Real hypersurfaces in complex two-plane Grassmannians

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Abstract. We give a characterization of real hypersurfaces in complex two-plane Grassmannians for which the shape operator $A$ satisfies a commutative relation with structure tensors $\varphi$ and $\varphi_1$.


Key words: real hypersurfaces; complex two-plane Grassmannians; shape operator; tube.

1 Introduction

We denote by $G_2(\mathbb{C}^{m+2}), m \geq 3$ the set of all complex 2–dimensional linear subspaces of $\mathbb{C}^{m+2}$. It is well known that the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is a unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure $J$ and a quaternionic Kähler structure $J$ not containing $J$ (see Berndt [1], Berndt and Suh [2, 3]. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperKähler manifold.

If we consider a $(4m - 1)$-dimensional real hypersurface $M$ in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, then the Kähler structure $J$ of $G_2(\mathbb{C}^{m+2})$ induces on $M$ an almost contact metric structure $(\varphi, \xi, \eta, g)$. The structure vector $\xi$ is defined as $\xi = -JN$, where $N$ denotes a local unit normal vector field of $M$ in $G_2(\mathbb{C}^{m+2})$. Also the quaternionic Kähler structure $J$ induces an almost contact 3-structure $(\varphi_\nu, \xi_\nu, \eta_\nu, g), \nu = 1, 2, 3$ on $M$. The almost contact 3-structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_\nu = -J_\nu N, \nu = 1, 2, 3$, where $J_\nu$ denotes a canonical local basis $J = \{J_1, J_2, J_3\}$ of $J$.

The real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ for which $[\xi] = \text{Span} \{\xi\}$ and $D^\perp = \text{Span} \{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator $A$ of $M$ have been studied by Berndt and Suh in [2], where they proved the following

Theorem 1.1. Let $M$ be a connected real hypersurface in $G_2(\mathbb{C}^{m+2}), m \geq 3$. Then both $[\xi]$ and $D^\perp$ are invariant under the shape operator of $M$ if and only if:

(A) $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$; or

(B) $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $H^n$ in $G_2(\mathbb{C}^{m+2})$. 


On the other hand, in [3] the same authors consider the geometric condition where the shape operator $A$ of real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor $\varphi$, i.e., $A\varphi = \varphi A$. Namely they proved the following

**Theorem 1.2.** Let $M$ be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the shape operator $A$ of $M$ commutes with the structure tensor $\varphi$, namely $A\varphi = \varphi A$ if and only if $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Also in [7] Suh proved that there aren’t real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ with $A\varphi_\nu = \varphi_\nu A$, $\nu = 1, 2, 3$. For more results on real hypersurfaces in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ see [6, 8, 9, 10, 5, 4].

In this paper we consider real hypersurfaces $M$ in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ satisfying the condition $\varphi\varphi_1 A = A\varphi_1 \varphi$. Namely we prove the following

**Theorem 1.3.** Let $M$ be a connected real hypersurface in $G_2(\mathbb{C}^{m+2}), m \geq 3$. Then the shape operator $A$ satisfies $\varphi\varphi_1 A = A\varphi_1 \varphi$ if and only if $M$ is of type A, namely an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

## 2 Related formulas

For basic material about Riemannian Geometry of $G_2(\mathbb{C}^{m+2})$ see [1, 2, 3]. For the Kähler structure $J$ and the quaternionic Kähler structure $\mathcal{J}$ on $G_2(\mathbb{C}^{m+2})$ it is known that $JJ_1 = J_1 J$ and $JJ_1$ is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\text{tr}(JJ_1) = 0$, where $J_1$ is any almost Hermitian structure in $\mathcal{J}$.

A canonical local basis $J_1$, $J_2$, $J_3$ of $\mathcal{J}$ consists of three local almost Hermitian structures $J_\nu$ in $\mathcal{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = J_{\nu+1}J_\nu$, where the index $\nu$ is taken modulo 3. Since $\mathcal{J}$ is parallel with respect to the Riemannian connection $\nabla$ of the Riemannian manifold $(G_2(\mathbb{C}^{m+2}), g)$, for any canonical local basis $J_1$, $J_2$, $J_3$ of $\mathcal{J}$ there exist three one-forms $q_1, q_2, q_3$ such that $\nabla_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2}$, for all vector fields $X$ on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\hat{R}$ of $(G_2(\mathbb{C}^{m+2}), g)$ is locally given by

$$\hat{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX$$

$$-g(JX,Z)JY - 2g(JX,Y)JZ$$

$$+ \sum_{\nu=1}^{3} \{ g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z \}$$

$$+ \sum_{\nu=1}^{3} \{ g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY \},$$

where $J_1$, $J_2$, $J_3$ is any canonical local basis of $\mathcal{J}$.

## 3 Real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we describe some fundamental formulas which will be used in the proof of our main theorem. Let $M$ be a $(4m - 1)$-dimensional real hypersurface in
(G_2(\mathbb{C}^{m+2}), g), that is a hypersurface of (G_2(\mathbb{C}^{m+2}), g) with codimension one. Then the Kahler structure J of (G_2(\mathbb{C}^{m+2}), g) induces on M an almost contact metric structure \((\varphi, \xi, \eta, g)\). Furthermore each \(J_\nu\), where \(J_1, J_2, J_3\) is a canonical local basis of \(\mathcal{J}\), induces an almost contact metric structure \((\varphi_\nu, \xi_\nu, \eta_\nu, g)\) on M. We also denote by \(g\) the induced Riemannian metric on \(M\). The Riemannian connection on \(M\) in denoted by \(\nabla\). We denote by \(N\) a local unit normal vector field of \(M\) and by \(A\) the shape operator of \(M\) with respect to \(N\).

For any local vector field \(X\) on a neighborhood of a point \(p \in M\) and the unit normal vector \(N\) the transformation under the Kahler structure \(J\) of \(G_2(\mathbb{C}^{m+2})\) can be given by \(JX = \varphi X + \eta(X)N\) and \(JN = -\xi\), where \(\varphi\) denotes a skew-symmetric transformation of the tangent bundle \(TM\) of \(M\), while \(n\) and \(\xi\) denote a 1-form and a vector field on a neighborhood in \(M\), respectively. Therefore \(g(\xi, X) = \eta(X)\). The tensors \((\varphi, \xi, \eta, g)\) define an almost contact metric structure on \(M\) and they satisfy the following relations

\[ \varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1, \]

for any tangent vector field \(X\) on \(M\).

Also, if \(J_1, J_2, J_3\) is a canonical local basis of \(\mathcal{J}\), then each \(J_\nu\) induces an almost contact metric structure \((\varphi_\nu, \xi_\nu, \eta_\nu, g)\) on \(M\). Using the above expression 2.1 for the curvature tensor \(\bar{R}\), the Gauss and Codazzi equations are respectively given by

\[
\bar{R}(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X \\
- g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z \\
+ \sum_{i=1}^{3} \{g(\varphi_i Y, Z)\varphi_i X - g(\varphi_i X, Z)\varphi_i Y - 2g(\varphi_i X, Y)\varphi_i Z\} \\
+ \sum_{i=1}^{3} \{g(\varphi_i Y, Z)\varphi_i \varphi X - g(\varphi_i \varphi X, Z)\varphi_i \varphi Y\} \\
- \sum_{i=1}^{3} \{\eta(Y)\eta_i (Z)\varphi_i \varphi X - \eta(X)\eta_i (Z)\varphi_i \varphi Y\} \\
- \sum_{i=1}^{3} \{\eta(X)g(\varphi_i Y, Z) - \eta(Y)g(\varphi_i \varphi X, Z)\} \xi_i \\
+ g(AY, Z)AX - g(AX, Z)AY,
\]

and

\[
(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi \\
+ \sum_{i=1}^{3} \{\eta_i (X)\varphi_i Y - \eta_i (Y)\varphi_i X - 2g(\varphi_i X, Y)\xi_i\} \\
+ \sum_{i=1}^{3} \{\eta_i (\varphi X)\varphi_i Y - \eta_i (\varphi Y)\varphi_i X\} + \sum_{i=1}^{3} \{\eta(X)\eta_i (\varphi Y) - \eta(Y)\eta_i (\varphi X)\} \xi_i,
\]

where \(\bar{R}\) denotes the curvature tensor of the real hypersurface \(M\) in \(G_2(\mathbb{C}^{m+2})\). Also, the following identities can be proved in a straightforward method.

\[
\varphi_{\nu+1} \xi_\nu = -\xi_{\nu+2}, \quad \varphi_{\nu} \xi_{\nu+1} = \xi_{\nu+2}, \quad \varphi_\nu \xi_\nu = \varphi_\nu \xi, \quad \eta_\nu (\varphi X) = \eta(\varphi_\nu X),
\]

\[
\varphi_\nu \varphi_{\nu+1} X = \varphi_{\nu+2} X + \eta_{\nu+1} (X) \xi_\nu, \quad \varphi_{\nu+1} \varphi_\nu X = -\varphi_{\nu+2} X + \eta_\nu (X) \xi_{\nu+1},
\]
where the index $\nu$ is taken modulo 3. We analyze $J_\nu X = \varphi_\nu X + \eta_\nu(X)N$, for any tangent vector $X$ of the real hypersurface $M$ in $G_2(C^{m+2})$, where $N$ denotes a unit normal vector field of $M$ in $G_2(C^{m+2})$. The tensors $(\varphi_\nu, \xi_\nu, \eta_\nu, g)$ satisfy the following

$$\varphi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \varphi_\nu \xi_\nu = 0, \quad \eta_\nu(\varphi_\nu X) = 0, \quad \eta_\nu(\xi_\nu) = 1,$$

for any tangent vector field $X$ on $M$. As well, the following formulas hold true (\cite{10})

$$\nabla_X \varphi = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \varphi AX,$$

$$\nabla_X \xi_\nu = q_{\nu+1}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \varphi_\nu AX,$$

$$\nabla_X \varphi_\nu Y = -q_{\nu+1}(X)\varphi_{\nu+2} Y + q_{\nu+2}(X)\varphi_{\nu+1} Y + \eta_\nu(AX) - g(AX, Y)\xi_\nu.$$

Also, from $JJ_\nu = J_{\nu}J$, $\nu = 1, 2, 3$, it follows that

$$\varphi_\nu \varphi_\nu X = \varphi_\nu \varphi X + \eta_\nu(X)\xi - \eta_\nu(\xi)\xi_\nu.$$

We will prove our main theorem in all cases. Specifically, these are:

(I) $\xi \notin D^\perp$ and $\xi \notin D$, (II) $\xi \in D^\perp$, (III) $\xi \in D$.

Case I: $\xi \notin D^\perp$ and $\xi \notin D$

This case, will be studied through the following subcases:

(i) dim $H^\perp \oplus A(\langle \xi_2, \xi_3 \rangle) = 9$, where $H^\perp = \langle \xi \rangle \oplus D \oplus D^\perp$

(ii) dim $H^\perp \oplus A\xi_2 = 8$ and $A\xi_3 \in H^\perp \oplus A\xi_2$,

(iii) dim $H^\perp = 7$, $A(\langle \xi_2, \xi_3 \rangle) \subseteq H^\perp$.

(iv) The remainder cases of the dimension of $H^\perp$

(v) $\eta_1(\xi) \neq \pm 1$ and $\eta_1(\xi) \neq 0$

Case II: $\xi \in D^\perp$

Similarly, this case will be studied through the subcases:

(i) $\eta_1(\xi) \neq \pm 1$

(ii) $\eta_1(\xi) = 1$

Case III: $\xi \in D$

4 The case $\xi \notin D^\perp$ and $\xi \notin D$

Here we will prove that the case $\xi \notin D^\perp$ and $\xi \notin D$ can not occur.

4.1 Some calculations

From the relation

$$(4.1) \quad \phi_1 AX = A\phi_1 X, \quad X \in TM,$$

for $X = \xi$, we have $A\xi = \rho\xi + \mu\phi\xi_1$, with $\rho$, $\mu$ locally definite differential functions, $\rho = \eta_1(A\xi)$ and $\mu = g(A\xi, \phi\xi_1)$. Also we have

$$\phi_1 A\xi = \phi_1(\rho\xi + \mu\phi\xi_1) = \mu\phi\phi_1 \xi = \mu\phi(-\xi + \eta_1(\xi)\xi_1) = \mu\eta_1(\xi)\phi\xi_1.$$

Since $A\phi_1\phi_1 = 0$, from (4.1) we take $\mu\eta_1(\xi)\phi_1 = 0$, or $\mu = 0$. Thus $A\xi = \rho \xi_1$. For $X = \xi_1$ the relation (4.1) gives $\phi_1A\xi_1 = -A\xi + \eta_1(\xi)A\xi_1$. Applying $\phi$ to this equation we take

\[ (4.2) \quad (1 - (\eta_1(\xi))^2)A\xi_1 = (\eta_1(A\xi_1) - \mu\eta_1(\xi))\xi_1 - \eta_1(\phi A\xi_1)\phi_1 + (\rho - \eta_1(\xi)\eta_1(A\xi_1))\xi \]

For $X = \phi \xi_1$, the relation (4.1) gives

\[ (4.3) \quad (1 - (\eta_1(\xi))^2)A\phi_1 = \eta_1(A\phi_1)\xi_1 - \eta_1(\phi A\phi_1)\phi_1 - \eta_1(\xi)\eta_1(A\phi_1)\xi \]

Since $1 - (\eta_1(\xi))^2 \neq 0$ let

\[ \alpha_1 = \frac{\eta_1(A\xi_1) - \mu\eta_1(\xi)}{1 - (\eta_1(\xi))^2}, \quad \beta_1 = \frac{-\eta_1(\phi A\xi_1)}{1 - (\eta_1(\xi))^2}, \quad \gamma_1 = \frac{\rho - \eta_1(\xi)\eta_1(A\xi_1)}{1 - (\eta_1(\xi))^2}, \]
\[ \alpha_2 = \frac{\eta_1(A\phi_1)}{1 - (\eta_1(\xi))^2}, \quad \beta_2 = \frac{-\eta_1(\phi A\phi_1)}{1 - (\eta_1(\xi))^2}, \quad \gamma_2 = \frac{-\eta_1(\xi)\eta_1(A\phi_1)}{1 - (\eta_1(\xi))^2}. \]

Then the equations (4.2) and (4.3) become

\[ (4.4) \quad A\xi_1 = \alpha_1 \xi_1 + \beta_1 \phi \xi_1 + \gamma_1 \xi, \quad A\phi_1 = \alpha_2 \xi_1 + \beta_2 \phi \xi_1 + \gamma_2 \xi. \]

Thus we obtain $\gamma_1 = \rho - \eta_1(\xi)\alpha_1$, $\beta_1 = \alpha_2$ and $\gamma_2 = -\eta_1(\xi)\beta_1$. Taking into account the above calculations we finally have the next equations

\[ (5.1) \quad A\xi = \rho \xi_1 \]
\[ A_\xi = \alpha_1 \xi_1 + \beta_1 \phi \xi_1 + (\rho - \eta_1(\xi)\alpha_1)\xi \]
\[ A\phi_1 = \beta_1 \xi_1 + \beta_2 \phi \xi_1 - \eta_1(\xi)\beta_1 \xi. \]

The Codazzi equation, for $Y = \xi$, takes the form

\[ (\nabla_X A)\xi - (\nabla_\xi A)X = \eta(X)\phi \xi - \eta(\xi)\phi_X - 2g(\phi_X, \xi) \]
\[ + \sum_{\nu=1}^{3} \{\eta_\nu(X)\phi_\nu \xi - \eta_\nu(\xi)\phi_\nu X - 2g(\phi_\nu X, \xi)\xi_\nu \} \]
\[ + \sum_{\nu=1}^{3} \{\eta_\nu(\phi X)\phi_\nu \phi \xi - \eta_\nu(\phi)\phi_\nu \phi X \} \]
\[ + \sum_{\nu=1}^{3} \{\eta(X)\eta_\nu(\phi) - \eta(\xi)\eta_\nu(\phi X)\}\xi_\nu \]

from which we obtain

\[ (5.2) \quad (X\rho)\xi_1 + \rho(q_3(\xi)\xi_2 - q_2(\xi)\xi_3 + \phi_1 A\xi) - A\phi AX - (\nabla_\xi X) = \]
\[ -\phi X + \sum_{\nu=1}^{3} \eta_\nu(X)\phi_\nu \xi - \sum_{\nu=1}^{3} \eta_\nu(\xi)\phi_\nu X - 3 \sum_{\nu=1}^{3} \eta_\nu(\phi_\nu X)\xi_\nu. \]

For $X = \xi_1$, the equation (4.6) gives

\[ (5.3) \quad (\xi_1\rho - \rho\beta_1\eta_1(\xi) - \xi\alpha_1)\xi_1 + (\rho q_3(\xi)\alpha_1 q_3(\xi))\xi_2 \]
\[ + (\alpha_1 q_2(\xi) - \rho q_2(\xi_1))\xi_3 + (-\alpha_1 \beta_1 + \beta_1^1 - \xi \beta_1)\phi \xi_1 \]
\[ + (\beta_1 \rho - \xi(\eta_1(\xi)\alpha_1))\xi - \beta_1 q_3(\xi)\phi_2 + \beta_1 q_2(\xi)\phi_3 \]
\[ + q_3(\xi)A\phi_2 + q_2(\xi)A\phi_3 = 2\eta_1(\xi)\xi_2 - 2\eta_2(\xi)\xi_3. \]
We denote by $\mathcal{H}^\perp = \langle \xi \rangle \oplus \mathcal{D} \oplus \phi\mathcal{D}^\perp$ and by $\mathcal{H}$ the orthogonal complement of $\mathcal{H}^\perp$, namely $\mathcal{H}^\perp \oplus \mathcal{H} = TM$. We notice that for our case the vectors $\xi, \xi_1, \xi_2, \xi_3, \phi \xi_1, \phi \xi_2$ and $\phi \xi_3$ which determine the space $\mathcal{H}^\perp$, are linearly independent. Indeed, let

$$\lambda \xi + \sum_{\nu=1}^{3} (\lambda_i \xi_i + \lambda_{i+3} \phi \xi_i) = 0$$

If we take the inner product of this successively by $\xi, \xi_1, \xi_2, \xi_3, \phi \xi_1, \phi \xi_2$ and $\phi \xi_3$ we obtain a homogeneous system with determinant

$$\begin{vmatrix}
1 & \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 \\
\alpha_1 & 1 & 0 & 0 & 0 & \alpha_3 & -\alpha_2 \\
\alpha_2 & 0 & 1 & 0 & -\alpha_3 & 0 & \alpha_1 \\
\alpha_3 & 0 & 0 & 1 & \alpha_2 & -\alpha_1 & 0 \\
0 & -\alpha_3 & \alpha_2 & 1 - \alpha_1^2 & -\alpha_1 \alpha_2 & -\alpha_1 \alpha_3 & 0 \\
0 & \alpha_3 & 0 & -\alpha_1 & -\alpha_1 \alpha_2 & 1 - \alpha_2^2 & -\alpha_2 \alpha_3 \\
0 & -\alpha_2 & \alpha_1 & 0 & -\alpha_1 \alpha_3 & -\alpha_2 \alpha_3 & 1 - \alpha_3^2 \\
\end{vmatrix} = (1 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2)^4,$$

where $\alpha_i = \eta_i(\xi), i = 1, 2, 3$. But $1 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2 = 1 - |\text{proj}_{\mathcal{D}^\perp}|^2 \neq 0$, where $\text{proj}_{\mathcal{D}^\perp}$ is the projection of $\xi$ on $\mathcal{D}^\perp$.

We shall further study some special cases on the dimension of $\mathcal{H}^\perp \oplus A(\langle \xi, \xi_3 \rangle)$.

1. $\dim \mathcal{H}^\perp \oplus A(\langle \xi_2, \xi_3 \rangle) = 9$
2. $\dim \mathcal{H}^\perp \oplus A(\langle \xi_3 \rangle) = 8$ and $A \xi_3 \not\subset \mathcal{H}^\perp \oplus A(\langle \xi_2 \rangle)$
3. $\dim \mathcal{H}^\perp = 7$ and $A \xi_2, A \xi_3 \not\subset \mathcal{H}^\perp$

4.2 $\dim \mathcal{H}^\perp \oplus A(\langle \xi_2, \xi_3 \rangle) = 9$

In this case, the vector fields $\{\xi, \xi_1, \xi_2, \xi_3, \phi \xi_1, \phi \xi_2, \phi \xi_3, A \xi_2, A \xi_3\}$ are linearly independent, at each point $p \in M$. Thus through the relation (4.7) it occurs that

$$\eta_1(\xi)(\xi_1) = \beta_1 + (1 - (\eta_1(\xi))^2) - \xi_1 = 0, \quad \rho q_3(\xi_1) = 2 \eta_3(\xi)$$

$$\rho q_2(\xi_1) = 2 \eta_2(\xi), \quad \beta_1(1 - \beta_1^2 + \alpha_1 \beta_2 = 0$$

$$\beta_1 \rho + \xi_1 + \eta_1(\xi) \xi_1 = 0, \quad q_2(\xi) = 0, \quad q_3(\xi) = 0.$$

Hence

$$\nabla_\xi \xi_1 = 0, \quad \xi \eta_1(\xi) = 0, \quad \nabla_\xi \phi \xi_1 = \rho \eta_1(\xi) \xi_1 - \rho \xi.$$

For $X = \phi \xi_1$ the equation (4.6) becomes

$$(\phi^3 \rho) \xi_1 + \rho (\eta_3(\phi \xi_1) \xi_2 - q_2(\phi \xi_1) \xi_3 + \phi_1 A \phi \xi_1) - A \phi A \phi \xi_1 - (\nabla_\xi A) \phi \xi_1 =$$

$$-\phi^2 \xi + \sum_{\nu=1}^{3} (\eta_\nu(\phi \xi_1) \phi \xi_1 - \sum_{\nu=1}^{3} (\eta_\nu(\phi \xi_1) \phi \xi_1 - 3 \sum_{\nu=1}^{3} (\eta_\nu(\phi \xi_1) \phi \xi_1).$$

Then by direct calculations we assert the following
Under our assumptions, the vector field $\xi$ cannot occur.

Now we have the following

**Lemma 4.1.** Under our assumptions, the vector field $\xi$ can be decomposed as

$$\xi = \eta(e) e + \eta_1(\xi) \xi_1,$$

where $\eta(e) e \in H$ is the projection of $\xi$ on the distribution $H$.

**Proof.** Let $\xi = \kappa_1 \xi_1 + \kappa_2 \xi_2 + \kappa_3 \xi_3 + \kappa_4 \phi \xi_1 + \kappa_5 \phi \xi_2 + \kappa_6 \phi \xi_3 + \kappa_7 e$, where $\eta(e) e \in H$ is the projection of $\xi$ on $H$ and $\kappa_i$, $i = 1, \cdots, 7$ locally definite differential functions. If we take all the inner products we obtain $\kappa_1 = \eta(e)$, $\kappa_7 = \eta(e) e$ and $\kappa_i = 0$, $i = 2, \cdots, 6$, which completes the proof.

Summing up the above we have the following

**Proposition 4.2.** There are not real hypersurfaces $M^{4m-1}$ of $G_2(C^{m+2})$ satisfying $\phi \phi_1 A = A \phi_1 \phi$ with $\xi \notin D^\perp$, $\xi \notin D$ and dim $H^\perp \oplus A(<\xi_2, \xi_3>) = 9$.

**Proof.** The covariant derivative of (4.12) with respect to $\xi$ gives

$$\phi A \xi = \nabla_\xi \xi = (\xi \eta_1(\xi)) \xi_1 + \eta_1(\xi) \nabla_\xi \xi_1 + (\xi \eta(e)) e + \eta(e) \nabla_\xi e.$$

Combining (4.9) and the above relation we obtain $\rho \phi \xi_1 = (\xi \eta(e)) e + \eta(e) \nabla_\xi e$. From the inner product of this with $\phi \xi_1$ it follows that

$$\rho ||\phi \xi_1||^2 = \eta(e) g(\nabla_\xi e, \phi \xi_1), \quad \text{or} \quad \rho ||\phi \xi_1||^2 = -\eta(e) g(e, \nabla_\xi \phi \xi_1).$$

Therefore $\rho(1 - (\eta_1(\xi))^2) = 0$, or $\rho = 0$, due to the fact that $1 - (\eta_1(\xi))^2 \neq 0$. Now, applying $A \phi_1$, from the left, on (4.12) we take $A \phi_1 e = \eta(e) A \phi_1 e$. Hence

$$\beta_1 \xi_1 + \beta_2 \phi \xi_1 - \beta_1 \eta_1(\xi) \xi_1 = \eta(e) A \phi_1 e.$$ (4.13)

The inner product of (4.13) with $\xi_1$ results in $\beta_1(1 - (\eta_1(\xi))^2) = 0$, from which we take $\beta_1 = 0$. Similarly, the inner product of (4.13) with $\phi \xi_1$ gives $\beta_2(1 - (\eta_1(\xi))^2) = 0$, from which we take $\beta_2 = 0$. Summing up the above, we have $\rho = 0$, $\beta_1 = 0$, $\beta_2 = 0$. Substituting these in the first equation of (4.11), we obtain $1 - (\eta_1(\xi))^2 = 0$, which cannot occur.
4.3 The case $\dim \mathcal{H}^{\bot} < A\xi_2 = 8$ and $A\xi_3 \in \mathcal{H}^{\bot} < A\xi_2$

In this case we let

$$A\xi_3 = \kappa_1 \xi_1 + \kappa_2 \phi_1 + \kappa_3 \xi + \kappa_4 \xi_2 + \kappa_5 \xi_3 + \kappa_6 \phi_2 + \kappa_7 \phi_3 + \kappa_8 A\xi_2$$

The Codazzi equation (4.7) becomes

$$
(\xi_1 \rho - \rho^2 \eta_1 (\xi_1) - \xi_1 \eta_1 (\xi_1) \rho_1 + \kappa_1 \eta_2 (\xi_1) + \kappa_1 \eta_2 (\xi_1) \rho_2 - \xi_2 \eta_1 (\xi_2)) + (\eta_2 (\xi_1) - \rho_2 \eta_1 (\xi_2) + \kappa_2 \eta_3 (\xi_1) + \kappa_2 \eta_3 (\xi_2) + \kappa_2 \eta_3 (\xi_3)) \phi_1
$$

$$+ (\beta_1 \rho - \xi_2 \eta_1 (\xi_2) + \kappa_3 \eta_4 (\xi_1) + \kappa_3 \eta_4 (\xi_2) + \kappa_3 \eta_4 (\xi_3) + \kappa_3 \eta_4 (\xi_4)) \phi_2 +

(\beta_3 \eta_2 (\xi_2) + \kappa_4 \eta_5 (\xi_1) + \kappa_4 \eta_5 (\xi_2) + \kappa_4 \eta_5 (\xi_3) + \kappa_4 \eta_5 (\xi_4)) \phi_3 + [\eta_3 (\xi_1) + \kappa_5 \eta_6 (\xi_1) + \kappa_5 \eta_6 (\xi_2) + \kappa_5 \eta_6 (\xi_3) + \kappa_5 \eta_6 (\xi_4)] A\xi_2 = 2\eta_2 (\xi_1) \xi_2 - 2\eta_2 (\xi) \xi_3
$$

The equation (4.6), for $X = \phi \xi_1$ becomes

$$
\begin{align*}
[\phi \xi_1 \rho - \beta_1 \eta_1 (\xi_1) - \beta_1 \rho_1 + \kappa_1 \eta_2 (\xi_1) + \kappa_1 \eta_2 (\xi_1) \rho_2 - \xi_2 \eta_1 (\xi_2)] & \xi_1 + [\rho_2 (\phi \xi_1) - \rho_2 (\xi_1) \beta_1 + 4\eta_1 (\xi_1) \eta_2 (\xi_2) \xi_2 + [\beta_1 \xi_2 (\xi_1) - \rho_2 (\phi \xi_2) + 4\eta_1 (\xi_1) \eta_2 (\xi_2)] \xi_3 \\
+ [\eta_2 (\xi_2) \beta_1 - \xi_2 \beta_1 \rho_1 + \rho_2 (\xi_1) \eta_2 (\xi_2) + \eta_2 (\xi_2) \rho_1 + \kappa_2 \eta_3 (\xi_1) \beta_1 - \xi_2 (\eta_3 (\xi_2) \beta_1) + \eta_2 (\xi_2) \rho_1]
\end{align*}
$$

(4.15)

We will prove the following result

**Lemma 4.3.** If $\dim \mathcal{H}^{\bot} < A\xi_2 = 8$ and $A\xi_3 \in \mathcal{H}^{\bot} < A\xi_2$, the vectors $
\{\xi_1, \xi_2, \xi_3, \xi, \phi_1, \phi_2, \phi_3, A\phi_2\}$ are linearly independent, or $q_3(\xi) = 0$.

**Proof.** We consider the linear combination

$$
\lambda_1 \xi_1 + \lambda_2 \phi_1 + \lambda_3 \xi + \lambda_4 \xi_2 + \lambda_5 \xi_3 + \lambda_6 \phi_2 + \lambda_7 \phi_3 + \lambda_8 A\phi_2 = 0.
$$

We have

$$
\lambda_1 \xi_1 + \lambda_2 \phi_1 + \lambda_3 \xi + \lambda_4 \xi_2 + \lambda_5 \xi_3 + \lambda_6 \phi_2 + \lambda_7 \phi_3 + \lambda_8 A\phi_2 = 0.
$$

or

$$
\lambda_1 \xi_1 + \lambda_2 \phi_1 + \lambda_3 \xi + \lambda_4 \xi_2 + \lambda_5 \xi_3 + \lambda_6 \phi_2 + \lambda_7 \phi_3 + \lambda_8 A\phi_2 = 0
$$

Applying $\phi_1 \phi$, from the left, we take finally $\lambda_8 = 0$, or $\kappa_8 = 0$. If $\kappa_8 = 0$, we have $q_3(\xi) = 0$. If $q_3(\xi) \neq 0$, we have $\lambda_1 = 0$, for any $i \in \{1, \cdots, 8\}$.

According to the above lemma we describe the following cases:

**A.** $q_3(\xi) = 0$. By applying $\phi \phi_1$ from the left to the equation (4.15), the $A\xi_2$-component gives $q_3(\xi) = 0$. Something that leads us to the proof of the previous paragraph, where $\dim \mathcal{H}^{\bot} + A(\xi_2, \xi_3) = 9$. This case has been proved that can not occur.

**B.** $q_3(\xi) \neq 0$. By applying $\phi \phi_1$ from the left to the equation (4.14) we have the relation

$$
\[ A\phi \xi_2 = [\eta_1(\xi)\alpha_1 + \kappa - \eta_2(\xi)\kappa_8\alpha_1] \xi_1 - \kappa \xi_2 + \kappa_6 \xi_3 + \]
\[ + [\eta_3(\xi) - \eta_1(\xi)\eta_3(\xi)\alpha_1 - \eta_1(\xi)\kappa_3 - \eta_3(\xi)\kappa_6 + \]
\[ \eta_2(\xi)\kappa_7 - \eta_2(\xi)\rho s_8 + \kappa_8 \eta_2(\xi)\alpha_1] \xi + \]
\[ [\eta_3(\xi)\beta_1 - \eta_1(\xi)\kappa_2 - \eta_2(\xi)\kappa_6 - \eta_3(\xi)\kappa_7 - \eta_2(\xi)\kappa_8 \beta_1] \phi \xi_1 + \]
\[ \kappa_5 \phi \xi_2 - \kappa_4 \phi \xi_3 - \kappa_4 A \phi \xi_3 \]

Multiplying this relation with \( \frac{(q_2(\xi))^2}{q_3(\xi)} \), substituting it in (4.15) and taking the \( A \phi \xi_1 \)-component we have \( (q_2(\xi))^2 + (q_3(\xi))^2 = 0 \), which is a contradiction.

4.4 The case \( \dim H^\perp = 7 \), \( A(< \xi_2, \xi_3 >) \subseteq H^\perp \)

In this section we will prove that \( \dim H^\perp = 7 \) and \( A(< \xi_2, \xi_3 >) \subseteq H^\perp \) can not occur.

4.4.1 The distribution \( H \)

We notice that the distribution \( H \), namely the orthogonal complement of the distribution \( H^\perp = < \xi > \oplus D^\perp \oplus Q^\perp \) is \( A \)-invariant. Indeed, for any \( U \in H^\perp \) and \( X \in H \), we have \( g(AX, U) = g(X, AU) = 0 \). Also, the distribution \( H \), is generally non-integrable. It is easy to see that, at any point \( p \in M \), the restriction of the operator \( \phi \xi_1 \), in the space \( H \) is \( 1 - 1 \) an orthogonal selfadjoint endomorphism. Thus the eigenvalues are \( \pm 1 \) and there is an orthonormal basis of eigenvectors. The eigenspaces are \( W(1) \subseteq H \) and \( W(-1) \subseteq H \).

4.4.2 The Codazzi equation

The Codazzi equation for \( X = \xi \) and \( Y = e \), where \( e \in H \) and \( Ae = \lambda e \) takes the form

\[
(\xi \lambda) e + \lambda \nabla \xi e - A \nabla \xi e - (ep) \xi_1 - \rho q_3(e) \xi_2 + \rho q_2(e) \xi_3 - \rho \lambda \phi_1 e + \lambda A \phi e = \]
\[ \phi e + \eta_1(\xi) \phi_1 e + \eta_2(\xi) \phi_2 e. \]

The tangent-component and normal-component of this equation with respect to the distribution \( H \) gives

\[
(4.16) \quad H^\perp: \quad ep = 0, \quad q_2(e) = 0, \quad q_3(e) = 0
\]

\[
H: \quad (\xi \lambda) e + \lambda \nabla \xi e - A \nabla \xi e - \rho \lambda \phi_1 e + \lambda A \phi e = \]
\[ \phi e + \eta_1(\xi) \phi_1 e + \eta_2(\xi) \phi_2 e. \]

(4.17)

Taking the inner product of (4.17) with \( e \) we obtain \( \xi \lambda = 0 \). The relation (4.17) becomes

\[
(4.18) \quad \lambda \nabla \xi e - A \nabla \xi e - \rho \lambda \phi_1 e + \lambda A \phi e = \phi e + \eta_1(\xi) \phi_1 e + \eta_2(\xi) \phi_2 e. \]

The inner product with \( \phi \phi_1 e \) gives \( g(\phi \phi_3, e) = 0 \). Thus \( tr \phi_3 = 0 \) on the distribution \( H \).
Let $e, \varepsilon \in \mathcal{H}$ with $||e|| = ||\varepsilon|| = 1$, $Ae = \lambda e$ and $A\varepsilon = \mu \varepsilon$. The Codazzi equation becomes

$$\begin{align*}
(\varepsilon \mu) e + \mu \nabla e \varepsilon - A \nabla e \varepsilon - (\varepsilon \lambda) e - \lambda \nabla e \varepsilon + A \nabla e \varepsilon = \\
-2g(\phi e, e) \xi - 2g(\phi e, \varepsilon) \xi_1 - 2g(\phi \varepsilon e, \varepsilon) \xi_2 - 2g(\phi \varepsilon e, \varepsilon) \xi_3.
\end{align*}$$
(4.19)

We take the inner product of (4.19) with $\xi$. After some calculations we obtain

$$2\lambda A\phi e - \rho A\phi_1 e = 2\phi e + [2\eta_1(\xi) + \rho \lambda] \phi_1 e + 2\eta_2(\xi) \phi_2 e.\quad (4.20)$$

Taking the inner product of (4.19) with $\xi_1$ we obtain

$$2(\lambda - \alpha_1) A\phi_1 e + [-\rho + \eta_1(\xi) \lambda_1] A\phi e = [2\eta_1(\xi) + \rho \lambda - \eta_1(\xi) \alpha_1 \lambda] \phi e + (2 + \alpha_1 \lambda) \phi_1 e.\quad (4.21)$$

Taking the inner product of (4.19) with $\phi_1 e$ we get

$$-\beta_1 A\phi_1 e + \eta_1(\xi) \beta_1 A\phi e = -\eta_1(\xi) \lambda \beta_1 \phi e + \lambda \beta_1 \phi_1 e + \eta_2(\xi) \phi_3 e.\quad (4.22)$$

Multiplying (4.20) with $\eta_1(\xi) \beta_1$ and (4.22) with $-2\lambda$ we obtain

$$\begin{align*}
[2\lambda \beta_1 - \eta_1(\xi) \beta_1 \rho] A\phi_1 e &= [2\eta_1(\xi) \beta_1 + 2\eta_1(\xi) \lambda^2 \beta_1] \phi e + \\
[2\eta_1(\xi)]^2 \beta_1 + \eta_1(\xi) \beta_1 \rho \lambda - 2\lambda^2 \beta_1 \phi_1 e + \\
2\eta_1(\xi) \beta_1 \eta_2(\xi) \phi_2 e - 2\eta_2(\xi) \lambda \phi_3 e.
\end{align*}$$
(4.23)

Similarly, multiplying (4.20) with $\beta_1$ and (4.22) with $-\rho$ we obtain

$$\begin{align*}
[2\lambda \beta_1 - \eta_1(\xi) \beta_1 \rho] A\phi e &= [2\beta_1 + \eta_1(\xi) \lambda \rho \beta_1] \phi e + \\
2\eta_1(\xi) \beta_1 \phi_1 e + 2\beta_1 \eta_2(\xi) \phi_2 e - \eta_2(\xi) \rho \phi_3 e.
\end{align*}$$
(4.24)

we prove the following

**Lemma 4.4.** Let $e \in \mathcal{H}$ be an eigenvector of the shape operator $A$ with $Ae = \lambda e$. Then $\dim <e, \phi e, \phi_1 e, \phi_2 e, \phi_3 e> \geq 3$.

**Proof.** Let $e \in \mathcal{H}$ be an eigenvector of $A$ with $Ae = \lambda e$ such that $\phi e = \omega_1 \phi_1 e + \omega_2 \phi_2 e + \omega_3 \phi_3 e$. The inner product of this with $\phi_3 e$ results in $\omega_3 = 0$. Thus $\phi e = \omega_1 \phi_1 e + \omega_2 \phi_2 e$.

By applying $\phi$ from the left, we have $-e = -\omega_1 \phi_1 e + \omega_2 \phi_2 e$. So $e \in <e, \phi_1 e, \phi_2 e >$. Also $g(\phi_1 e, \phi_2 e) = -g(\phi_2 e, \phi_2 e) = -g(e, \phi_1 e) = -g(e, \phi_3 e) = 0$.

Thus $\dim <\phi_1 e, \phi_2 e> = 2$. It is obvious that $\dim <e, \phi_3 e> = 2$. Since $\phi e = \omega_1 \phi_1 e + \omega_2 \phi_2 e$, we have $\phi_1 e \in <e, \phi_3 e>$ and $\phi_2 e \in <e, \phi_3 e>$. Moreover, from $\phi e = \omega_1 \phi_1 e + \omega_2 \phi_2 e$, we take $\phi_3 e \in <e, \phi_2 e>$. But $\phi e \in <\phi_1 e, \phi_2 e>$ and $\dim <\phi_1 e, \phi_2 e> = 2$.

Summing up all the above, we have

$$\dim <\phi_1 e, \phi_2 e> = \dim <e, \phi_3 e> = \dim <\phi_1 e, \phi_2 e> = 2$$

and $\phi <e, \phi_3 e> = <\phi_1 e, \phi_2 e>$, $<e, \phi_3 > \perp <\phi_1 e, \phi_2 e>$. This completes the proof. \hfill \Box

Now we work on the subspace $<e, \phi_3 e> = <\phi_1 e, \phi_2 e>$. If we assume that the vector $e$ is not parallel to the vector $\phi_1 e$, then

$$<e, \phi_3 e> = <\phi_1 e, \phi_2 e> = <e, \phi_1 e>, \quad Ae = \lambda e$$

and $A\phi_1 e = \lambda \phi_1 e$. Thus $<e, \phi_3 e> \subseteq V(\lambda)$, where $V(\lambda)$ is the eigenspace of the eigenvalue $\lambda$ and $A\phi_3 e = \lambda \phi_3 e$. From the inner product of (4.22) with $\phi_3 e$ we have
\begin{align*}
\lambda \eta_1(\xi)\beta_1 g(\phi e, \phi_3 e) - \lambda \beta_1 g(\phi_1 e, \phi_3 e) &= 2 \eta_2(\xi).
\end{align*}
Therefore it occurs that \( \eta_2(\xi) = 0 \), which is impossible. So \( \phi_1 e = \nu e \) with \( \nu \neq 0 \), or \( \phi e = -\frac{1}{\nu} \phi_1 e \). Finally, we have \( \omega_2 = 0 \) and thus \( \phi e = \omega_1 \phi_1 e \). Let \( B = \beta_1(2\lambda - \eta_1(\xi)\rho) \).
If \( B = 0 \), from the equation (4.23) we take \( \beta_1 = 0 \) and from the \( \phi_3 e \)-component \( \lambda = 0 \). From (4.24) we have \( \rho = 0 \), which is a contradiction. Thus \( B \neq 0 \). We substitute the relations (4.23) and (4.24) in the equation (4.21) and we take the \( \phi_2 e \)-component
\begin{align*}
[-\rho + \eta_1(\xi)\alpha_1] \frac{2\beta_1 \eta_2(\xi)}{B} + (2\lambda - \alpha_1) \frac{2\eta_1(\xi)\beta_1 \eta_2(\xi)}{B} &= 0
\end{align*}
or
\begin{align*}
-\rho + \eta_1(\xi)\alpha_1 + 2\eta_1(\xi)\lambda - \alpha_1 \eta_1(\xi) &= 0.
\end{align*}
Thus \( \alpha_1 = \frac{\rho}{2\eta_1(\xi)} \), and therefore the submanifold which has tangent bundle \( H \) is totally umbilical.
Since \( B \neq 0 \), it is obvious that \( \beta_1 \neq 0 \) and so \( A\phi e = \lambda \phi e \) and \( A\phi_1 e = \lambda \phi_1 e \). We substitute these in (4.24) and taking the \( \phi_3 e \)-component, we have \( \eta_2(\xi) = 0 \), which is a contradiction. Thus the case \( \dim H^\perp = 7 \) and \( A(\xi_2, \xi_3) \subseteq H^\perp \) cannot occur.

5 The dimension of \( H^\perp \)

In this paragraph we examine the remaining cases of the dimension of the distribution \( H^\perp \); namely \( \dim H^\perp = 6 \), \( \dim H^\perp = 5 \) and \( D^\perp = \phi D^\perp \).

\textbf{A.} \( \dim H^\perp = 6 \). We consider the one dimensional subspace \( < \vec{x} > = (\xi > \oplus D^\perp) \cap \phi D^\perp \). The vector field \( \xi_1 \) is well defined from the relation \( \phi \phi_1 A = A\phi_1 \phi \). For the basis of the space \( D^\perp \) we may choose the vectors \( \xi_2 \) and \( \xi_3 \) with the following way. We put \( \xi_2 = \vec{x} - \eta_1(\vec{x})\xi_1 - \eta(\vec{x})\xi \). So \( \xi_2 \in D^\perp \), because \( \vec{x} \in < \xi > \oplus D^\perp \). Also
\begin{align*}
||\xi_2'||^2 &= g(\vec{x} - \eta_1(\vec{x})\xi_1 - \eta(\vec{x})\xi, \vec{x} - \eta_1(\vec{x})\xi_1 - \eta(\vec{x})\xi) \\
&= 1 - (\eta_1(\vec{x}))^2 - (\eta(\vec{x}))^2 + 2\eta_1(\vec{x})\eta(\vec{x})\eta_1(\xi).
\end{align*}
Let \( B = \sqrt{||\xi_2'||^2} = \sqrt{1 - (\eta_1(\vec{x}))^2 - (\eta(\vec{x}))^2 + 2\eta_1(\vec{x})\eta(\vec{x})\eta_1(\xi)} \) and
\begin{align*}
\xi_2 = \frac{1}{B} [\vec{x} - \eta_1(\vec{x})\xi_1 - \eta(\vec{x})\xi].
\end{align*}
Thus \( \vec{x} = \eta_1(\vec{x})\xi_1 + B\xi_2 + \eta(\vec{x})\xi \). We complete the basis of \( D^\perp \) with a vector \( \xi_3 \in D^\perp \). Thus
\begin{align}
\phi \vec{x} &= \eta_1(\vec{x})\phi \xi_1 + B\phi \xi_2 \\
\phi \vec{x} &= c_1 \xi_1 + c_2 \phi \xi_2 + c_3 \phi \xi_3,
\end{align}
on the other hand \( \vec{x} \in \phi D^\perp \), so \( \vec{x} = c_1 \phi \xi_1 + c_2 \phi \xi_2 + c_3 \phi \xi_3 \), or
\begin{align*}
\phi \vec{x} &= -c_1 \xi_1 - c_2 \xi_2 - c_3 \xi_3 + (c_1 \eta_1(\xi) + c_2 \eta_2(\xi) + c_3 \eta_3(\xi))\xi.
\end{align*}
Thus \( \phi \vec{x} \in < \xi > \oplus D^\perp \) and from (5.1) we have \( ||\vec{x}|| = 1 \) and \( \vec{x} \perp \phi \vec{x} \). From the above we have \( \vec{x} = \xi \) and thus \( \xi \in \phi D^\perp \), which is impossible. Thus \( \dim H^\perp \neq 6 \).

\textbf{B.} \( \dim H^\perp = 5 \). In this case we have \( (\xi > \oplus D^\perp) \cap \phi D^\perp = < \vec{x}, \phi \vec{x} > \), because otherwise we have \( \xi \in \phi D^\perp \). Something that is a contradiction. Therefore the only remaining case that we should consider is the case where there exists \( \vec{x} \) such that \( (\xi > \oplus D^\perp) \cap \phi D^\perp = < \vec{x}, \phi \vec{x} > \) with \( ||\vec{x}|| = 1 \). Since \( < \vec{x}, \phi \vec{x} > \subseteq \phi D^\perp \) and
<\bar{x},\phi\bar{x}>\subset\subset<\xi>\oplus\mathcal{D}^\bot we have \bar{x} \perp \xi and thus \bar{x},\phi\bar{x} \in \mathcal{D}^\bot. We choose a basis \mathcal{D}^\bot by the following way \xi_1 = \bar{x}, \xi_2 = \phi\bar{x} and we complete with a vector \xi_3. So \\
\phi\xi_1 = \phi\bar{x} = \xi_2. Thus \phi\xi_1 = \xi_2. So \phi_2\phi\xi_1 = 0, or \phi_2\phi\xi_1 = 0, or \phi_2\xi_3 = \tilde{\eta}_1(\xi)\xi_2. \\
Since \phi\xi_1 = \xi_2, we have \phi(\tilde{\eta}_1(\xi)\xi_1) = \tilde{\eta}_1(\xi)\xi_2. Thus \phi(\xi_3 - \tilde{\eta}_1(\xi)\xi_1) = \lambda_\xi, or \lambda = 0, or \xi_3 = \tilde{\eta}_1(\xi)\xi_1, something which is impossible. Thus \\
\dim H^\bot \neq 5.

\textbf{C.} \mathcal{D}^\bot = \phi\mathcal{D}^\bot. The distribution \mathcal{D}^\bot is \phi--invariant. Thus \forall X \in \mathcal{D}^\bot, \exists Y \in \mathcal{D}^\bot with \phi Y = X, because \ker\phi \notin \mathcal{D}^\bot. So \phi is an 1 - 1 and onto the subspace \mathcal{D}^\bot.

Thus \forall \xi_1 \in \mathcal{D}^\bot, \tilde{\eta}_1(\xi) = 0 and so, for the vector \xi_1, we have \eta_1(\xi) = 0, which is impossible.

6 The case \eta_1(\xi) \neq \pm 1 and \eta_1(\xi) \neq 0

Let \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7 be a permutation of the set \{\xi, \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3\}. Let

\begin{align*}
\varepsilon_1 &= \alpha_3\varepsilon_3 + \cdots + \alpha_7\varepsilon_7 + \alpha_1\mathcal{A}_2 + \alpha_2\mathcal{A}_3 \\
\varepsilon_2 &= \beta_\xi\varepsilon_3 + \cdots + \beta_\xi\varepsilon_7 + \beta_\xi\mathcal{A}_2 + \beta_\xi\mathcal{A}_3
\end{align*}

with \varepsilon_3, \cdots, \varepsilon_7, \mathcal{A}_2, \mathcal{A}_3 linear independent vectors. Thus \dim < \varepsilon_3, \cdots, \varepsilon_7, \mathcal{A}_2, \mathcal{A}_3 >= 7 and \dim < \varepsilon_3, \cdots, \varepsilon_7 >= 5.

\textbf{A.} Let \varepsilon_1 \in < \varepsilon_3, \cdots, \varepsilon_7 >. We have 4 \leq \dim < \xi, \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3 >= 6.

So \dim < \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3 >= 4 and according to the previous proof we have < \varepsilon_3, \cdots, \varepsilon_7 >\subset< \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3 >, from which we take \dim < \varepsilon_3, \cdots, \varepsilon_7 >\leq 4, which concludes to a contradiction. Thus \dim < \varepsilon_3, \cdots, \varepsilon_7 >= 6.

Similarly it is easy to see that \varepsilon_2 \in < \varepsilon_3, \cdots, \varepsilon_7 > is impossible. Thus \dim < \varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots, \varepsilon_7 >= 7. But \varepsilon_1, \varepsilon_2 \in < \varepsilon_3, \cdots, \varepsilon_7, \mathcal{A}_2, \mathcal{A}_3 > and \{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7\} is a basis of the subspace, so \mathcal{A}_2 = \sum_i \gamma_i\varepsilon_i and \mathcal{A}_3 = \sum_i \delta_i\varepsilon_i. This case has been studied in \S4.4.

\textbf{B.} Let \mathcal{A}_3 = \alpha_3\varepsilon_3 + \cdots + \alpha_7\varepsilon_7 + \alpha_1\mathcal{A}_2, \varepsilon_1 = \beta_\xi\varepsilon_3 + \cdots + \beta_\xi\varepsilon_7 + \beta_\xi\mathcal{A}_2 with \varepsilon_3, \cdots, \varepsilon_7, \mathcal{A}_2 \text{ linearly independent vectors. We have } \varepsilon_1 \notin < \varepsilon_3, \cdots, \varepsilon_7 >, \text{ because otherwise } \dim < \varepsilon_3, \cdots, \varepsilon_7 >= 4, \text{ something which is impossible. Thus }

\mathcal{A}_3 = \tilde{\alpha}_1\varepsilon_1 + \tilde{\alpha}_2\varepsilon_2 + \cdots + \tilde{\alpha}_7\varepsilon_7 + \tilde{\alpha}_8\mathcal{A}_2.

Let \varepsilon_1 = \alpha_2\varepsilon_2 + \cdots + \alpha_7\varepsilon_7 + \omega_2\mathcal{A}_2 + \omega_3\mathcal{A}_3, \text{ with } \dim < \varepsilon_2, \cdots, \varepsilon_7, \mathcal{A}_2, \mathcal{A}_3 >= 8.

Then we have \dim < \varepsilon_2, \cdots, \varepsilon_7 >= 6. If \varepsilon_1 \in < \varepsilon_2, \cdots, \varepsilon_7 >, then

\dim((\mathcal{D}^\bot < \xi >) + \phi\mathcal{D}) = 4 \quad \text{and} \quad < \varepsilon_2, \cdots, \varepsilon_7 >\subset< (\mathcal{D}^\bot < \xi >) + \phi\mathcal{D}),

which is impossible. Thus \dim < \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7 >= 7. The case < \mathcal{A}_2, \mathcal{A}_3 >=< \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7 > has been studied in \S4.4. If \mathcal{A}_3 \notin < \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7 >, then \dim < \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7, \mathcal{A}_3 >= 8. We have < \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7, \mathcal{A}_3 >=< \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7, \mathcal{A}_2, \mathcal{A}_3 >, and hence

\mathcal{A}_2 \in < \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_7, \mathcal{A}_3 >,

which case has been studied in \S4.3.
7 The case $\xi \in D^\perp$

In this case we have the two subcases: $\eta_1(\xi) \neq \pm 1$ and $\eta_1(\xi) = 1$.

7.1 The subcase $\eta_1(\xi) \neq \pm 1$

If $\eta_1(\xi) = 0$, then $A\xi_2, A\xi_3 \in D^\perp$ and finally $H^\perp = D^\perp = <\xi, \xi_1, \phi\xi_1> = \phi D^\perp$. Hence we can continue the proof as in §4.4. Let now $\eta_1(\xi) \neq 0$ and $\xi_2 = \xi - \eta_1(\xi)\xi_1$. Then

$$g(\xi_2, \xi_2) = g(\xi - \eta_1(\xi)\xi_1, \xi - \eta_1(\xi)\xi_1) = 1 - (\eta_1(\xi))^2$$

We define

$$\xi_2 = \frac{\xi - \eta_1(\xi)\xi_1}{\sqrt{1 - (\eta_1(\xi))^2}}$$

and thus $\xi = \eta_1(\xi)\xi_1 + \sqrt{1 - (\eta_1(\xi))^2}\xi_2$.

The inner product of the last relation with $\xi_2$ gives $\eta_2(\xi) = \sqrt{1 - (\eta_1(\xi))^2}$. We complete the basis of the subspace $D^\perp$ with a vector $\xi_3$. We observe that $0 = \phi\xi = \eta_1(\xi)\phi\xi_1 + \eta_2(\xi)\phi\xi_2$, so $\phi\xi_1 \parallel \phi\xi_2$. Also $\phi\xi_2 = \eta_1(\xi)\xi_3$, so $\phi\xi_2 \parallel \xi_3$.

Moreover,

$$\phi_3\xi = \eta_1(\xi)\phi_3\xi_1 + \eta_2(\xi)\phi_3\xi_2, \quad \text{or} \quad \phi_3\xi = \eta_1(\xi)\xi_2 - \eta_2(\xi)\xi_1,$$

so $\phi_3\xi \in <\xi_1, \xi_2>$. Summing up these formulas we have

$$<\xi_1, \xi_2> = <\xi, \phi\xi_3> \quad \text{and} \quad <\xi_3> = <\phi\xi_1> = <\phi\xi_2>.$$

Namely $\xi \in D^\perp = \phi D^\perp$. Thus $A\xi_3 = A\phi\xi_1 = \beta_1\xi_1 + \beta_2\phi\xi_1 - \eta_1(\xi)\beta_1\xi$. Also $\eta_2(\xi)\xi_2 = \xi - \eta_1(\xi)\xi_1$, so $\eta_2(\xi)A\xi_2 = A\xi - \eta_1(\xi)A\xi_1$, or

$$\eta_2(\xi)A\xi_2 = [-\eta_1(\xi)\alpha_1 + \rho]\xi_1 - \eta_1(\xi)\beta_2\phi\xi_1 + [-\eta_1(\xi)\rho + (\eta_1(\xi))^2 a_1]\xi,$$

with $\eta_2(\xi) \neq 0$. Thus $A\xi_2, A\xi_3 \in D^\perp$ and finally $H^\perp = D^\perp = <\xi, \xi_1, \phi\xi_1> = \phi D^\perp$.

Now we can continue the proof in a similar way as the proof in the case §4.4.

7.2 The subcase $\eta_1(\xi) = 1$

In this case we will prove that the hypersurface $M^{4m-1}$ is of type $A$, namely $M^{4m-1}$ is locally congruent to a tube around a totally geodesic $G_2(C^{m+1})$ in $G_2(C^{m+2})$.

From $\eta_1(\xi) = 1$, we have $\xi = \xi_1$ and $\phi\xi = \phi\xi_1$. Also we have the relations

$$\phi\phi_1\xi = 0 = \phi_1\phi\xi, \quad \phi\phi_1\xi_2 = \phi\xi_3 = \xi_2 = -\phi_1\phi_3 = \phi_1\phi_2,$$

$$\phi\phi_1\xi_3 = -\phi_2\xi_3 = \phi_1\xi_3 = \phi_1\phi_3,$$

$$\phi\phi_1\xi = \phi_1\phi\xi + \eta_1(\xi)\xi - \eta(\xi)\xi_1 = \phi_1\phi, \quad \text{for} \ e \in \mathcal{D}$$

Thus, in our case, the relation $\phi\phi_1 A = A\phi_1\phi$, is equivalent with $\phi\phi_1 A = A\phi_1$.

We have the following transformations of the distributions

$$\phi <\xi_1> = \{0\}, \quad \phi <\xi_2> = <\xi_2>, \quad \phi <\xi_3> = <\xi_2>, \quad <\xi > = <\xi_1>.$$

Also we have $\phi\phi_1\xi = \phi\phi_1\xi_1 = 0$, $\phi\phi_1\xi_2 = \phi\phi_1\xi_3 = \xi_2$, $\phi\phi_1\xi_3 = -\phi\phi_1\xi_2 = \xi_3$.

The restriction of the operator $\phi\phi_1$ on the subspace $\mathcal{D}$ is an orthogonal selfadjoint
The spaces \( W(1) \) and \( W(-1) \) are \( \phi - \)invariant.

By differentiating \( \eta_2(\xi) = 0 \) and \( \eta_3(\xi) = 0 \) covariantly, we have \( g(\nabla_X \xi_2, \xi) + g(\xi_2, \nabla_X \xi) = 0 \), or \( q_3(X) = 2\eta_2(AX) \). Also \( g(\nabla_X \xi_3, \xi) + g(\xi_3, \nabla_X \xi) = 0 \), or \( q_2(X) = 2\eta_2(AX) \). From these we obtain

\[
q_2(\xi) = q_3(\xi) = q_2(\xi_1) = q_3(\xi_1) = 0, \quad q_2(\xi_3) = q_3(\xi_3).
\]

Let \( A\xi_2 = \kappa_1 \xi_1 + \kappa_2 \xi_2 + \kappa_3 \xi_3 + \kappa_4 e \), with \( e \in \mathcal{D} \cap W(1), ||e|| = 1 \). From which it occurs that \( 2A\xi_2 = q_2(\xi_2) \xi_2 + q_2(\xi_3) \xi_3 + q_2(e) e \). Similarly, if we put \( A\xi_3 = \lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_3 \xi_3 + \lambda_4 e \), with \( e \in \mathcal{D} \cap W(1), ||e|| = 1 \), we obtain \( 2A\xi_3 = q_3(\xi_2) \xi_2 + q_3(\xi_3) \xi_3 + q_3(e) e \). Now, differentiating covariantly the relation \( \phi_1 \phi_1 Y = \phi_1 \phi_Y \), we take \( \nabla_X \phi_\phi_1 \phi_1 Y = \nabla_X \phi_\phi_1 \phi_1 Y, (\nabla_X \phi_\phi_1 \phi_1 Y) + \phi_\phi X \phi_\phi_1 Y = (\nabla_X \phi_\phi_1 \phi_1 Y) + \phi_\phi X Y \phi_\phi_1 Y \). So

\[
(\nabla_X \phi_\phi_1 \phi_1 Y - \phi_\phi X (\nabla_X A) Y = g(\phi_1 AX, AY) \xi - q_2(X) \phi_\phi_3 AY + q_1(X) A \phi_\phi Y + q_3(X) \phi_\phi_2 AY - q_3(X) A \phi_\phi Y +
\]

\[
+ \rho g(AX, \phi Y) \xi + \rho Y(\phi \phi X) - \eta Y(\phi Y) \phi_\phi X
\]

We take the inner product of this with \( Z \) and we have

\[
g((\nabla_X A)Z, \phi_\phi Y) - g((\nabla_X A)Y, \phi_\phi Z) =
\eta(Z) g(\phi_\phi AX, AY) - q_2(X) g(\phi_\phi_3 AY, Z) + q_3(X) g(\phi_\phi_2 AY, Z) - q_3(X) g(\phi_\phi_3 AY, Z) +
\rho g(AX, \phi Y) \eta() +
\]

\[
\rho \eta(\phi Y) g(\phi AX, Z) - \eta(\phi Y) g(\phi AX, Z)
\]

We take a cyclic permutation and we add these equations. Then the left side becomes

\[
\theta = \eta(X) \phi_\phi_1 \phi_1 Y - g(\phi_\phi X, \phi_\phi_1 Y) \xi +
\]

\[
\sum_{\nu=1}^{3} \{- \eta_\nu(X) \phi_\phi_\nu_1 \phi_1 Y - g(\phi_\phi_\nu X, \phi_\phi_1 Y) \xi_\nu - 2 \eta_\nu(\phi_\phi_1 Y) \phi_\phi_\nu X\} +
\]

\[
\sum_{\nu=1}^{3} \{ \eta_\nu(\phi X) \phi_\phi_\nu_1 \phi_1 Y + g(\phi_\phi_\nu X, \phi_\phi_1 Y) \phi_\phi_\nu_\xi_\nu\} +
\]

\[
\sum_{\nu=1}^{3} \{- \eta(X) \eta_\nu(\phi_\phi_1 Y) \phi_\phi_\nu_\xi_\nu - \eta(X) \eta_\nu(\phi_\phi_1 Y) \xi_\nu\} +
\]

\[
\eta(Y) \phi_\phi_1 X - \eta(X) \phi_\phi_1 Y +
\]
From this we obtain
\[
\sum_{\nu=1}^{3} \left\{ \eta_{\nu}(Y) \phi_1 \phi_{\nu} X - \eta_{\nu}(X) \phi_1 \phi_{\nu} Y - 2g(\phi_{\nu} Y, X) \phi_1 \xi_{\nu} \right\} +
\sum_{\nu=1}^{3} \left\{ \eta_{\nu}(Y) \phi_1 \phi_1 \phi_{\nu} \phi X - \eta_{\nu}(X) \phi_1 \phi_1 \phi_{\nu} \phi Y \right\} +
\sum_{\nu=1}^{3} \left\{ \eta(Y) \eta_{\nu}(X) \phi_1 \phi_{\nu} \xi_{\nu} - \eta(X) \eta_{\nu}(Y) \phi_1 \phi_{\nu} \xi_{\nu} \right\} +
g(\phi Y, \phi_1 \phi X) \xi + \eta(Y) \phi_1 \phi X +
\sum_{\nu=1}^{3} \left\{ -g(\phi_{\nu} Y, \phi_1 \phi X) \phi \xi_{\nu} - \eta_{\nu}(X) \phi_1 \phi_{\nu} \phi \phi X \right\} +
\sum_{\nu=1}^{3} \left\{ -g(\phi_{\nu} Y, \phi_1 \phi X) \phi \xi_{\nu} - \eta_{\nu}(Y) \phi_1 \phi_{\nu} \phi \phi X \right\} +
\sum_{\nu=1}^{3} \left\{ \eta_{\nu}(Y) \eta_{\nu}(\phi_1 \phi X) \xi + \eta(Y) \eta_{\nu}(\phi_1 \phi X) \phi \xi_{\nu} \right\}
\]

The right side of the equation which occurs by adding (7.3) and its cyclic permutation, can be written \(g(A, Z)\) where

\[
A = g(\phi_1 AX, AY) \xi - q_2(AX) \phi_3 AY + q_2(X) A \phi_3 Y + q_3(X) \phi_2 AY - q_3(Y) A \phi_2 Y + \rho g(AX, AY) \xi + \rho g(X) \phi AY - \eta(Y) A \phi_1 AX +
+ \eta(X) A \phi_1 AY - q_2(Y) A \phi_3 Y + q_2(X) A \phi_3 X + q_3(Y) A \phi_2 X - q_3(Y) \phi \phi_2 AY - \eta(X) \rho \phi AY + \rho g(\phi AY, X) \xi -
-g(A \phi_3 AY, X) \xi - \eta(Y) A \phi_1 AX - g(\phi_3 AX, Y) 2A \xi_2 +
+g(\phi_3 \phi_3 Y, X) 2A \xi_2 + +g(\phi_2 Y, Y) 2A \xi_2 - g(A \phi_2 X, Y) 2A \xi_3 +
+\rho \eta(Y) A \phi X - \rho \eta(X) A \phi Y + \eta(X) A \phi_1 AX
\]

From all the above we obtain \(\Theta = \Lambda\). For \(Y = \xi\) the equation \(\Theta = A\) is written

\[
\phi_1 \phi X + \phi_1 \phi_2 X - \eta_2(X) \phi_1 \phi_2 \xi - \eta_3(X) \phi_1 \phi_3 \xi -
-2g(\phi_2 \xi, X) \phi_1 \xi_2 - 2g(\phi_3, X) \phi_1 \xi_3 + \eta_2(X) \phi_1 \xi_2 +
\eta_1(\phi X) \phi_1 \xi_3 + \phi_1 \phi X + g(\phi_2 \xi, \phi_1 \phi X) \xi_2 +
+g(\phi_3, \phi_1 \phi X) \xi_3 + \phi_1^2 \phi X + 2\eta_2(\phi_1 \phi X) \phi \xi_2 +
+2\eta_3(\phi_1 \phi X) \phi \xi_3 + \eta_3(\phi_1 \phi X) \phi \xi_3 =
-\rho q_2(X) \phi_1 \phi_3 + \rho \eta_3(X) \phi_1 \phi_2 \xi + \rho \phi A X - 2A \phi_1 AX +
2\rho g(\phi_3, \phi X, \xi) A \xi_2 - 2\rho g(\phi_2 \phi X, \xi) A \xi_3 + \rho A \phi X
\]

From this we obtain

\[
2A \phi_1 AX = 2\phi_1 X + 2\phi X + 4\eta_2(X) \xi_3 - 4\eta_3(X) \xi_2 + \rho q_2(X) \xi_3 -
-\rho q_3(X) \xi_2 + \rho \phi A X + \rho A \phi X - 2\eta_3(X) \rho A \xi_2 + 2\rho \eta_2(X) A \xi_3.
\]
We have $\nabla_\xi \xi = \phi A \xi = 0$, thus $M$ is a hypersurface with geodesic Reeb flow and thus
\[
\rho A \phi X + \rho \phi AX - 2\rho A \phi X + 2\phi X =
\]
\[
2\sum_{\nu=1}^{3} \left( -\eta_\nu(X) \phi \xi_\nu - \eta_\nu(\phi X) \xi_\nu - \eta_\nu(\xi) \phi_\nu X + 2\eta(X) \eta_\nu(\xi) \phi_\nu + 2\eta_\nu(\phi X) \eta_\nu(\xi) \right).
\]
From this we obtain
\begin{equation}
(7.5) \quad -2\rho A \phi X = -2\rho X - 2\phi_1 X - \rho A \phi X - \rho \phi AX + 4\eta_2(X) \xi_3 - 4\eta_3(X) \xi_2.
\end{equation}

If $X \in W(1)$, then $AX \in W(1)$ and $\phi_1 Y = -\phi Y$, for any $Y \in W(1)$. Thus the relation (7.5) becomes
\begin{equation}
(7.6) \quad 2\rho A \phi X = -\rho A \phi X - \rho \phi AX + 4\eta_2(X) \xi_3 - 4\eta_3(X) \xi_2.
\end{equation}

Substituting the relation (7.6) in (7.4) we have
\[
\rho [2\rho A \phi X + \phi AX - 2\eta_2(X) A \xi_2 + 2\eta_2(X) A \xi_3 + q_2(X) \xi_3 - q_3(X) \xi_2] = 0
\]
Let $\rho \neq 0$. Then $2\rho A \phi X - 2\eta_2(X) A \xi_2 + 2\eta_2(X) A \xi_3 + q_2(X) \xi_3 - q_3(X) \xi_2 = 0$. Substituting $X = \xi_2$ we take $2\rho A \xi_2 + q_2(\xi_2) \xi_3 - q_3(\xi_2) \xi_2 = 0$. This gives $2\rho \xi_2 = q_2(\xi_2) \xi_2 + q_3(\xi_2) \xi_3$. Thus $A \xi_2 \in D^\perp$. Similarly, for $X = \xi_3$, we have $2\rho \xi_3 = q_2(\xi_3) \xi_2 + q_3(\xi_3) \xi_3$. So $A \xi_3 \in D^\perp$. Thus $A \xi = \rho \xi$ and $AD^\perp \subseteq D^\perp$, which means that the hypersurface $M$ is locally of type $A$.

Now we examine the case where $\rho = 0$, at a neighborhood of a point $p \in M$. Let $X \in W(1)$, with $AX = \lambda X$. According to (7.5) we have
\[
-2\lambda A \phi X = 4\eta_2(X) \xi_3 - 4\eta_3(X) \xi_2.
\]
If $\lambda = 0$, then $\eta_2(X) \xi_3 - 4\eta_3(X) \xi_2 = 0$ and thus $X \in D \cap W(1)$. If $\lambda \neq 0$, then
\[
A \phi X = 2\eta_2(X) \lambda \xi_2 - 2\eta_3(X) \lambda \xi_3.
\]
Let $X \in W(-1)$ with $AX = \lambda X$. According to (7.5) we have $A \phi X = \frac{2}{\lambda} \phi X$.

If $X \in W(1)$, then $\phi AX = 2\eta_2(X) \xi_2 - 2\eta_3(X) \xi_3$. Let $\{v_1, \cdots, v_k\} \subseteq W(1)$ be an orthonormal basis of eigenvectors of $A$, namely $Av_i = \lambda_i v_i$. Then $A \phi v_i = 2\eta_2(v_i) \xi_2 - 2\eta_3(v_i) \xi_3$. We have $\lambda_i A \phi v_i = 2\eta_2(v_i) \xi_2 - 2\eta_3(v_i) \xi_3$. If there is a $\lambda_i = 0$, then $\eta_2(v_i) = 0$, $\eta_3(v_i) = 0$, so $v_i \in D \cap W(1)$.

If $\lambda_i \neq 0$, then
\[
A \phi v_i = 2\eta_2(v_i) \lambda_i \xi_2 - 2\eta_3(v_i) \lambda_i \xi_3 \in C^\perp \xi.
\]
Thus $\xi_2 = \sum_i \rho_i v_i$, $\xi_3 = \sum_i \mu_i v_i$ and so $< A \xi_2, A \xi_3 > \subseteq D^\perp$. This means that $AD^\perp \subseteq D^\perp$ and $A(< \xi >) \subseteq < \xi >$. According to Theorem 1.1 the real hypersurface $M^{4m-1}$ is a tube of type $A$. 

8 The case $\xi \in D$

8.1 The distribution $H^\perp$

Since $\xi \in D$, we have $\eta_i(\xi) = 0$, $i = 1, 2, 3$ and thus for any $X \in TM$ and $i = 1, 2, 3$ we have $X\eta_i(\xi) = 0$, so $g(\nabla_X \xi_i, \xi) + g(\xi_i, \nabla_X \xi) = 0$. Thus $A\phi\xi_i = 0$, for $i = 1, 2, 3$.

From the relation $A\phi\xi_i = 0$, for $i = 2, 3$, we take $\phi\phi_1 A\phi\xi_i = 0$ and thus $A\phi_1 \phi^2 \xi_i = 0$, or $A\phi_1 \xi_i = 0$. Thus $A\xi_2 = 0$ and $A\xi_3 = 0$.

Now we calculate $A\xi$ and $A\xi_i$. We have $\phi\phi_1 A\xi = A\phi_1 \phi\xi$, so $\phi\phi_1 A\xi = 0$. Thus $\phi_1 A\xi \in \text{Ker} \phi$. Hence there is locally a smooth function $\mu$ with $\phi_1 A\xi = \mu \xi$. Applying $\phi_1$ on this we have $\phi_1^2 A\xi = \mu \phi\xi_1$, or $-A\xi + \eta_1 (A\xi) \xi_1 = \mu \phi\xi_1$, or $A\xi = \eta_1 (A\xi) \xi_1 - \mu \phi\xi_1$. We put $\rho = \eta_1 (A\xi)$, so $A\xi = \rho \xi_1 - \mu \phi\xi_1$. Also, $\phi\phi_1 A\xi_1 = A\phi_1 \phi\xi_1$, or $\phi\phi_1 A\xi_1 = A\phi_1^2 \xi_1$, or $\phi\phi_1 A\xi_1 = -A(\xi + \eta_1 (A\xi) \xi_1)$.

Thus $\phi\phi_1 A\xi_1 = -A\xi$ and $\phi\phi_1 A\xi_1 = -\rho \xi_1 + \mu \phi\xi_1$, or $\phi^2 \phi_1 A\xi_1 = -\rho \phi\xi_1 + \mu \phi^2 \xi_1$, or $\phi_1 A\xi_1 = -\rho \phi\xi_1 - \mu \xi_1$, or $\phi_1 A\xi_1 = \rho \phi\xi_1 + \mu \xi_1$, or $\phi_1^2 A\xi_1 = \rho \phi^2 \xi_1$.

Finally, putting $\alpha_1 = \eta_1 (A\xi_1)$ we have

$$A\xi_1 = \alpha_1 \xi_1 + \rho \xi_i.$$

Also, since $\phi\phi_1 A\xi_1 = A\phi_1 \phi\xi_1$, by using the relation (8.1) we finally have $\mu = 0$.

Summarising these cases we have $A\xi = \rho \xi_1$, $A\xi_1 = \alpha_1 \xi_1 + \rho \xi_i$, $A\xi_2 = 0$, $i = 2, 3$ and $A\phi\xi_j = 0$, $j = 1, 2, 3$. Thus, the distribution $H^\perp$ is $A$--invariant. For $X = \xi$ and $Y = \xi_i$, $i = 1, 2, 3$ the Codazzi equation is

$$(\nabla_\xi A) \xi_i - (\nabla_{\xi_i} A) \xi = 0.$$

For $i = 1$ we have $\nabla_\xi A\xi_1 = A\nabla_\xi \xi_1 - \nabla_{\xi_1} A\xi + A\nabla_{\xi_1} \xi = 0$. Hence

$$(\xi \alpha_1 - \xi_1 \rho) \xi_1 + (\alpha_1 q_3(\xi) - \rho q_3(\xi_1)) \xi_2 + (q_2(\xi_1) \rho - \alpha_1 q_2(\xi)) \xi_3 + (\xi \rho) \xi = 0.$$

From this we obtain $\xi \alpha_1 = \xi_1 \rho$, $\alpha_1 q_3(\xi) - \rho q_3(\xi_1) = 0$, $q_2(\xi_1) \rho - \alpha_1 q_2(\xi) = 0$, $\xi \rho = 0$ in a neighborhood of every point $p \in M$.

Also, for $X = \xi$ and $Y = \xi_i$, $i = 2, 3$, from the Codazzi equation, we get

$$-q_{i+1}(\xi) A\xi_{i+1} + q_{i+1}(\xi) A\xi_{i+2} - (\xi \rho) \xi_1 - \rho (q_3(\xi) \xi_2 - q_2(\xi) \xi_3) = 0$$

where the index $i$ is taken modulo 3.

From these we have $q_3(\xi) = 0$, $q_j(\xi) = 0$, $\xi_j \rho = 0$, $i, j = 2, 3$.

For $X = \xi$ and $Y = \phi\xi_i$, $i = 1, 2, 3$ the Codazzi equation gives

$$(\nabla_\xi A) \phi\xi_i - (\nabla_{\phi\xi_i} A) \xi = -4 \xi_i.$$

From this we have

$$\rho^2 \eta_1 (\xi_1) - (\phi \xi_i, \rho) \xi_1 - \rho q_3(\phi\xi_i) \xi_2 + \rho q_2(\phi\xi_i) \xi_3 = -4 \xi_i.$$

For $i = 1$ we have $\phi\xi_1 \rho = \rho^2 + 4$, $q_3(\phi\xi_1) = 0$, $q_2(\phi\xi_1) = 0$.

For $i = 2$ we have $\phi\xi_2 \rho = 0$, $q_3(\phi\xi_2) = \frac{4}{\rho}$, $q_2(\phi\xi_2) = 0$.

For $i = 3$ we have $\phi\xi_3 \rho = 0$, $q_3(\phi\xi_3) = 0$, $q_2(\phi\xi_2) = -\frac{2}{\rho}$.
For \( X = \xi_1 \) and \( Y = \xi_i \), \( i = 2, 3 \) the Codazzi equation is
\[
(\nabla_{\xi_1}A)\xi_i - (\nabla_{\xi_i}A)\xi_1 = \phi_1 \xi_i - \phi_i \xi_1 - 2g(\phi_2 \xi_1, \xi_i)\xi_2 - 2\eta_2(\xi_i)\xi_3
\]
from which we have
\[
-q_{i+2}(\xi_1)A\xi_{i+1} + q_{i+1}(\xi_1)A\xi_{i+2} - (\xi_i\alpha_1)\xi_1 - \alpha_1 q_3(\xi_i)\xi_2 + \alpha_1 q_2(\xi_i)\xi_3 - (\xi_i\rho)\xi = 0,
\]
where the index \( i \) is taken modulo 3. For \( i = 2 \) we have \( \xi_2\alpha_1 = q_3(\xi_1)\alpha_1 \), \( \xi_2\rho = \rho q_3(\xi_1) \), \( \alpha_1 q_1(\xi_2) = 0 \), \( \alpha_1 q_2(\xi_2) = 0 \).

For \( i = 3 \), we have \( \xi_3\alpha_1 = -q_2(\xi_1)\alpha_1 \), \( \xi_3\rho = -\rho q_2(\xi_1) \).

From the above it is obvious that \( q_2(\xi) = q_3(\xi) = 0 \) and \( q_2(\xi_1) = q_3(\xi_1) = 0 \). Thus we have \( \xi_2\alpha_1 = 0 \), \( \xi_2\rho = 0 \), \( \alpha_1 q_3(\xi_2) = 0 \), \( \alpha_1 q_2(\xi_2) = 0 \), \( \xi_3\alpha_1 = 0 \), \( \xi_3\rho = 0 \).

For \( X = \xi_1 \) and \( Y = \phi\xi_i \), \( i = 1, 2, 3 \) the Codazzi equation is
\[
(\nabla_{\xi_1}A)\phi\xi_i - (\nabla_{\phi\xi_i}A)\xi_1 = -2\eta_1(\xi_i)\xi + \phi_1 \phi\xi_i + \phi_i \phi\xi_1
\]
from which it occurs that
\[
(\alpha_1 \rho + \rho \alpha_1 \delta_{i1})\xi_1 + \rho A\xi_i - (\phi\xi_i \alpha_1)\xi_1 - \alpha_1 q_3(\phi\xi_i)\xi_2 + \alpha_1 q_2(\phi\xi_i)\xi_3 - (\phi\xi_i \rho)\xi = -2\eta_1(\xi_i)\xi + \phi_1 \phi\xi_i + \phi_i \phi\xi_1
\]
From the above we have \( \alpha_1 = 0 \), \( \phi\xi_2\rho = 0 \), \( \phi\xi_3\rho = 0 \).

### 8.2 The distribution \( \mathcal{H} \)

In our case the distribution \( \mathcal{H} \) is \( A \)-invariant. Let \( U \in \mathcal{H}^\perp \) and \( X \in \mathcal{H} \). Then \( 0 = g(AU, X) = g(U, AX) \). From this we obtain \( AX \in \mathcal{H} \). Thus the restriction \( A \mid \mathcal{H} \) of the shape operator \( A \) in the distribution \( \mathcal{H} \) is selfadjoint endomorphism at every point \( p \in M^{4m-1} \). Thus there is an orthonormal basis of eigenvectors of \( A \mid \mathcal{H} \), which can be extended smoothly in a neighborhood of the point \( p \in M^{4m-1} \).

In order to study the functions \( dp \) and \( q_i \), \( i = 1, 2, 3 \) we apply the Codazzi equation for \( X = \xi \) and \( Y = e \) with \( e \in \mathcal{H} \) and \( Ae = \lambda e \), where \( \lambda \) is a principal curvature. We have \( (\nabla_{\xi}A)e - (\nabla_{e}A)\xi = \phi e \). From this it occurs that \( (\xi\lambda)e + \lambda \nabla_{\xi}e - (e\rho)\xi_1 - \rho q_3(e)\xi_3 + \rho q_2(e)\xi_2 - \lambda \rho \phi_1 e + \lambda A\phi e = \phi e \). This can be decomposed in two equations
\[
(\xi\lambda)e + (\lambda I - A)\nabla_{\xi}e - (\rho \phi_1 e + \lambda A\phi e = \phi e \) and \( -(e\rho)\xi_1 - \rho q_2(e)\xi_3 + \rho q_3(e)\xi_2 = 0 \).

The last equation gives \( e\rho = 0 \), \( q_2(e) = 0 \), \( q_3(e) = 0 \), in a neighborhood of the point \( p \in M^{4m-1} \). So, for the function \( \rho \) we have \( X\rho = 0 \), for any \( X \perp \phi\xi_1 \) and \( \phi\xi_1\rho = \rho^2 + 4 \). Thus the function \( \rho \) depends only along the direction \( \phi\xi_1 \).

Let \( \gamma : (-\epsilon, \epsilon) \to M^{4m-1} \) be the geodesic with initial conditions \( \gamma(0) = p \), \( \dot{\gamma}(0) = \phi\xi_1 \), parameterized by arc length \( s \), in a neighborhood of the point \( p \).

From this we obtain \( \rho = 2\tan(2s + c) \). Also
\[
q_2(X) = 0, \quad \forall X \perp \phi\xi_3 \quad \text{and} \quad q_2(\phi\xi_3) = -\frac{4}{\rho}
\]
\[
q_3(X) = 0, \quad \forall X \perp \phi\xi_2 \quad \text{and} \quad q_2(\phi\xi_2) = \frac{4}{\rho}
\]
We have the following
Proposition 8.1. The transformation \( T = \phi \phi_1 : \mathcal{H} \to \mathcal{H} \) is an orthogonal, selfadjoint endomorphism and commutative with \( A \mid \mathcal{H} \). Thus there is a basis which diagonalize simultaneously \( T \) and \( A \mid \mathcal{H} \) in \( \mathcal{H} \).

Corollary 8.2. For every eigenspace \( V(\lambda) \subseteq \mathcal{H} \) of the shape operator \( A \) we have \( V(\lambda) \subseteq W(1) \), or \( V(\lambda) \subseteq W(-1) \).

If we put \( \kappa = \pm 1 \), then \( W(\kappa) = \{ X \in \mathcal{H} \mid \phi_1 X = -\kappa \phi X \} \).

Proposition 8.3. We have \( \phi W(\kappa) \subseteq W(\kappa) \), \( \phi_1 W(\kappa) \subseteq W(\kappa) \), \( \phi_2 W(\kappa) \subseteq W(-\kappa) \) and \( \phi_3 W(\kappa) \subseteq W(-\kappa) \).

Proof. Let \( e \) be locally defined vector field of the distribution \( W(\kappa) \). Then \( (\phi \phi_1) e = \kappa e \). Also it is obvious that \( e \in \mathcal{H} \). Thus \( e \in W(\kappa) \).

Similarly \( (\phi \phi_1) e = \kappa e \), so \( e \in W(\kappa) \).

Finally \( (\phi \phi_1) e = -\kappa e \), for \( i = 2, 3 \). \( \Box \)

Remark 8.1. If \( e \in W(\kappa) \), then \( Ae \in W(\kappa) \).

Proposition 8.4. Every eigenspace \( W(\kappa) \) of the distribution \( \mathcal{H} \) can be decomposed at most in two eigenspaces \( V(\lambda) \) and \( V(\mu) \) of the shape operator \( A \mid \mathcal{H} \).

Proof. i) Let \( e, \varepsilon \in W(1) \) with \( Ae = \lambda e \) and \( A\varepsilon = \mu \varepsilon \).

Then, the Codazzi equation has the form

\[
\nabla^c Ae - A\nabla^c e - \nabla^c Ae + A\nabla^c e = -2g(\phi e, \varepsilon)\xi - 2g(\phi_1 e, \varepsilon)\xi_1.
\]

Taking the inner product of this with \( \xi \) we have

\[
(-2\lambda - \rho)g(\xi, Ae) + (2 - \rho \lambda)g(\xi, \phi e) = 0
\]

It is obvious that \( Ae, \phi e \in W(1) \). Thus \( (-2\lambda - \rho)A\phi e + (2 - \rho \lambda)\phi e = 0 \). If we assume that \( -2\lambda - \rho = 0 \), equivalently \( \lambda = -\frac{\rho}{2} \), then \( 2 - \lambda \rho = 0 \), or \( \rho^2 + 4 = 0 \), which is impossible. Thus

\begin{equation}
A\phi e = \frac{-\rho \lambda}{\rho + 2\lambda} e, \quad e \in W(1).
\end{equation}

ii) Let \( e \in W(1) \) with \( Ae = \lambda e \) and \( \varepsilon \in W(-1) \) with \( A\varepsilon = \mu \varepsilon \).

The Codazzi equation, for \( X = e \) and \( Y = \varepsilon \), has the form

\begin{equation}
\nabla^c Ae - A\nabla^c e - \nabla^c Ae + A\nabla^c e = -2g(\phi_2 e, \varepsilon)\xi_2 - 2g(\phi_3 e, \varepsilon)\xi_3.
\end{equation}

Taking the inner product with \( \xi_2 \) we have \( \lambda g(A\phi_2 e, \varepsilon) = g(\phi_2 e, \varepsilon) \). But \( \phi_2 e \in W(-1) \) and \( W(-1) \) is \( A \)-invariant, thus \( A\phi_2 e \in W(-1) \). So

\[
A\phi_2 e = \frac{1}{\lambda} \phi_2 e, \quad e \in W(1).
\]

Similarly, taking the inner product of (8.3) with \( \xi_3 \), we have

\[
A\phi_3 e = \frac{1}{\lambda} \phi_3 e, \quad e \in W(1).
\]
Also, from (8.3), we have
\[ A\phi_2 \varepsilon = \frac{1}{\mu} \phi_2 \varepsilon, \quad \varepsilon \in W(-1). \]

Similarly, taking the inner product of (8.3) with \( \xi_3 \), we have
\[ A\phi_3 \varepsilon = \frac{1}{\mu} \phi_3 \varepsilon, \quad \varepsilon \in W(-1). \]

(iii) Let \( e, \varepsilon \in W(-1) \) with \( Ae = \lambda e \) and \( A\varepsilon = \mu \varepsilon \). The Codazzi equation gives
\[
(e\mu)\varepsilon + \mu \nabla e \varepsilon - A\nabla e \varepsilon - (\varepsilon\lambda)e - \lambda \nabla e \varepsilon + A\nabla e \varepsilon = -2g(\phi e, \varepsilon)\xi - 2g(\phi_1 e, \varepsilon)\xi_1.
\]
Taking the inner product with \( \xi \), we obtain
\[
(-2\lambda + \rho)g(e, A\phi e) = (-2 - \rho\lambda)g(e, \phi e),
\]
for any \( e \in W(-1) \). Also \( A\phi e, \phi e \in W(-1) \). Hence \((-2\lambda + \rho)A\phi e = (-2 - \rho\lambda)\phi e\). If \(-2\lambda + \rho = 0\), equivalently \( \lambda = \frac{\rho}{2} \), then \(-2 - \lambda\rho = 0\), so \( \rho^2 + 4 = 0 \), which is impossible. Thus
\[ A\phi e = \frac{2 + \rho\lambda}{-\rho + 2\lambda} \phi e, \quad e \in W(-1). \]

Now, according to the above, if \( e \in W(1) \) with \( Ae = \lambda e \), then \( A\phi_2 e = \frac{1}{\lambda} \phi_2 e \) with \( \phi_2 e \in W(-1) \). But
\[ A\phi_2 e = \frac{\rho + 2\lambda}{2\lambda - \rho} \phi_2 e, \]
from which we take
\[ A\phi_3 e = \frac{\rho + 2\lambda}{2 - \rho\lambda} \phi_3 e. \]

But \( A\phi_3 e = \frac{1}{\lambda} \phi_3 e \) and thus we have
\[ \lambda^2 + \rho \lambda - 1 = 0, \]
which provide two distinguishable principal curvatures
\[
\lambda_{1,2} = \frac{-\rho \pm \sqrt{\rho^2 + 4}}{2}.
\]
Since \( e \in W(-1) \) with \( Ae = \lambda e \), we have \( A\phi_2 e = \frac{1}{\lambda} \phi_2 e \) with \( \phi_2 e \in W(1) \). But
\[ A\phi_2 e = \frac{-\rho + 2\lambda}{2\lambda + \rho} \phi_2 e, \]
from which we have \( A\phi_3 e = \frac{-\rho + 2\lambda}{2 + \rho\lambda} \phi_3 e \). But \( A\phi_3 e = \frac{1}{\lambda} \phi_3 e \) and thus we have
\[ \lambda^2 - \rho \lambda - 1 = 0. \]
From this equation, we get two distinct principal curvatures
\[
\tilde{\lambda}_{1,2} = \frac{\rho \pm \sqrt{\rho^2 + 4}}{2}.
\]

It is obvious that \( \tilde{\lambda}_1 \tilde{\lambda}_1 = 1 \) and \( \tilde{\lambda}_2 \tilde{\lambda}_2 = 1 \) which implies that \( \lambda_1, \tilde{\lambda}_1 \), and \( \lambda_2, \tilde{\lambda}_2 \) are respectively pairwise inverse. \( \square \)
**Remark 8.2.** Since the function $\rho$ changes only along the geodesic $\gamma$ with $\dot{\gamma}(s) = \phi \xi_1$, the principal curvatures $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ change only along $\gamma$.

**Corollary 8.5.** If $e \in V(\lambda)$ and $e \in H$, then $\phi e \in V(\lambda)$ and $\phi_2 e, \phi_3 e \in V(\tilde{\lambda})$.

**Proof.** If $e \in V(\lambda) \cap W(1)$, then $A\phi e = \frac{2-\lambda \rho}{\rho+2 \lambda} \phi e$ and $\lambda^2 + \rho \lambda - 1 = 0$. So $(2 - \rho \lambda) = \lambda(\rho + 2 \lambda)$, namely $\lambda = \frac{2 - \rho \lambda}{\rho + 2 \lambda}$. Thus $A\phi e = \lambda e$, $A\phi_2 e = \frac{1}{\lambda} \phi e$ and $A\phi_3 e = \frac{1}{\lambda} \phi e$. Thus $A\phi_2 e = \tilde{\lambda} \phi_2 e$ and $A\phi_3 e = \tilde{\lambda} \phi_3 e$.

Similarly, if $e \in V(\lambda) \cap W(-1)$, then $A\phi e = \frac{2+\lambda \rho}{\rho+2 \lambda} \phi e$ and $\lambda^2 - \rho \lambda - 1 = 0$. So $\lambda(2 \lambda - \rho) = 2 + \rho \lambda$, namely $\lambda = \frac{2+\rho \lambda}{\rho+2 \lambda}$. Thus $A\phi e = \lambda e$, $A\phi_2 e = \frac{1}{\lambda} \phi e$ and $A\phi_3 e = \frac{1}{\lambda} \phi e$. Hence $A\phi_2 e = \tilde{\lambda} \phi_2 e$ and $A\phi_3 e = \tilde{\lambda} \phi_3 e$. \hfill $\square$

**Proposition 8.6.** For any eigenspace $V(\lambda)$ of the shape operator $A$ with $V(\lambda) \subseteq W(\kappa)$, $\kappa = \pm 1$, we have:

1. The $V(\lambda)$ is a non integrable distribution.
2. The $V(\lambda) \oplus <\xi, \xi_1>$ is an integrable distribution.

**Proof.** From the Codazzi equation, for $X = \xi$ and $Y = e \in H$ with $Ae = \lambda e$, the tangent component to $H$ gives $(\xi_1 e) + \lambda \nabla_\xi e - A \nabla_\xi e - \rho \lambda \phi_1 e + \lambda A \phi e = \phi e$.

Since $\lambda$ is a function of $\rho$ we have $\lambda_1 = 0$. Thus

$$ (8.6) \quad (\lambda I - A) \nabla_\xi e - \rho \lambda \phi_1 e + \lambda A \phi e = \phi e. $$

We examine the cases:

1. If $V(\lambda) \subseteq W(1)$, then $\phi_1 e = -\phi e$. The relation $(8.6)$ is $(\lambda I - A) \nabla_\xi e + (\lambda^2 + \rho \lambda - 1) \phi e = 0$. Hence $A \nabla_\xi e \in V(\lambda)$, because $\nabla_\xi e \in H$ and $\lambda_1, \lambda_2, \lambda_4$ are distinguishable.

Convex $2$. If $V(\lambda) \subseteq W(-1)$, then $\phi_1 e = \phi e$. The relation $(8.6)$ is $(\lambda I - A) \nabla_\xi e + (\lambda^2 - \rho \lambda - 1) \phi e = 0$. Hence $A \nabla_\xi e \in V(\lambda)$. From the Codazzi equation for $X = \xi_1$ and $Y = e \in H$ with $Ae = \lambda e$ we have

$$ (8.7) \quad (\lambda I - A) \nabla_{\xi_1} e - \rho \lambda \phi_1 e + \lambda A \phi_1 e = \phi_1 e $$

We now examine it through the subcases:

2i. $e \in W(1)$ The relation $(8.7)$ gives $(\lambda I - A) \nabla_{\xi_1} e = (\lambda^2 + \rho \lambda - 1) \phi_1 e = 0$. Thus $A \nabla_{\xi_1} e = \lambda \nabla_{\xi_1} e$. Namely $\nabla_{\xi_1} e \in V(\lambda)$.

2ii. $e \in W(-1)$

The relation $(8.7)$ gives $(\lambda I - A) \nabla_{\xi_1} e = (\lambda^2 - \rho \lambda - 1) \phi_1 e = 0$. Hence $A \nabla_{\xi_1} e = \lambda \nabla_{\xi_1} e$. Namely $\nabla_{\xi_1} e \in V(\lambda)$.

From the Codazzi equation for $X = e, Y = \phi e$ with $Ae = \lambda e$ we take

$$ A[\phi e, e] = \lambda[\phi e, e] - 2 \xi - 2g(\phi_1 e, \phi e) \xi_1. $$

Now, when $e \in W(1)$, we have $A[\phi e, e] = \lambda[\phi e, e] - 2 \xi + 2 \xi_1$. If $e \in W(-1)$, we have $A[\phi e, e] = \lambda[\phi e, e] - 2 \xi - 2 \xi_1$. Thus, for $e \in W(\kappa)$ we infer

$$ (8.8) \quad A[\phi e, e] = \lambda[\phi e, e] - 2 \xi + \kappa \xi_1. $$
Let \([\phi e, e] = u + u^\perp\) with \(u \in \mathcal{H}\) and \(u^\perp \in \mathcal{H}^\perp\). From (8.8) we get \(Au = \lambda u\) and \(Au^\perp = \lambda u^\perp - 2\xi + 2\xi_1\). Let
\[
(8.9) \quad u^\perp = \nu_1 \xi + \nu_2 \xi_1 + \nu_3 \xi_2 + \nu_4 \xi_3 + \nu_5 \phi \xi_1 + \nu_6 \phi \xi_2 + \nu_7 \phi \xi_3.
\]
Then
\[
(8.10) \quad Au^\perp = \rho \nu_1 \xi_1 + \rho \nu_2 \xi.
\]
The relation (8.2), according to (8.9) and (8.10), for \(e \in W(1)\), gives
\[
[\phi e, e] = u + 2\lambda + \rho \lambda - \rho \xi + 2\lambda - \rho \xi_1 + 2\lambda - \rho \xi_1
\]
Also, if \(e \in W(-1)\) then
\[
[\phi e, e] = u + 2\lambda - \rho \lambda - \rho \xi + 2\lambda - \rho \xi_1
\]
Thus \([\phi e, e] \in V(\lambda) < \xi, \xi_1 >\), in any one of the two cases.

The Codazzi equation for \(X = e, Y = \epsilon, e \perp \epsilon, Ae = \lambda \epsilon, \epsilon \perp \phi e\) and \(A \epsilon = \lambda \epsilon\) gives \((\nabla \epsilon A) \epsilon - (\nabla \epsilon A) e = 0\). From this we obtain \(A[\epsilon, e] = \lambda [\epsilon, e]\). Finally \([\xi, \xi_1] = \nabla \xi \xi_1 - \nabla \xi_1 \xi = \phi_1 \mathcal{A} \xi - \phi \mathcal{A} \xi_1 = 0\).

### 8.3 Variations by geodesics on the direction \(\phi \xi_1\)

Let \(u \in TM^{4m-1}\) be a locally definite vector field. For the curvature tensor \(\mathcal{R}\) of the space \(G_2(\mathbb{C}^{m+2})\) we have
\[
\begin{align*}
\mathcal{R}(u, \phi \xi_1) \phi \xi_1 &= u - g(u, \phi \xi_1) \phi \xi_1 + 3g(u, J \phi \xi_1) J \phi \xi_1 \\
&+ 3 \sum_{\nu=1}^{3} g(u, J_{\nu} \phi \xi_1) J_{\nu} \phi \xi_1 + 3 \sum_{\nu=1}^{3} g(J_{\nu} J \phi \xi_1, \phi \xi_1) J_{\nu} J u \\
&- 3 \sum_{\nu=1}^{3} g(u, J J_{\nu} \phi \xi_1) J_{\nu} J \phi \xi_1
\end{align*}
(8.11)
\]
Hence the Jacobi equation, for variations along the geodesics with direction \(\phi \xi_1\), takes the form
\[
u'' + \mathcal{R}(u, \phi \xi_1) \phi \xi_1 = 0,
\]
which in our case can be rewritten as
\[
u'' + u - g(u, \phi \xi_1) \phi \xi_1 + 3g(u, J \phi \xi_1) J \phi \xi_1 \\
+ 3 \sum_{\nu=1}^{3} g(u, J_{\nu} \phi \xi_1) J_{\nu} \phi \xi_1 + 3 \sum_{\nu=1}^{3} g(J_{\nu} J \phi \xi_1, \phi \xi_1) J_{\nu} J u \\
- 3 \sum_{\nu=1}^{3} g(u, J J_{\nu} \phi \xi_1) J_{\nu} J \phi \xi_1 = 0
(8.12)
\]
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We have the relations
\[ J_0 = -\xi, \quad J_1 = -\xi, \quad J_2 = \xi, \quad J_3 = \phi \xi \]
\[ JJ_0 = N, \quad JJ_1 = \xi, \quad JJ_3 = -\xi \]
By substituting these relations in (8.12) we have
\[ u'' + u + 3\eta(u)\xi + 3\eta_1(u)\xi_1 - \eta_2(u)\xi_2 - \eta_3(u)\xi_3 \]
\[ + \eta_1(\phi u)\phi \xi_1 - 3\eta_2(\phi u)\phi \xi_2 - 3\eta_3(\phi u)\phi \xi_3 - g(u, N)N = 0 \]
(8.13)

We recall the following

**Theorem 8.7.** Let \((M, g, J)\) be a quaternion Kähler manifold of real dimension \(\geq 8\) with non zero scalar curvature. Then every isometry \(f\) of \((M, g, J)\) is an automorphism of \((M, g, J)\). Namely, for any \(p \in M\) and any almost Hermitian structure \(J_1 \in J_f(p)\), there is an almost Hermitian structure \(J'_1 \in J_f(p)\) such that \(f_*J_1 = J'_1f_*\).

It is obvious that the above theorem is valued on the space \(G_2(C^{m+2})\).

**Proposition 8.8.** On the integral distribution \(V(\lambda) \oplus<\xi, \xi_1>\), where \(V(\lambda) \subset \mathcal{H}\) is an eigenspace of the shape operator \(A\) with eigenvalue \(\lambda\), the Jacobi equation has locally the solutions
\[ Y_1(t) = (\cos 2t - \frac{t}{2} \sin 2t)\xi \]
\[ Y_2(t) = (\cos 2t - \frac{t}{2} \sin 2t)\xi_1 \]
\[ Y_3(t) = (\cos t + \frac{t}{2} \lambda \sin t)B_y(t), \quad e \in V(\lambda), \]
where \(B_y(t)\) is the parallel displacement of \(e \in V(\lambda)\) along the geodesic with velocity vector \(\phi \xi_1\).

First we will prove the following

**Lemma 8.9.** If \(B_y(t)\) is the parallel displacement of \(e \in V(\lambda)\) along the geodesic \(\gamma(t)\) with initial conditions \(\gamma(0) = p\) and \(\gamma'(0) = \phi \xi_1\), then \(B_y(t) \in \mathcal{H}\).

**Proof.** We assume that \(B_y(t) = \alpha(t)\xi + \sum_{i=1}^{3} \beta_i(t)\xi_i + \sum_{i=1}^{3} \delta_i(t)\phi \xi_i + \zeta(t)N + u(t)\), where \(u(t) \in \mathcal{H}\). Differentiating along the direction \(\phi \xi_1\) we have
\[ 0 = \alpha'(t)\xi + \beta_1'(t)\xi_1 + \beta_2'(t)\xi_2 + \beta_3(\phi \xi_1)\xi_3 + \beta_3'(t)\xi_3 - \beta_3(\phi \xi_1)\xi_2 + \beta_3'(t)\xi_2 + \beta_3(\phi \xi_1)\phi \xi_3 + \beta_3'(t)\phi \xi_3 + \beta_3(\phi \xi_1)\phi \xi_2 + \zeta'(t)N + \nabla_{\phi \xi_1} u(t) \]
If \(X \in \mathcal{H}^\perp\), then \(g(X, \nabla_{\phi \xi_1} u(t)) = -g(\nabla_{\phi \xi_1} X, u(t)) = 0\). Namely \(\nabla_{\phi \xi_1} u(t) = 0\), so \(u(t)\) is a parallel vector field with \(u(0) = B_y(0) = e\) and thus it is identified with \(B_y(t)\).

We extend this identification to the interval \([0, r]\) in such a way that the geodesic is in the hypersurface \(M\). Since \(|u(0)| = 1\), there is an open neighborhood such that \(u(t) \neq 0\). It is obvious that \(\alpha(t) = 0\), \(\beta_1(t) = 0\), \(\zeta(t) = 0\), and \(\delta_1(t) = 0\). Also \(\phi \xi_1 g(u(t), u(t)) = 2g(\nabla_{\phi \xi_1} u(t), u(t)) = 0\). Thus in a neighborhood of zero we have \(|u(t)| = 1\). Hence \(u(t)\) can be extended to \([0, r]\), because \(B_y(t)\) extended on \([0, r]\) and \(|B_y(t)| = 1\). Thus \(B_y(t) \equiv u(t)\). So \(B_y(t) \in \mathcal{H}\).

The proof of Proposition 8.8. We will check that the vector fields \(Y_1(t), Y_2(t)\) and \(Y_3(t)\) are solutions of the Jacobi equation. For \(Y_1(t)\) we have \(Y_1'(t) = (-2\sin(2t) - 3\eta_{11}(u)\xi_1 - 3\eta_{21}(u)\xi_2 - 3\eta_{31}(u)\xi_3)\).
\( \rho \cos(2t))\xi \) and \( Y_{e}''(t) = (-\rho \sin(2t))\xi \). Replacing these in the Jacobi equation we notice that it is satisfied. For \( Y_{e}(t) \), we have

\[
Y_{e}'(t) = (- \sin t + \lambda \cos t B_{e}(t) \quad \text{and} \quad Y_{e}''(t) = (- \cos t - \lambda \sin t B_{e}(t)).
\]

Replacing these in the Jacobi equation we see that it is satisfied as well.

Now we extend the solutions along the geodesic and outside the hypersurface \( M^{4m-1} \) in the space \( \mathbb{G}_2(\mathbb{C}^{m+2}) \). We know that every isometry is an automorphism in the space \( \mathbb{G}_2(\mathbb{C}^{m+2}) \) (see Theorem 8.7). We consider the isometry \( f \) which transfers the vectors of \( T_{p}M \) at any point along the geodesics which start from \( p \). Thus we consider the geodesic of the space \( \mathbb{G}_2(\mathbb{C}^{m+2}) \) with velocity vector the parallel extension of \( \phi \xi_{1} \), namely \( \tilde{\phi} \xi_{1} \). Then at the point \( f(p) \) there exist three structures \( J_1' |_{f(p)} \), \( J_2' |_{f(p)} \), \( J_3' |_{f(p)} \) such that the vectors \( J_1' |_{f(p)} (Z) \), \( J_2' |_{f(p)} (Z) \), \( J_3' |_{f(p)} (Z) \) with \( Z \in \{ \xi, \xi_{1}, \phi \xi_{1} \} \) are the parallel transports of the vectors \( J_1 |_{p} (Z) \), \( J_2 |_{p} (Z) \), \( J_3 |_{p} (Z) \) with \( Z \in \{ \xi, \xi_{1}, \phi \xi_{1} \} \). Thus the solutions of the proposition 8.8 extend along the geodesic \( \gamma \) and continue to be solutions of the equation (8.12).

Now we chose the focal map \( \Phi \mathcal{f}_{p} = \exp_{p}(r \phi \xi_{1}) \) with \( r \neq 0 \), such that \( Y_{e}(r) = 0 \), for any \( e \in V(\lambda) \). We distinguish the cases:

Case a

\[
\lambda = \lambda_{1} = -\rho + \sqrt{\rho^{2} + 4}
\]
equivalently \( e \in W(1) \cap \mathcal{H} \). We have \( Y_{e}(r) = 0 \), therefor \( \cos(r) + \lambda \sin(r) = 0 \). Thus

\[
\lambda = -\cot(r).
\]

In this case we have \( Y_{\xi}(r) \neq 0 \) and \( Y_{\xi_{1}}(r) \neq 0 \). Indeed, let

\[
\cos(2r) - \frac{\rho}{2} \sin(2r) = 0, \quad \text{or} \quad \rho = 2 \cot(2r).
\]

On the other hand we have

\[
\rho = 2 \tan(2s).
\]

Thus

\[
2s = \frac{\pi}{2} - 2r.
\]

Using (8.16) in (8.14), we have

\[
\lambda = \frac{1 - \sin(2s)}{\cos(2s)} \quad \text{with} \quad -\frac{\pi}{2} < s < \frac{\pi}{4}.
\]

Using (8.15) in (8.18) we have

\[
\tan(r) = \frac{-\cos(2s)}{1 - \sin(2s)}.
\]
Also from (8.17) we have
\[ \tan(r) = \frac{-\cos(\frac{\pi}{2} - 2r)}{1 - \sin(\frac{\pi}{2} - 2r)} \]
or \[ \sin^2(r) + \cos^2(r) = 0, \] which makes a contradiction.

**Case b** We suppose that there is a pair of eigenvalues \( \lambda_2, \tilde{\lambda}_2 \). We check what happens when \( V(\tilde{\lambda}_2) \subseteq W(-1) \cap \mathcal{H} \) with
\[
(8.19) \quad \tilde{\lambda}_2 = \rho - \sqrt{\rho^2 + 4}.
\]
The relation \( Y_\xi(r) = 0 \) is equivalent with
\[
(8.20) \quad \lambda = -\cot(r).
\]
In this case we have \( Y_\xi(r) \neq 0 \) and \( Y_{\xi_1}(r) \neq 0 \). Equivalently, \( \cos(2r) - \frac{\rho}{2} \sin(2r) \neq 0 \).

In fact, let \( \cos(2r) - \frac{\rho}{2} \sin(2r) = 0 \). Then
\[
(8.21) \quad \rho = 2 \cot(2r).
\]
On the other hand, we have \( \rho = 2 \tan(2s) \), and hence
\[
(8.22) \quad 2s = \frac{\pi}{2} - 2r.
\]
We substitute (8.21) in (8.19) and we get
\[
(8.23) \quad \tilde{\lambda}_2 = \frac{\sin(2s) - 1}{\cos(2s)}.
\]
From (8.20) and (8.23) we infer
\[
-\cot(r) = \frac{\sin(2s) - 1}{\cos(2s)},
\]
which is valid only for \( s = \frac{\pi}{2} \), which is a contradiction.

Thus for \( s \) near \( s_0 = 0 \) there is not exist horizontal lift with \( \ker(\Phi_r) = V(\lambda) \oplus <\xi_1, \xi> \). Locally, in a neighborhood of \( s_0 = 0 \), for a given \( r \) we have \( \ker(\Phi_r) = V(\lambda) \), which is a contradiction, since it is not an integrable distribution, according to the implicit function theorem. \( \square \)

**Remark 8.3.** The case where we take the eigenvalues \( \lambda_2, \tilde{\lambda}_1 \) has the same termination, because if there exists the eigenvalue \( \lambda_2 \), then there exists and the eigenvalue \( \lambda_1 \) and thus we have the same contradiction. Also, if we have the eigenvalue \( \lambda_1 \), then we have the eigenvalue \( \lambda_2 \), too, so we have a contradiction, too.
8.4 The case \( m = 2 \)

If \( m = 2 \), the tangent bundle \( TM \) of the hypersurface \( M^7 \) locally is given by \( TM = \langle \xi, \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3 \rangle \). We will prove that the submanifold \( P \) with tangent bundle the integral distribution \( T_0 = \langle \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3 \rangle \) is totally geodesic. The second fundamental form \( \overline{B} \) of \( P \) in the space \( \mathbb{G}_2(\mathbb{C}^{m+2}) \) is given by

\[
(8.24) \quad \nabla_X Y = \tilde{\nabla}_X Y + \overline{B}(X, Y),
\]

for any \( X, Y \in T_0 \), where \( \tilde{\nabla} \) is the covariant derivative of \( P \). Also we have

\[
(8.25) \quad \nabla_X Y = \nabla_X Y + g(AX, Y)N,
\]

for any \( X, Y \in T_0 \). From (8.24) and (8.25), we have

\[
(8.26) \quad \nabla_X Y + g(AX, Y)N = \tilde{\nabla}_X Y + \overline{B}(X, Y),
\]

for any \( X, Y \in T_0 \). It is easy to see that \( g(AX, Y) = 0 \), for any \( X, Y \in T_0 \). Also, we have \( \nabla_X Y = \tilde{\nabla}_X Y \) for any \( X, Y \in T_0 \). Thus, from (8.26), \( \overline{B}(X, Y) = 0 \), for any \( X, Y \in T_0 \). So, \( P \) is totally geodesic. But, we know that the only maximal totally geodesic submanifolds, for \( m = 2 \), are \( \mathbb{R}^+ \mathbb{G}_2(3) \) (embedding as a complex submanifold), \( (S^n \times S^3)/\mathbb{Z}_2 \) with \( \alpha + \beta = 4 \) and \( CP^2 \), which have real dimension \( \leq 4 \), a contradiction. Thus, for \( m = 2 \), there not exist hypersurfaces \( M \) with our assumption.

9 Tubes of type \( A \)

It is easy to see that every tube of type \( A \) satisfies the relation \( \phi\phi_1 A = A\phi_1 \phi \). Indeed, for real hypersurfaces of type \( A \), we have

\[ A\xi = \alpha \xi \quad \text{and} \quad \xi \in D^\perp \quad \text{with} \quad \xi = \xi_1, \quad JN = J_1 N, \quad AD \subseteq D. \]

Also we have that the eigenspaces are: \( T_{\alpha} = \langle \xi > = \langle \xi_1 > \) with \( \dim T_{\alpha} = 1 \) and eigenvalue \( \alpha = \sqrt{8} \cot(\sqrt{8} r) \), \( T_{\beta} = \langle \xi_2, \xi_3 \rangle > \) with \( \dim T_{\beta} = 2 \) and eigenvalue \( \beta = \sqrt{2} \cot(\sqrt{2} r) \), \( T_{\lambda} = \{ X | X \perp D^\perp, \ JX = J_1 X \} \) with \( \dim T_{\lambda} = 2m - 2 \) and eigenvalue \( \lambda = -\sqrt{2} \tan(\sqrt{2} r), \quad T_{\mu} = \{ X | X \perp D^\perp, \ JX = -J_1 X \} \) with \( \dim T_{\mu} = 2m - 2 \) and eigenvalue \( \mu = 0 \).

We can check that for every eigenspace \( T_i, \ i = \alpha, \beta, \lambda, \mu \) we have \( \phi\phi_1 T_i = A\phi_1 \phi T_i \). Thus \( \phi\phi_1 AX = A\phi_1 \phi X \), for any \( X \in TM \). So the hypersurfaces of type \( A \) satisfy the relation \( \phi\phi_1 A = A\phi_1 \phi \).

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