principal vector fields $X$ and $Y$ so that

$$AX = \lambda X, \quad AY = \mu Y.$$  

Then from Proposition 4.1,

$$A\phi X = \frac{L}{\lambda} \phi X, \quad A\phi Y = \frac{L}{\mu} \phi Y.$$
On the other hand, since $M$ is of type (A$_2$) or (B), both the eigenspaces of $\lambda$ and $\mu$ are invariant under $\phi$. Hence we get

$$\frac{L}{\lambda} = \lambda, \quad \frac{L}{\mu} = \mu.$$  

Namely, $L = \lambda^2 = \mu^2 > 0$. Hence we get $\mu = \pm \lambda$. Next from (4.1), we have $L = c/4$, so

$$\frac{c}{4} = L = \lambda^2 = \mu^2.$$  

Hence $c > 0$. Next, from Proposition 4.1 again, we have

$$S X = k_\lambda X, \quad S Y = k_\mu Y,$$

where

$$k_\lambda = \frac{c}{4}(2n+1) + (h-\lambda)\lambda, \quad k_\mu = \frac{c}{4}(2n+1) + (h-\mu)\mu.$$  

and $k_\lambda = k_\mu$. The equation $k_\lambda = k_\mu$ implies that $h\lambda = h\mu = -h\lambda$. Thus $M$ satisfies $h = 0$, that is, $M$ is a minimal hypersurface.

Now we check this condition for type A$_2$-hypersurfaces and type B-hypersurfaces.

**Example 4.7 (Type B in $\mathbb{C}P_n(c)$).** Let $M$ be a tube of radius $r = 2u/\sqrt{c}$ over a complex quadric or a totally geodesic Lagrangian imbedded $\mathbb{R}P_n$ $(0 < u < \pi/4)$ in $\mathbb{C}P_n(c)$. In this case the principal curvatures are given by

$$\lambda = \lambda_1 = -\frac{\sqrt{c}}{2} \cot u, \quad \mu = \lambda_2 = \frac{\sqrt{c}}{2} \tan u, \quad \alpha = \sqrt{c} \tan(2u)$$

with multiplicities $n-1$, $n-1$ and 1, respectively. It is clear that type B hypersurfaces do not satisfy $\alpha = 0$.

**Example 4.8 (Type A$_2$ in $\mathbb{C}P_n(c)$).** Let $M$ be a tube of radius $r = 2u/\sqrt{c}$ over totally geodesic subspace $\mathbb{C}P_p$ $(1 < p < n-2, 0 < u < \pi/2)$. In this case the principal curvatures are given by

$$\lambda = \lambda_1 = -\frac{\sqrt{c}}{2} \tan u, \quad \mu = \lambda_2 = \frac{\sqrt{c}}{2} \cot u, \quad \alpha = \sqrt{c} \cot(2u)$$

with multiplicities $2p$, $2q$ and 1, respectively. Here $q$ is a positive integer such that $p + q = n - 1$.

The principal curvature $\alpha$ vanishes identically if and only if $u = \pi/4$. In this case, the principal curvatures of the tube are

$$\lambda = \lambda_1 = -\frac{\sqrt{c}}{2}, \quad \mu = \lambda_2 = \frac{\sqrt{c}}{2}, \quad \alpha = 0$$

with multiplicities $2p$, $2q$ and 1. Then $h$ is computed as

$$(4.3) \quad h = 2p\lambda_1 + 2q\lambda_2 = \frac{\sqrt{c}}{2}(-2p + 2q) = -(p - q)\sqrt{c}.$$
Now let us look for type A₂ real hypersurfaces with pseudo-parallel Ricci operator in $\mathbb{C}P_n(c)$.

As we saw before, a type A₂ real hypersurface $M$ with pseudo-parallel Ricci operator is minimal and $A\xi = 0$. The equation (4.3) shows that a type A₂ real hypersurface $M$ with $A\xi = 0$ is minimal if and only if $p = q$. Hence $p = (n - 1)/2$. This implies that the multiplicities of the principal curvatures are given by $2p = n - 1$, $2q = n - 1$ and 1 for $\lambda_1$, $\lambda_2$ and $\alpha = 0$, respectively.

From Theorem 3.8, we get the following fact.

**Proposition 4.9.** Let $M$ be a tube of radius $r = 2u/\sqrt{c}$ over totally geodesic subspace $C\mathbb{P}_p(1 < p < n - 2$, $0 < u < \pi/2)$. Then the following three properties are mutually equivalent.

1. $M$ is pseudo-Einstein.
2. $\cot^2 u = p/q$.
3. $h = \text{tr } A = \alpha$.

In such a case, principal curvatures are given by

$$
\lambda_1 = -\frac{\sqrt{c}}{2} \sqrt{\frac{q}{p}}, \quad \lambda_2 = \frac{\sqrt{c}}{2} \sqrt{\frac{p}{q}}, \quad \alpha = \frac{\sqrt{c}(p - q)}{2\sqrt{pq}}.
$$

One can see that type A₂ real hypersurfaces with pseudo-parallel Ricci operator satisfies $\cot^2 u = p/q = 1$ and hence pseudo-Einstein.

**Theorem 4.10.** Let $M \subset \tilde{M}_n(c)$ be a real hypersurface in a complex space form of nonzero constant holomorphic sectional curvature $c$ ($n > 2$) with distinct 3 principal curvatures. Then $M$ has pseudo-parallel Ricci operator if and only if it is locally holomorphically congruent to a tube of radius $r = \frac{\pi}{2\sqrt{c}}$ over a totally geodesic $C\mathbb{P}_{(n-1)/2}$. In this case $M$ is minimal and pseudo-Einstein. The principal curvatures of $M$ are given by

$$
\lambda_1 = -\frac{\sqrt{c}}{2}, \quad \lambda_2 = \frac{\sqrt{c}}{2}, \quad \alpha = 0
$$

with multiplicities $n - 1$, $n - 1$, 1, respectively.

# 5 Three dimensional real hypersurfaces

Recently, pseudo-Einstein real hypersurfaces in $\mathbb{C}P_2(c)$ and $\mathbb{C}H_2(c)$ are classified by T. Ivey, H. S. Kim and Ryan (This gives a complete answer to [34, Question 9.5] posed by Niebergall and Ryan). In particular it is shown that every pseudo-Einstein real hypersurface is a Hopf hypersurface.

**Theorem 5.1 ([14], [17]).** The pseudo-Einstein real hypersurfaces in $\mathbb{C}P_2(c)$ and $\mathbb{C}H_2(c)$ are locally holomorphically congruent to one of the following hypersurfaces:

- a geodesic hypersphere in $\mathbb{C}P_2(c)$ or $\mathbb{C}H_2(c)$,
- a horosphere in $\mathbb{C}H_2(c)$,
• a tube of totally geodesic $CH_1 \subset CH_2(c)$,

• a non-homogeneous tube of a certain holomorphic curve in $CP_2(c)$ of radius $\pi/\sqrt{4c}$ or

• a Hopf hypersurface in $CH_2(c)$ with $A\xi = 0$ which are constructed by a pair of Legendre curves in the unit 3-sphere.

Clearly every 3-dimensional pseudo-Einstein real hypersurface is pseudo-symmetric (see Proposition 2.3). In our previous paper [9], we obtained the following classification.

**Theorem 5.2.** The Hopf hypersurfaces in $CP_2(c)$ or $CH_2(c)$ with pseudo-parallel Ricci operator are locally holomorphically congruent to a horosphere in $CH_2(c)$, a geodesic sphere in $CP_2(c)$ or $CH_2(c)$, a homogeneous tube over $CH_1(c)$ in $CH_2$, a non-homogeneous real hypersurface which is realized as a tube over a certain holomorphic curve in $CP_2(c)$ with radius $\pi/\sqrt{4c}$, or a Hopf hypersurface in $CH_2(c)$ with $A\xi = 0$.

A real hypersurface $M \subset \tilde{M}_n(c)$ of a non-flat complex space form is said to be a ruled real hypersurface if its holomorphic distribution

$$x \mapsto T_x^2 M = \{ X \in T_x M \mid \eta_x(X) = 0 \}$$

is integrable and each of its maximal integral manifold is locally holomorphically congruent to a totally geodesic complex hypersurface in $\tilde{M}_n(c)$.

Every ruled real hypersurface can be constructed in the following way: Take a regular curve $\gamma$ in $\tilde{M}_n(c)$ with tangent vector field $\gamma'$ at each point of $\gamma$, there exits a unique complex hyperplane of $\tilde{M}_n(c)$ cutting $\gamma$ so as to be orthogonal not only $\gamma'$ but also to $J\gamma'$. The union of these hyperplanes is a ruled hypersurface (away from singularities). Ruled hypersurfaces can be characterised as follows (see [30]):

**Proposition 5.3.** Let $M \subset \tilde{M}_n(c)$ be a real hypersurface ($n \geq 2, c \neq 0$). Then the following conditions are mutually equivalent:

1. $M$ is a ruled real hypersurface.

2. The shape operator $A$ satisfies $g(AV,W) = 0$ for all vector fields $V$ and $W$ orthogonal to $\xi$.

3. If we define functions $\mu$ and $\nu$ by

$$\mu = g(A\xi, \xi), \quad \nu = |A\xi - \mu\xi|.$$  

Then these functions are related to $A$ by

(a) The set $M^o = \{ x \in M \mid \nu(x) \neq 0 \}$ is an open dense subset of $M$

(b) There exits a unit vector field $U$ on $M^o$ orthogonal to $\xi$ such that

$$A\xi = \mu\xi + \nu U, \quad AU = \nu\xi, \quad AX = 0$$

for all $X$ orthogonal to both $\xi$ and $U$. 
Lemma 5.4 ([21], [31]). Let $M \subset \mathbb{C}H_n(c)$ be a ruled real hypersurface in a complex hyperbolic $n$-space of constant holomorphic sectional curvature $c < 0$ and $n \geq 2$. Then the integral curves of $\phi U$ are geodesics in $M$. On a trajectory $\delta(s)$ of $\phi U$, the function $\nu$ is given by

$$
(5.2) \quad \nu(\delta(s)) = -\frac{\sqrt{-c}}{2} \tanh\left(\frac{\sqrt{-c}(s + a)}{2}\right), \text{ or } \nu(\delta(s)) = \frac{\sqrt{-c}}{2}
$$

for some constant $a$.

Now let us study ruled hypersurfaces in non-flat complex space form $\widetilde{M}_2(c)$ with pseudo-parallel Ricci operator. In our previous paper [9], the following result was obtained.

Proposition 5.5. 1. A ruled real hypersurface $M$ in $\mathbb{C}P_2(c)$ does not have pseudo-parallel Ricci operator.

2. A ruled real hypersurface $M$ in $\mathbb{C}H_2(c)$ has pseudo-parallel Ricci operator if and only if $0 < \nu^2 = -3c/4$.

Now let $M \subset \mathbb{C}H_2(c)$ be a ruled real hypersurface. Then $M$ has pseudo-parallel Ricci operator if and only if $0 < \nu^2 = -3c/4$. On the other hand, by Lemma 5.4, the only possibility for $\nu$ be a constant is $\nu = \sqrt{-c}/2$. This contradicts to $\nu^2 = -3c/4$. Thus the Ricci operator of $M$ can not be pseudo-parallel.

Theorem 5.6. Ruled real hypersurfaces in $\mathbb{C}H_2(c)$ can not have pseudo-parallel Ricci operator.

Corollary 5.7. Ruled real hypersurfaces in non-flat complex space form $\widetilde{M}_2(c)$ can not have pseudo-parallel Ricci operator.

Conjecture. Real hypersurfaces in a non-flat complex space form $\widetilde{M}_2(c)$ with pseudo-parallel Ricci operator are Hopf.

6 Concluding remarks

6.1

Every real hypersurface $M \subset \widetilde{M}_n(c)$ satisfies

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi$$

for all vector fields $X$ and $Y$ on $M$. It is easy to check the nonexistence of real hypersurfaces with parallel $\phi$ in non-flat complex space forms.

Problem 6.1. Classify real hypersurfaces in a non-flat complex space form $\widetilde{M}_n(c)$ with pseudo-parallel $\phi$. 
6.2
A real hypersurface in $\tilde{M}_n(c)$ is said to be \textit{pseudo-parallel} if its shape operator is pseudo-parallel. Pseudo-parallel real hypersurfaces in non-flat complex space form $\tilde{M}_n(c)$ are classified by G. A. Lobos and M. Ortega [28] for $n \geq 2$. Note that the case $n = 2$ is independently done in [9].

\textbf{Theorem 6.1 ([28], [9])}. A real hypersurface $M \subset \tilde{M}_n(c)$ with $c \neq 0$ and $n \geq 2$ is pseudo-parallel if and only if it is locally holomorphically congruent to either

(A1) \textit{a geodesic sphere of radius } $r \in (0, \pi/\sqrt{c})$ \textit{in } $\mathbb{C}P_n(c)$,

(A0) \textit{a horosphere in } $\mathbb{C}H_n(c)$,

(A1) \textit{a geodesic sphere } $\mathbb{C}H_n(c)$, or

(A1) \textit{a tube of a complex hyperbolic hyperplane } $\mathbb{C}H_{n-1} \subset \mathbb{C}H_n(c)$.

This result implies that a real hypersurface is pseudo-parallel if and only if it is $\eta$-umbilical.

6.3
On real hypersurfaces in complex space forms, several generalisations of parallelism are known.

\textbf{Definition 6.2}. An endomorphism field $P$ on a real hypersurface is said to be

- $\eta$-parallel if $g((\nabla_X P)Y, Z) = 0$ for all vector fields $X$, $Y$ and $Z$ orthogonal to $\xi$,

- semi $\eta$-parallel if $g((R(X, Y)P)Z, W) = 0$ for all vector fields $X$, $Y$, $Z$ and $W$ orthogonal to $\xi$,

- pseudo $\eta$-parallel if there exists a function $L$ such that

$$g((R(X, Y)P)Z, W) = Lg(((X \wedge Y)P)Z, W)$$

for all vector fields $X$, $Y$, $Z$ and $W$ orthogonal to $\xi$.

Real hypersurfaces with $\eta$-parallel shape operator are classified by Kimura-Maeda and Suh.

\textbf{Theorem 6.2 ([23], [37])}. Let $M \subset \tilde{M}_n(c)$, where $n \geq 2$ and $c \neq 0$, be a Hopf hypersurface. Then $M$ has $\eta$-parallel shape operator if and only if it is locally holomorphically congruent to a type A or type B hypersurface in $\mathbb{C}P_n(c)$ or $\mathbb{C}H_n(c)$.

Note that every ruled real hypersurface is $\eta$-parallel. For the shape operator $A$, the following problem naturally arises.

\textbf{Problem 6.3}. Classify real hypersurfaces with pseudo $\eta$-parallel shape operator in $\mathbb{C}P_n(c)$ and $\mathbb{C}H_n(c)$ with $n \geq 2$. 
Add in Proof:

1. On a real hypersurface $M \subset \tilde{M}_n(c)$, the structure Jacobi operator $\ell$ is defined by $\ell(X) = R(X, \xi)\xi$ for all vector field $X$ on $M$. In [35], K. Panagiotidou and P. J. Xenos classified real hypersurfaces in non-flat complex space forms with pseudo-parallel structure Jacobi operator.

2. Mayuko Kon classified real hypersurfaces in non-flat complex space forms with pseudo $\eta$-parallel Ricci operator. She proved that a real hypersurface $M \subset \tilde{M}_n(c)$ ($n > 2$, $c \neq 0$) has pseudo $\eta$-parallel Ricci operator if and only if it is pseudo-Einstein. As an application of the result, she also obtained Theorem 1 and Theorem 2 of the present paper. Her results imply that the class of real hypersurfaces with pseudo $\eta$-parallel Ricci operator is strictly larger than that of real hypersurfaces with pseudo-parallel Ricci operator.

In addition, Kon classified real hypersurfaces in non-flat complex space form $\tilde{M}_n(c)$ ($n > 2$) with pseudo $\eta$-parallel shape operator. As like in the case of Ricci operator, Kon showed that the class of real hypersurfaces in $\tilde{M}_n(c)$ ($n > 2$) with pseudo $\eta$-parallel shape operator is strictly larger than that of real hypersurfaces with pseudo-parallel shape operator [26].

Acknowledgements. The author would like to express his sincere thanks to professor Sadahiro Maeda for his constant encouragement and invaluable comments to this work. The author would also like to thank Dr. Mayuko Kon for useful discussion.

References


Real hypersurfaces in complex space forms


Author’s address:
Jun-ichi Inoguchi
Department of Mathematical Sciences,
Faculty of Science, Yamagata University,
Yamagata, 990-8560, Japan.
E-mail: inoguchi@sci.kj.yamagata-u.ac.jp