 Derived curvature tensors in generalized Finsler space

Svetislav M. Minčić and Milan Lj. Zlatanović

Abstract. Using the Shamhoke’s idea given by (1.3), we consider generalized Finsler space \((\mathbb{GF}_N)\). Based on two kinds of covariant derivative of a tensor and non-symmetric connection \(P^*\) we have 10 Ricci type identities, 3 curvature tensors and 15 magnitudes which we have called "curvature pseudotensors" in \(\mathbb{GF}_N\). In the present work, using above mentioned Ricci type identities, we obtain "combined" Ricci type identities and 8 new curvature tensors-"derived" curvature tensors.


Key words: generalized Finsler spaces, Ricci type identities, covariant derivative of the first kind, covariant derivative of the second kind, non-symmetric connection, curvature tensor, derived curvature tensor, curvature pseudotensor.

1 Introduction

The generalized Finsler space \((\mathbb{GF}_N)\) is a differentiable manifold with non-symmetric basic tensor

\[
g_{ij}(x^1, \ldots, x^N, \dot{x}^1, \ldots, \dot{x}^N) \equiv g_{ij}(x, \dot{x}),
\]

where

\[
g_{ij}(x, \dot{x}) \neq g_{ji}(x, \dot{x}), \quad (g = \det(g_{ij}) \neq 0, \ \dot{x} = dx/dt).
\]

Based on (1.1), one can defined the symmetric respectively anti-symmetric part of \(g_{ij}\)

\[
g_{ij} = \frac{1}{2}(g_{ij} + g_{ji}), \quad g_{ij} = \frac{1}{2}(g_{ij} - g_{ji}),
\]

where, following [13], is

\[
a) \ g_{ij}(x, \dot{x}) = \frac{1}{2} \frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j}, \quad b) \ \frac{\partial g_{ij}(x, \dot{x})}{\partial \dot{x}^k} = 0,
\]

where \(F(x, \dot{x})\) is a metric function in \(\mathbb{GF}_N\), having the properties known from the theory of usual Finsler space \((\mathbb{F}_N)\) (see e.g. [12, 14, 16]). That means that the distance \(ds\) between two neighboring points \(x\) and \(x + dx\) is given by \(ds = F(x, dx)\) and


1. $F(x, \dot{x})$ is continuously differentiable at least four times in its $2N$ arguments.

2. $F(x, \dot{x}) > 0$ providing all $dx^i$ are not 0.

3. $F(x, x)$ is positively homogeneous of the 1st degree in $\dot{x}$, i.e. $F(x, k\dot{x}) = k\dot{F}(x, \dot{x})$, $k > 0$.

4. $\frac{\partial^2 F^2(x, \dot{x})}{\partial x^i \partial x^j} \xi^i \xi^j > 0$ for any given $\dot{x}$, and $\sum_i (\xi^i)^2 > 0$, $\xi^i \in R$.

The lowering and the raising of indices one defines by the tensors $g_{ij}$ and $h^{ij}$ respectively, where $h^{ij}$ is defined as follows

\begin{equation}
   g_{ij} h^{jk} = \delta^k_i, \quad (g = \det(g_{ij}) \neq 0).
\end{equation}

We can define generalized Cristoffel symbols of the 1st and the 2nd kind:

\begin{align}
   \gamma_{i, jk} & = \frac{1}{2} (g_{ji,k} - g_{jk,i} + g_{ik,j}) \neq \gamma_{i, kj}, \\
   \gamma^i_{jk} & = h^{ip} g_{pj,k} = \frac{1}{2} h^{ip} (g_{jp,k} - g_{jk,p} + g_{pk,j}) \neq \gamma^i_{kj},
\end{align}

where, e.g., $g_{ji,k} = \partial g_{ij} / \partial x^k$.

Then we have

\begin{equation}
   \gamma^i_{jk} g_{kp} = \gamma_{s, jk} h^{ps} g_{ip} = \gamma_{s, jk} \delta^s_i = \gamma_{i, jk}.
\end{equation}

Introducing a tensor $C_{ij,k}$ as like at $F_N$, we have

\begin{equation}
   C_{ijk}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2} g_{ij, \dot{x}^k} \overset{\text{def}}{=} \frac{1}{2} g_{ij, \dot{x}^k} = \frac{1}{4} F^2 \dot{x}^i \dot{x}^j \dot{x}^k,
\end{equation}

where "$\overset{\text{def}}{=}"$ signifies "equal based on (1.3b)". We see that $C_{ijk}$ is symmetric in relation to each pair of indices. Also, we have

\begin{equation}
   C_{ijk} \overset{\text{def}}{=} h^{ip} C_{pj,k} \overset{\text{(1.8)}}{=} h^{ip} C_{jp,k} = h^{ip} C_{j kp}.
\end{equation}

With help of coefficients

\begin{equation}
   P^i_{jk} = \gamma^i_{jk} - C^i_{kp,j} \dot{x}^p \neq P^i_{kj},
\end{equation}

one obtains coefficients of non-symmetric affine connections in the Rund’s sense [15, 16]:

\begin{align}
   P_{jk}^i & = \gamma^i_{jk} - h^{sl} (C_{jlp}^k P_{ps}^p + C_{ksp}^p P_{js}^p - C_{jkp}^p P_{qs}^p) \dot{x}^s \neq P_{kj}^i, \\
   P_{i,jk}^* & = P_{jk}^* g_{ij} = \gamma_{i,jk} - (C_{i,jp}^k P_{ps}^p + C_{ikp}^p P_{js}^p - C_{jkp}^p P_{js}^p) \dot{x}^s \neq P_{i,kj}^*.
\end{align}

In GF$_N$ we denote double anti-symmetric and double symmetric part for connection $P^*$ respectively:

\begin{align}
   & a) \quad T_{jk}^*(x, \dot{x}) = P_{jk}^* \overset{\text{def}}{=} P_{jk}^* - P_{kj}^*, \quad b) \quad P_{ij}^* = P_{jk}^* + P_{kj}^*.
\end{align}
where $T^*_k$ is the torsion tensor.

We define two kinds of covariant derivative of a tensor in the space $\mathbb{G}F_N$. For example, for a tensor $a_{t_1...t_n}^{r_1...r_u}(x, \xi)$ we have covariant derivative of the $1^{st}$ and $2^{nd}$ kind (1.14)

$$a_{t_1...t_n}^{r_1...r_u} = a_{...,m} + a_{...,\xi} \xi_m + \sum_{\alpha=1}^n P^\alpha_{m \nu} a_{t_1...t_\nu}^{r_1...r_{\alpha-1} p r_{\alpha+1}...r_u} - \sum_{\beta=1}^v P^\beta_{m \nu} a_{t_1...t_\nu}^{r_1...r_{\beta-1} p \beta_{\beta+1}...r_u}$$

where $\xi(x)$ is an arbitrary tangent vector in the tangent space $T_N(x)$, and $a_{...,\xi} = \partial a_{...,\xi}/\partial \xi^p$.

2 Ricci type identities in $\mathbb{G}F_N$

It is known that in Finsler space one Ricci identity exists for $\delta$-derivation, corresponding to alternating covariant derivative of the $2^{nd}$ order. In the case of generalized Finsler space there exist 10 possibilities to form the difference $a_{t_1...t_n |m| n}^{r_1...r_u} - a_{t_1...t_n |m| n}^{r_1...r_u} (\lambda, \mu, \nu, \omega = 1, 2)$, where $\mid 1, 2 \mid$ denotes two kinds of covariant derivative based on (1.14), and from that one obtains 10 Ricci type identities, three curvature tensors $\tilde{K}, \tilde{K}, \tilde{K}$, and 15 magnitudes $\tilde{A}_1, \ldots, \tilde{A}_{15}$ which will be called "curvature pseudotensors". The mentioned possibilities are obtained for

$$(\lambda, \mu; \nu, \omega) \in \{(1, 1; 1, 1), (2, 2; 2, 2), (1, 2; 1, 2), (2, 1; 2, 1), (1, 1; 2, 2), (1, 1; 2, 2), (1, 2; 1, 2), (2, 2; 2, 1), (1, 2; 2, 1)\}$$

i.e.

$$(2.2) \ a_{t_1...t_n \lambda |m| n}^{r_1...r_u} - a_{t_1...t_n \lambda |m| n}^{r_1...r_u} = \sum_{\alpha=1}^u \tilde{K}^\alpha_{nm} (p_{r_\alpha}) a_{...,p} - \sum_{\beta=1}^v \tilde{K}^\beta_{mn} (t_{\beta}) a_{...,p} = \mp (-1)^\lambda T^{mp}_{mn} a_{t_1...t_n \lambda \lambda |m| n}^{r_1...r_u} , \ \lambda = 1, 2,$$

$$(2.3) \ a_{t_1...t_n 1|1}^{r_1...r_u} - a_{t_1...t_n 1|1}^{r_1...r_u} = \sum_{\alpha=1}^u \tilde{A}^\alpha_{nm} (p_{r_\alpha}) a_{...,p} - \sum_{\beta=1}^v \tilde{A}^\beta_{mn} (t_{\beta}) a_{...,p} + a_{...,p} a_{...,p}^{-1} = T^{mp}_{mn} a_{t_1...t_n 1|1}^{r_1...r_u} , \ a_{...,p}^{mn} = T^{mp}_{mn},$$

$$(2.4) \ a_{t_1...t_n 1|1}^{r_1...r_u} - a_{t_1...t_n 1|1}^{r_1...r_u} = \sum_{\alpha=1}^u \tilde{A}^\alpha_{nm} (p_{r_\alpha}) a_{...,p} - \sum_{\beta=1}^v \tilde{A}^\beta_{mn} (t_{\beta}) a_{...,p} - a_{...,p} a_{...,p}^{-1} = - T^{mp}_{mn} a_{t_1...t_n 1|1}^{r_1...r_u} ,$$
\[
(2.5) \quad a_{t_1 \ldots t_u}^{r_1 \ldots r_u} - a_{t_1 \ldots t_u}^{r_1 \ldots r_u} = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \frac{\bar{F}_{\alpha\beta}^{r_\alpha p} (p)}{r_{\alpha}} a^{..} - \sum_{\beta=1}^{v} \frac{\bar{F}_{\beta}^{r_{\beta}} (p)}{r_{\beta}} a^{..} \\
+ a^{..} \langle mn \rangle > + a^{..} \langle mn \rangle > - P^{pp}_{mn} (a^{..} | p) a^{..} | p, \\
\]

\[
(2.6) \quad a_{t_1 \ldots t_u}^{r_1 \ldots r_u} - a_{t_1 \ldots t_u}^{r_1 \ldots r_u} = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \frac{\bar{F}_{\alpha\beta}^{r_\alpha p} (p)}{r_{\alpha}} a^{..} - \sum_{\beta=1}^{v} \frac{\bar{F}_{\beta}^{r_{\beta}} (p)}{r_{\beta}} a^{..} \\
+ a^{..} \langle mn \rangle > + a^{..} \langle mn \rangle > , \\
\]

\[
(2.7) \quad a_{t_1 \ldots t_u}^{r_1 \ldots r_u} - a_{t_1 \ldots t_u}^{r_1 \ldots r_u} = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \frac{\bar{F}_{\alpha\beta}^{r_\alpha p} (p)}{r_{\alpha}} a^{..} - \sum_{\beta=1}^{v} \frac{\bar{F}_{\beta}^{r_{\beta}} (p)}{r_{\beta}} a^{..} \\
+ a^{..} \langle mn \rangle > + a^{..} \langle mn \rangle > - P^{pp}_{mn} a^{..} | p + P^{pp}_{mn} a^{..} | p, \\
\]

\[
(2.8) \quad a_{t_1 \ldots t_u}^{r_1 \ldots r_u} - a_{t_1 \ldots t_u}^{r_1 \ldots r_u} = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \frac{\bar{F}_{\alpha\beta}^{r_\alpha p} (p)}{r_{\alpha}} a^{..} - \sum_{\beta=1}^{v} \frac{\bar{F}_{\beta}^{r_{\beta}} (p)}{r_{\beta}} a^{..} \\
- a^{..} \langle mn \rangle > + a^{..} \langle mn \rangle > + P^{pp}_{mn} a^{..} | p - P^{pp}_{mn} a^{..} | p, \\
\]

\[
(2.9) \quad a_{t_1 \ldots t_u}^{r_1 \ldots r_u} - a_{t_1 \ldots t_u}^{r_1 \ldots r_u} = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \frac{\bar{F}_{\alpha\beta}^{r_\alpha p} (p)}{r_{\alpha}} a^{..} - \sum_{\beta=1}^{v} \frac{\bar{F}_{\beta}^{r_{\beta}} (p)}{r_{\beta}} a^{..} \\
- a^{..} \langle mn \rangle > + a^{..} \langle mn \rangle > , \\
\]

\[
(2.10) \quad a_{t_1 \ldots t_u}^{r_1 \ldots r_u} - a_{t_1 \ldots t_u}^{r_1 \ldots r_u} = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \frac{\bar{F}_{\alpha\beta}^{r_\alpha p} (p)}{r_{\alpha}} a^{..} - \sum_{\beta=1}^{v} \frac{\bar{F}_{\beta}^{r_{\beta}} (p)}{r_{\beta}} a^{..} \\
- P^{pp}_{mn} (a^{..} | p - a^{..} | p). \\
\]

The identity (2.10) can be written in another form. Namely, counting the difference in the last brackets in (2.10), one obtains

\[
(2.11) \quad a_{t_1 \ldots t_u}^{r_1 \ldots r_u} - a_{t_1 \ldots t_u}^{r_1 \ldots r_u} = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \frac{\bar{K}_{\alpha\beta}^{r_\alpha p} (p)}{r_{\alpha}} a^{..} - \sum_{\beta=1}^{v} \frac{\bar{K}_{\beta}^{r_{\beta}} (p)}{r_{\beta}} a^{..} . \\
\]

The induced denotations are

\[
(2.12) \quad \bar{K}_{1}^{i} = P_{jmn}^{i} - P_{jmn}^{i} + P^{pp}_{mn} p^{i} - P^{pp}_{mn} p^{i} + P^{pp}_{mn} \xi^{s} - P^{pp}_{mn} \xi^{s}, \\
(2.13) \quad \bar{K}_{2}^{i} = P_{m}^{i} - P_{mn}^{i} + P^{pp}_{mn} p^{i} - P^{pp}_{mn} p^{i} + P^{pp}_{mn} \xi^{s} - P^{pp}_{mn} \xi^{s}, \\
(2.14) \quad \bar{K}_{3}^{i} = \bar{A}_{jmn}^{i} + P_{mn}^{p} p^{i} - P_{jmn}^{i} - P_{i}^{n} - P^{pp}_{mn} p^{i} - P^{pp}_{mn} p^{i} + P^{pp}_{mn} \xi^{s} - P^{pp}_{mn} \xi^{s} . \\
\]
Derived curvature tensors in generalized Finsler space

(2.15) $\tilde{A}_1^{jm,n} = p^{*i}_{jm,n} - p^{vi}_{jn,m} + p^{vp}_{jm,n} - p^{vp}_{jn,m} p^{*i}_{jm,n} + p^{*p}_{jm,n} s^s_{jn,s,n} - p^{*p}_{jn,s,n} s^s_{jm,s,n};$

(2.16) $\tilde{A}_2^{jm,n} = p^{*i}_{jm,n} - p^{vi}_{jn,m} + p^{vp}_{jm,n} - p^{vp}_{jn,m} p^{*i}_{jm,n} + p^{*p}_{jm,n} s^s_{jn,s,n} - p^{*p}_{jn,s,n} s^s_{jm,s,n};$

(2.17) $\tilde{A}_3^{jm,n} = p^{*i}_{mj,n} - p^{vi}_{nj,m} + p^{vp}_{mj,n} - p^{vp}_{nj,m} p^{*i}_{mj,n} + p^{*p}_{mj,n} s^s_{nj,s,m} - p^{*p}_{nj,s,m} s^s_{mj,s,m};$

(2.18) $\tilde{A}_4^{jm,n} = p^{*i}_{mj,n} - p^{vi}_{nj,m} + p^{vp}_{mj,n} - p^{vp}_{nj,m} p^{*i}_{mj,n} + p^{*p}_{mj,n} s^s_{nj,s,m} - p^{*p}_{nj,s,m} s^s_{mj,s,m};$

(2.19) $\tilde{A}_5^{jm,n} = p^{*i}_{jm,n} - p^{vi}_{nj,m} + p^{vp}_{jm,n} - p^{vp}_{nj,m} p^{*i}_{jm,n} + p^{*p}_{jm,n} s^s_{nj,s,n} - p^{*p}_{nj,s,n} s^s_{jm,s,n};$

(2.20) $\tilde{A}_6^{jm,n} = p^{*i}_{jm,n} - p^{vi}_{nj,m} + p^{vp}_{jm,n} - p^{vp}_{nj,m} p^{*i}_{jm,n} + p^{*p}_{jm,n} s^s_{nj,s,n} - p^{*p}_{nj,s,n} s^s_{jm,s,n};$

(2.21) $\tilde{A}_7^{jm,n} = p^{*i}_{jm,n} - p^{vi}_{nj,m} + p^{vp}_{jm,n} - p^{vp}_{nj,m} p^{*i}_{jm,n} + p^{*p}_{jm,n} s^s_{nj,s,n} - p^{*p}_{nj,s,n} s^s_{jm,s,n};$

(2.22) $\tilde{A}_8^{jm,n} = p^{*i}_{jm,n} - p^{vi}_{nj,m} + p^{vp}_{jm,n} - p^{vp}_{nj,m} p^{*i}_{jm,n} + p^{*p}_{jm,n} s^s_{nj,s,n} - p^{*p}_{nj,s,n} s^s_{jm,s,n};$

(2.23) $\tilde{A}_9^{jm,n} = p^{*i}_{mj,n} - p^{vi}_{nj,m} + p^{vp}_{mj,n} - p^{vp}_{nj,m} p^{*i}_{mj,n} + p^{*p}_{mj,n} s^s_{nj,s,m} - p^{*p}_{nj,s,m} s^s_{mj,s,m};$

(2.24) $\tilde{A}_{10}^{jm,n} = p^{*i}_{mj,n} - p^{vi}_{nj,m} + p^{vp}_{mj,n} - p^{vp}_{nj,m} p^{*i}_{mj,n} + p^{*p}_{mj,n} s^s_{nj,s,m} - p^{*p}_{nj,s,m} s^s_{mj,s,m};$

(2.25) $\tilde{A}_{11}^{jm,n} = p^{*i}_{mj,n} - p^{vi}_{nj,m} + p^{vp}_{mj,n} - p^{vp}_{nj,m} p^{*i}_{mj,n} + p^{*p}_{mj,n} s^s_{nj,s,m} - p^{*p}_{nj,s,m} s^s_{mj,s,m};$

(2.26) $\tilde{A}_{12}^{jm,n} = p^{*i}_{mj,n} - p^{vi}_{nj,m} + p^{vp}_{mj,n} - p^{vp}_{nj,m} p^{*i}_{mj,n} + p^{*p}_{mj,n} s^s_{nj,s,n} - p^{*p}_{nj,s,n} s^s_{mj,s,n};$

(2.27) $\tilde{A}_{13}^{jm,n} = p^{*i}_{mj,n} - p^{vi}_{nj,m} + p^{vp}_{mj,n} - p^{vp}_{nj,m} p^{*i}_{mj,n} + p^{*p}_{mj,n} s^s_{nj,s,n} - p^{*p}_{nj,s,n} s^s_{mj,s,n};$

(2.28) $\tilde{A}_{14}^{jm,n} = p^{*i}_{mj,n} - p^{vi}_{nj,m} + p^{vp}_{mj,n} - p^{vp}_{nj,m} p^{*i}_{mj,n} + p^{*p}_{mj,n} s^s_{nj,s,n} - p^{*p}_{nj,s,n} s^s_{mj,s,n};$

(2.29) $\tilde{A}_{15}^{jm,n} = p^{*i}_{jm,n} - p^{vi}_{nj,m} + p^{vp}_{jm,n} - p^{vp}_{nj,m} p^{*i}_{jm,n} + p^{*p}_{jm,n} s^s_{nj,s,n} - p^{*p}_{nj,s,n} s^s_{jm,s,n};$

\[
\begin{pmatrix} p \\ r_\alpha \end{pmatrix} a^{r_{1\ldots r_n}} = a^{r_{1\ldots r_n}}_{1\ldots f_o}, \quad \begin{pmatrix} t_\beta \\ p \end{pmatrix} a^{t_{1\ldots t_o}} = a^{t_{1\ldots t_o}}_{1\ldots f_{o+1}} \quad \begin{pmatrix} l_\gamma \\ s \end{pmatrix} a^{l_{1\ldots l_o}} = a^{l_{1\ldots l_o}}_{1\ldots f_{o+1}}.
\]

(2.30) $\sum a^{r_{1\ldots r_n}}_{1\ldots f_m} = \sum \sum T^{s_{r_n}}_{p_{m}} \begin{pmatrix} p \\ r_\alpha \end{pmatrix} (a_{\ldots n}^{s_{1\ldots s_n}} + a_{\ldots n}^{s_{1\ldots s_n}}) - \sum \sum T^{t_{l_m}}_{p_{o}} \begin{pmatrix} t_\beta \\ p \end{pmatrix} (a_{\ldots n}^{s_{1\ldots s_n}} + a_{\ldots n}^{s_{1\ldots s_n}}), \quad a^{t_{1\ldots t_o}}_{1\ldots f_o} < mn \rangle$
\[ a^{r_1 \ldots r_u}_{t_1 \ldots t_v \leq mn} = \sum_{\alpha=1}^{u-1} \sum_{\beta=2}^{u} \sum_{\alpha < \beta} \frac{p^{r_{\alpha}}_{pm} P^{r_{\beta}}_{[ns]}}{a_{r_{\alpha}} \left( \frac{s_{r_{\beta}}}{p_{r_{\beta}}} \right) a_{...}} \]

(2.32)

\[ - \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \frac{p^{r_{\alpha}}_{[pm]} P^{s_{\beta}}_{[nt \alpha]}}{a_{r_{\alpha}} \left( \frac{t_{\beta}}{p_{r_{\beta}}} \right) a_{...}} \]

\[ + \sum_{\alpha=1}^{v-1} \sum_{\beta=2}^{v} \sum_{\alpha < \beta} \frac{P^{p_{r}}_{[t, m]} P^{s_{\beta}}_{[nt \alpha]}}{a_{r_{\alpha}} \left( \frac{t_{\beta}}{s_{r_{\beta}}} \right) a_{...}} \]

(2.33)

\[ a^{r_1 \ldots r_u}_{t_1 \ldots t_v \leq mn} = \sum_{\alpha=1}^{u-1} \sum_{\beta=2}^{u} \sum_{\alpha < \beta} \frac{P^{r_{\alpha}}_{[pm]} P^{r_{\beta}}_{[sn]} + P^{r_{\alpha}}_{[pm]} P^{r_{\beta}}_{[sn]}}{a_{r_{\alpha}} \left( \frac{s_{r_{\beta}}}{p_{r_{\beta}}} \right) a_{...}} \]

(2.34)

\[ - \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \frac{P^{r_{\alpha}}_{[pm]} P^{s_{\beta}}_{[nt \alpha]} + P^{r_{\alpha}}_{[pm]} P^{s_{\beta}}_{[nt \alpha]}}{a_{r_{\alpha}} \left( \frac{t_{\beta}}{p_{r_{\beta}}} \right) a_{...}} \]

\[ + \sum_{\alpha=1}^{v-1} \sum_{\beta=2}^{v} \sum_{\alpha < \beta} \frac{P^{p_{r}}_{[t, m]} P^{s_{\beta}}_{[nt \alpha]} + P^{p_{r}}_{[t, m]} P^{s_{\beta}}_{[nt \alpha]}}{a_{r_{\alpha}} \left( \frac{t_{\beta}}{s_{r_{\beta}}} \right) a_{...}} \]

(2.35)

\[ P^{r_{\alpha}}_{[pm]} P^{r_{\beta}}_{[ns]} = P^{r_{\alpha}}_{[pm]} P^{r_{\beta}}_{[ns]} - P^{r_{\alpha}}_{[mp]} P^{r_{\beta}}_{[sn]} \]

(2.36)
3 Combined Ricci type identities and derived curvature tensors in $\mathbb{GF}_N$

In this section we demonstrate how it is possible to obtain combined Ricci type identities by virtue of identities from the Sec. 2. In these identities appear only curvature tensors, besides the $\tilde{K}_1, \tilde{K}_2, \tilde{K}_3$ also new curvature tensors $\tilde{K}_1^i, \ldots, \tilde{K}_8^i$, obtained with help of curvature pseudotensors $\tilde{A}_1, \ldots, \tilde{A}_8$.

1. Taking the sum of the two equations (2.2) (for $\theta = 1, 2$), we obtain

$$\mathcal{L}_1 = a_{\ldots}[mn] - a_{\ldots}[nm] + a_{\ldots}[m] - a_{\ldots}[n],$$

i.e.

$$\mathcal{L}_1 = \sum_{\alpha=1}^{u} (\tilde{K}_1 + \tilde{K}_2)_{jmn}^\alpha (p_{r\alpha}) a_{\ldots} - \sum_{\beta=1}^{u} (\tilde{K}_1 + \tilde{K}_2)_{t\alpha mn}^p (t_{\beta r}) a_{\ldots} - T^{*}_{mn}(a_{\ldots}[p] - a_{\ldots}[\frac{1}{2}])$$

where we note

$$(\tilde{K}_1 + \tilde{K}_2)_{jmn} = \tilde{K}_1^{i} jmn + \tilde{K}_2^{i} jmn.$$ But, the left side in (3.1) can be observed also as (2.5)[mn], i.e. as double antisymmetric part (an alternation) of (2.5) with respect to indices $m, n$ (analogously in other cassis). So,

$$\mathcal{L}_1 = \sum_{\alpha=1}^{u} \tilde{A}_{5}^{\alpha} j\mu mn (p_{r\alpha}) a_{\ldots} - \sum_{\beta=1}^{u} \tilde{A}_{6}^{\alpha} j\beta mn (t_{\beta p}) a_{\ldots} - T^{*}_{mn}(a_{\ldots}[p] - a_{\ldots}[\frac{1}{2}]).$$

Comparing (3.1) and (3.3), we obtain

$$\tilde{A}_{5}^{i} j\mu mn = \tilde{A}_{6}^{i} j\mu mn = (\tilde{K}_1 + \tilde{K}_2)_{jmn}.$$ 2. From the sum of (2.3) and (2.4), which we denote as (2.3+2.4), one obtains

$$\mathcal{L}_2 = \sum_{\alpha=1}^{u} (\tilde{A}_1 + \tilde{A}_2)_{j\mu mn}^\alpha (p_{r\alpha}) a_{\ldots} - \sum_{\beta=1}^{u} (\tilde{A}_1 + \tilde{A}_2)_{t\beta mn}^p (t_{\beta r}) a_{\ldots} + T^{*}_{mn}(a_{\ldots}[p] - a_{\ldots}[\frac{1}{2}]).$$

How the left side in (3.4) can be observed also as (2.10)[mn], we get

$$\mathcal{L}_2 = \sum_{\alpha=1}^{u} \tilde{A}_{15}^{\alpha} j\mu mn (p_{r\alpha}) a_{\ldots} - \sum_{\beta=1}^{u} \tilde{A}_{15}^{\alpha} j\beta mn (t_{\beta p}) a_{\ldots} - T^{*}_{mn}(a_{\ldots}[p] - a_{\ldots}[\frac{1}{2}]).$$

Comparing (3.4) and (3.5), we see that

$$\tilde{K}_1^i jmn = \frac{1}{2} (\tilde{A}_1 + \tilde{A}_2)^i jmn = \frac{1}{2} (\tilde{A}_1 + \tilde{A}_2)^i jmn = \frac{1}{2} \tilde{A}_{15}^i jmn.$$
is a tensor. So (3.4) can be written in the form

\[(3.7) \quad \frac{\mathcal{Q}}{2} = 2 \sum_{\alpha=1}^{u} \tilde{K}_{\alpha}^{*} p_{\alpha} (r_{\alpha}) a_{\cdots} = 2 \sum_{\beta=1}^{v} \tilde{K}_{\beta}^{*} t_{\beta} (p_{\beta}) a_{\cdots} + T_{\alpha}^{*} (a_{\cdots} - a_{\cdots \alpha} f).\]

3. By help of (2.6+2.9) one obtains

\[(3.8) \quad \frac{\mathcal{Q}}{3} = \sum_{\alpha=1}^{u} (\tilde{A} + \tilde{A})_{\alpha} p_{\alpha} (r_{\alpha}) a_{\cdots} = 2 \sum_{\beta=1}^{v} (\tilde{A} + \tilde{A})_{\beta} t_{\beta} (p_{\beta}) a_{\cdots} + (a_{\cdots \alpha} f + a_{\cdots \alpha} f).\]

As \(\mathcal{Q}\) can be obtained also by (2.7+2.8), we have

\[(3.9) \quad \frac{\mathcal{Q}}{3} = \sum_{\alpha=1}^{u} (\tilde{A} + \tilde{A})_{\alpha} p_{\alpha} (r_{\alpha}) a_{\cdots} = \sum_{\beta=1}^{v} (\tilde{A} + \tilde{A})_{\beta} t_{\beta} (p_{\beta}) a_{\cdots} + (a_{\cdots \alpha} f + a_{\cdots \alpha} f)\]

Based on (3.34, 3.35) is

\[a_{\cdots \alpha} f + a_{\cdots \alpha} f = \sum_{\alpha=1}^{u-1} \sum_{\beta=2}^{v} (T_{\alpha}^{*} P_{\alpha}^{*} + P_{\alpha}^{*} T_{\beta}^{*} + P_{\alpha}^{*} T_{\beta}^{*} + P_{\alpha}^{*} T_{\beta}^{*}) (p_{\alpha} (r_{\beta}) a_{\cdots}) \]

what we can write

\[a_{\cdots \alpha} f = a_{\cdots \alpha} f + a_{\cdots \alpha} f = \sum_{\alpha=1}^{u-1} \sum_{\beta=2}^{v} (T_{\alpha}^{*} P_{\alpha}^{*} + P_{\alpha}^{*} T_{\beta}^{*} + P_{\alpha}^{*} T_{\beta}^{*} + P_{\alpha}^{*} T_{\beta}^{*}) (p_{\alpha} (r_{\beta}) a_{\cdots}) \]

\[(3.10) \quad a_{\cdots \alpha} f = a_{\cdots \alpha} f + a_{\cdots \alpha} f = \sum_{\alpha=1}^{u-1} \sum_{\beta=2}^{v} (T_{\alpha}^{*} P_{\alpha}^{*} + P_{\alpha}^{*} T_{\beta}^{*} + P_{\alpha}^{*} T_{\beta}^{*} + P_{\alpha}^{*} T_{\beta}^{*}) (p_{\alpha} (r_{\beta}) a_{\cdots}) \]

from where we see that

\[a_{\cdots \alpha} f = a_{\cdots \alpha} f + a_{\cdots \alpha} f \]

is a tensor. Also, we conclude that

\[a_{\cdots \alpha} f = a_{\cdots \alpha} f. \]
Further, from (3.8, 3.9, 3.11), one gets
\begin{equation}
\frac{\mathcal{Q}}{3} = 2 \sum_{\alpha = 1}^{u} \tilde{K}_{jmn}^{r \alpha} \left( \frac{p}{r_{\alpha}} \right) a_{\alpha} - 2 \sum_{\beta = 1}^{v} \tilde{K}_{jmn}^{s \beta} \left( \frac{t_{\beta}}{p} \right) a_{\beta} + a_{\cdots mn},
\end{equation}
where the magnitudes
\begin{align}
\tilde{K}_{jmn}^{s i} &= \frac{1}{2} (\tilde{A}_{i} + \tilde{A}_{i})_{jmn} = \frac{1}{2} (\tilde{A}_{i} + \tilde{A}_{i})_{jmn}, \\
\tilde{K}_{jmn}^{s i} &= \frac{1}{2} (\tilde{A}_{i} + \tilde{A}_{i})_{jmn} = \frac{1}{2} (\tilde{A}_{i} + \tilde{A}_{i})_{jmn}
\end{align}
are tensors, obtained from curvature pseudotensors, and $a_{\cdots mn}$ is a tensor, given by virtue of (3.11), symmetric with respect of $m, n$.

4. If we add the equations (2.8) and (2.10), in (2.9) change the positions of indices $m, n$ and subtract from the sited sum, which we denote (2.8 + 2.10 − 2.9), one obtains
\begin{equation}
\mathcal{Q} = \sum_{\alpha = 1}^{u} \left( [\tilde{A}_{i} + \tilde{A}_{i}]_{jmn} - \tilde{A}_{jmn} \right) (r_{\alpha}) a_{\alpha} - \sum_{\beta = 1}^{v} \left( [\tilde{A}_{i} + \tilde{A}_{i}]_{jmn} - \tilde{A}_{jmn} \right) (t_{\beta}) a_{\beta} + T_{mn} a_{\cdots p} |_{i},
\end{equation}
By help of (2.25-2.27, 2.29) it is easy to prove the following:
\begin{equation}
(\tilde{A}_{i} + \tilde{A}_{i})_{jmn} - \tilde{A}_{jmn} = (\tilde{A}_{i} + \tilde{A}_{i})_{jmn} - \tilde{A}_{jmn}.
\end{equation}
Inducting a tensor
\begin{equation}
\tilde{K}_{jmn}^{s i} = (\tilde{A}_{i} + \tilde{A}_{i})_{jmn} - \tilde{A}_{jmn} = (\tilde{A}_{i} + \tilde{A}_{i})_{jmn} - \tilde{A}_{jmn},
\end{equation}
we can write $\mathcal{Q}$ in the form:
\begin{equation}
\mathcal{Q} = \sum_{\alpha = 1}^{u} \tilde{K}_{jmn}^{r \alpha} \left( \frac{p}{r_{\alpha}} \right) a_{\alpha} - \sum_{\beta = 1}^{v} \tilde{K}_{jmn}^{s \beta} \left( \frac{t_{\beta}}{p} \right) a_{\beta} + T_{mn} a_{\cdots p} |_{i}.
\end{equation}

5. From the combination (2.5-2.6 + 2.8) one obtains
\begin{equation}
\mathcal{Q} = \sum_{\alpha = 1}^{u} \left( \tilde{A}_{i} - \tilde{A}_{i} \right)_{jmn} \left( \frac{p}{r_{\alpha}} \right) a_{\alpha} - \sum_{\beta = 1}^{v} \left( \tilde{A}_{i} - \tilde{A}_{i} \right)_{jmn} \left( \frac{t_{\beta}}{p} \right) a_{\beta} + a_{\cdots mn}_{\geq} - a_{\cdots mn}_{\geq} + a_{\cdots mn}_{\geq} + T_{mn} a_{\cdots p} |_{i}.
\end{equation}
Based on (2.33, 2.34, 2.35), it is easy to prove that
\begin{equation}
a_{\cdots mn} = a_{\cdots mn_{\geq}} + a_{\cdots mn_{\geq}} = 0.
\end{equation}
Observing $\mathcal{L}_2^6$ as (2.2) we obtain

$$
(3.22) \quad \mathcal{L}_5^6 = \sum_{\alpha=1}^{u} \tilde{K}_2^{\alpha p_{mn}} \left( \frac{p}{r_{\alpha}} \right) a_{\cdots} - \sum_{\beta=1}^{v} \tilde{K}_2^{\beta t_{s mn}} \left( \frac{t_{\beta}}{p} \right) a_{\cdots}^* + T_{mn}^p a_{\cdots}^* |p|.
$$

Comparing last two equations we see that

$$
(3.23) \quad (\tilde{A}_{6} - \tilde{A}_{7} + \tilde{A}_{12})_{p mn} = (\tilde{A}_{6} - \tilde{A}_{8} + \tilde{A}_{12})_{p mn} = \tilde{K}_2^{\alpha i_{p mn}},
$$

where $\tilde{K}_2$ is given in (2.13).

6. a) From the combinations (2.3 – 2.6 – 2.9$_{nm}$) one obtains

$$
(3.24) \quad \mathcal{L}_6^6 = \sum_{\alpha=1}^{u} [(\tilde{A}_{1} - \tilde{A}_{4})_{p_{mn}} + \tilde{A}_{13}^{\alpha p_{mn}} + \tilde{A}_{14}^{\alpha p_{mn}}] \left( \frac{p}{r_{\alpha}} \right) a_{\cdots} - \sum_{\beta=1}^{v} [(\tilde{A}_{2} - \tilde{A}_{8})_{r_{s mn}} + \tilde{A}_{15}^{\beta p_{mn}} + \tilde{A}_{16}^{\beta p_{mn}}] \left( \frac{t_{\beta}}{p} \right) a_{\cdots}^* + a_{\cdots}^* |p|.
$$

Based on (2.32, 2.34, 2.35) we have

$$
(3.25) \quad a_{\cdots}^* |p| \geq a_{\cdots}^* |p| = -a_{\cdots}^* |p|.
$$

b) The next possibility is to observe $\mathcal{L}_6^6$ as (-2.8$_{nm}$ - 2.6 - 2.10$_{nm}$)

$$
(3.26) \quad \mathcal{L}_6^6 = -\sum_{\alpha=1}^{u} [(\tilde{A}_{1} - \tilde{A}_{4})_{p_{mn}} + \tilde{A}_{13}^{\alpha p_{mn}} + \tilde{A}_{14}^{\alpha p_{mn}}] \left( \frac{p}{r_{\alpha}} \right) a_{\cdots} - \sum_{\beta=1}^{v} [(\tilde{A}_{2} - \tilde{A}_{8})_{r_{s mn}} + \tilde{A}_{15}^{\beta p_{mn}} + \tilde{A}_{16}^{\beta p_{mn}}] \left( \frac{t_{\beta}}{p} \right) a_{\cdots}^* - a_{\cdots}^* |p| + T_{mn}^p a_{\cdots}^* |p|.
$$

If for tensor beside $(p_{r_{\alpha}}) a_{\cdots}$, and $(t_{\beta}) a_{\cdots}^*$ we introduce the designations

$$
(3.27) \quad \tilde{K}_6^{\alpha i_{s mn}} = (\tilde{A}_{1} - \tilde{A}_{4})_{j_{mn}} - \tilde{A}_{13}^{\alpha j_{mn}} = -(\tilde{A}_{1} + \tilde{A}_{4})_{j_{mn}} - \tilde{A}_{13}^{\alpha j_{mn}},
$$

$$
(3.28) \quad \tilde{K}_6^{\alpha i_{s mn}} = (\tilde{A}_{2} - \tilde{A}_{8})_{j_{mn}} - \tilde{A}_{14}^{\alpha j_{mn}} = -(\tilde{A}_{2} + \tilde{A}_{8})_{j_{mn}} - \tilde{A}_{14}^{\alpha j_{mn}},
$$

we can write

$$
(3.29) \quad \mathcal{L}_6^6 = \sum_{\alpha=1}^{u} \tilde{K}_6^{\alpha r_{\alpha}} \left( \frac{p}{r_{\alpha}} \right) a_{\cdots} - \sum_{\beta=1}^{v} \tilde{K}_6^{\beta t_{s mn}} \left( \frac{t_{\beta}}{p} \right) a_{\cdots}^* + T_{mn}^p a_{\cdots}^* |p|.
$$

7. a) Applying the combinations (2.4 + 2.6 + 2.9$_{nm}$), we get

$$
(3.30) \quad \mathcal{L}_7^7 = \sum_{\alpha=1}^{u} [(\tilde{A}_{3} + \tilde{A}_{4})_{p_{mn}} + \tilde{A}_{13}^{\alpha p_{mn}} + \tilde{A}_{14}^{\alpha p_{mn}}] \left( \frac{p}{r_{\alpha}} \right) a_{\cdots} - \sum_{\beta=1}^{v} [(\tilde{A}_{4} + \tilde{A}_{8})_{r_{s mn}} + \tilde{A}_{15}^{\beta p_{mn}} + \tilde{A}_{16}^{\beta p_{mn}}] \left( \frac{t_{\beta}}{p} \right) a_{\cdots}^* + a_{\cdots}^* |p| + a_{\cdots}^* |p| = -T_{mn}^p a_{\cdots}^* |p|.
$$
b) From \((2.7 + 2.9_{nm} - 2.10_{nm})\), we get

\[
\mathcal{Q} = \sum_{\alpha=1}^{u} [\tilde{A}_{pmn}^{\alpha} + (\tilde{A}_{14} - \tilde{A}_{15})_{pmn} \left( \frac{p}{r_{\alpha}} \right)] a_{\ldots} - \sum_{\beta=1}^{v} [\tilde{A}_{t_{p}mn}^{\beta} + (\tilde{A}_{14} - \tilde{A}_{15})_{t_{p}rmn} \left( \frac{t_{p}}{p} \right)] a_{\ldots}
\]

\[
+ a_{\ldots,\varepsilon_{nm}}\gamma - a_{\ldots,\varepsilon_{mn}}\gamma - a_{\ldots,\varepsilon_{nm}}\gamma - a_{\ldots,\varepsilon_{mn}}\gamma = a_{\ldots,\varepsilon_{mn}}\gamma.
\]

From the corresponding formulas we prove that

\[(3.30)\]

\[
a_{\ldots,\varepsilon_{[mn]}\gamma} + a_{\ldots,\varepsilon_{mn}\gamma} + a_{\ldots,\varepsilon_{mn}\gamma} = a_{\ldots,\varepsilon_{mn}\gamma} - a_{\ldots,\varepsilon_{mn}\gamma} = a_{\ldots,\varepsilon_{(mn)}}\gamma.
\]

Introducing the tensors

\[(3.31)\]

\[\tilde{K}_{7j_{mn}} = (\tilde{A}_{13} - \tilde{A}_{15})_{j_{mn}} + \tilde{A}_{15}^{j_{mn}},
\]

\[(3.32)\]

\[\tilde{K}_{8j_{mn}} = (\tilde{A}_{14} - \tilde{A}_{15})_{j_{mn}} + \tilde{A}_{15}^{j_{mn}},
\]

we have

\[
\mathcal{Q} = \sum_{\alpha=1}^{u} \tilde{K}_{7j_{pmn}} \left( \frac{p}{r_{\alpha}} \right) a_{\ldots} - \sum_{\beta=1}^{v} \tilde{K}_{8j_{p}mn} \left( \frac{t_{p}}{p} \right) a_{\ldots} + a_{\ldots,\varepsilon_{(mn)}} + T_{mn}^{p} a_{\ldots,\varepsilon_{p}}.
\]

Remark. The curvature tensors \(\tilde{K}, \tilde{K}, \tilde{K}, \tilde{K}, \tilde{K}, \tilde{K}, \tilde{K}\), derived curvature tensors \(\tilde{K}^{*}, \ldots, \tilde{K}^{*}\) and also curvature pseudotensors \(\tilde{A}_{1}, \ldots, \tilde{A}_{15}\) in generalized Finsler space reduce to one curvature tensor \(\tilde{K}\) in Finsler space \(F_{N}\) (in the symmetric case, see [12, ch. IV, equation (1.5)]). This is obvious from (1.11), because in \(F_{N}\) is \(\gamma_{i}^{j} = \gamma_{i}^{j}\).

Acknowledgements. The authors gratefully acknowledge for support from the research projects 174012 of the Serbian Ministry of Science.

References


Authors’ address:
Svetislav M. Minčić, Milan Lj. Zlatanović
University of Niš, Faculty of Science and Mathematics,
Višegradska 33, 18000 Niš, Serbia.
E-mail: svetislavmincic@yahoo.com, zlatmilan@yahoo.com