A classification on totally umbilical proper slant and hemi-slant submanifolds of a nearly trans-Sasakian manifold

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Abstract. In this paper, we study slant and hemi-slant submanifolds of nearly trans-Sasakian manifolds. We obtain the necessary and sufficient conditions on a totally umbilical proper slant submanifold and show that it is totally geodesic if the mean curvature vector $H \in \mu$. As well, we obtain the integrability conditions of the distributions of hemi-slant submanifolds and prove some characterization theorems.

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1 Introduction

S. Tanno [14] classified connected almost contact metric manifold whose automorphism group has maximum dimension. He classified them into the following three classes.

(i) Homogenous normal contact Riemannian manifolds with constant ϕ holomorphic sectional curvature if the sectional curvature of the plane section containing ξ, say $K(X, \xi) > 0$.

(ii) Global Riemannian product of a line (or a circle) and a Kaehlerian manifold with constant holomorphic sectional curvature, if $K(X, \xi) = 0$.

(iii) A warped product space $R \times_f C_n$, if $K(X, \xi) < 0$.

It is known that the manifolds of class (i) are characterized by some tensor equations, it has a Sasakian structure. The manifolds of class (ii) are characterized by a tensorial relation admitting a cosymplectic structure. The manifolds of class (iii) are characterized by some tensorial equations, it has Kenmotsu structure.

In 1985, Oubina introduced a new class of almost contact metric manifold known as trans- Sasakian manifold [11]. This class contains α-Sasakian and β-Kenmotsu manifold [7]. Recently, C. Gherghe [6] introduced a nearly trans-Sasakian structure

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of type (α, β), which generalize trans-Sasakian structure in the same sense as nearly Sasakian structure generalize Sasakian ones. A trans-Sasakian structure is always a nearly trans-Sasakian structure. Moreover, a nearly trans-Sasakian structure of type (α, β) is a nearly-Sasakian or nearly Kenmotsu or nearly cosymplectic if $\beta = 0$ or $\alpha = 0$ or $\alpha = \beta = 0$, respectively.

On the other hand, slant submanifolds in complex geometry were defined and introduced by B.Y. Chen as a natural generalization of both holomorphic and totally real submanifolds [5]. Since then many research articles have been appeared on the existence of these submanifolds in different known spaces. The slant submanifolds of an almost contact metric manifolds were defined and studied by A. Lotta [10]. After that, these submanifolds were studied by J.L. Cabrerizo et. al in the setting of Sasakian manifolds [3].

The notion of semi-slant submanifolds of an almost Hermitian manifold was introduced by N. Papaghiuc [12]. Later on many research articles on semi-slant submanifolds and their warped product also appeared in the setting of complex as well as contact setting ([2], [8]; [15]). Hemi-slant submanifolds first were introduced by A. Carriazo [4] and he called them pseudo-slant submanifolds. Recently, B. Sahin [13] studied these (hemi-slant) submanifolds in Kaehler setting for their warped product.

In this paper, we study slant and hemi-submanifolds of a nearly trans-Sasakian manifold. In section 3, we recall the definition and some basic results of a slant submanifold of an almost contact metric manifold. We obtain necessary and sufficient conditions on a totally umbilical proper slant submanifold M of a nearly trans-Sasakian manifold and then prove that a totally umbilical proper slant submanifold is totally geodesic when the mean curvature vector $H \in \mu$. Section 4 deals with the integrability of the distributions on the hemi-slant submanifolds of a nearly trans-Sasakian manifold and then we obtain some classification results for these submanifolds in the setting of nearly trans-Sasakian manifolds.

2 Preliminaries

Let $\bar{M}$ be a $(2m+1)-$dimensional almost contact metric manifold together with a metric tensor g, a tensor field ϕ of type $(1,1)$, a vector field ξ and a $1-$form η on $\bar{M}$, satisfying:

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1, \quad \eta(X) = g(X, \xi) \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) = -g(X, \phi Y) \tag{2.2}$$

for any vector fields X, Y on $\bar{M}$. If in addition, the following hold:

$$\bar{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.3}$$

then $\bar{M}$ is called a *trans-Sasakian manifold* [11].

If we weaken the condition (2.3) as :

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = \alpha(2g(X, Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y). \tag{2.4}$$

then the structure is called *nearly trans-Sasakian structure* . Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^{\perp}M$ be the Lie algebra of vector fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connections on M, then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \ \forall X, Y \in TM, \ \forall V \in T^{\perp}M, \tag{2.6}$$

where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_V is the Weingarten map associated with the vector field $V \in T^{\perp}M$ as

$$g(A_V X, Y) = g(h(X, Y), V). \tag{2.7}$$

For any $x \in M$ and $X \in T_x M$, we write

$$\phi X = TX + NX, \tag{2.8}$$

where $TX \in T_x M$ and $NX \in T_x^{\perp} M$. Similarly, for any $V \in T_x^{\perp} M$, we have

$$\phi V = tV + nV, \tag{2.9}$$

where tV (resp. nV) is the tangential component (resp. normal component) of ϕV.

Now, for any $X, Y \in TM$, let us denote the tangential and normal parts of $(\bar{\nabla}_X \phi)Y$ by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ respectively. Then by an easy computation, we obtain the following formulae

$$\mathcal{P}_X Y = (\bar{\nabla}_X T)Y - A_{NY} X - th(X, Y) \tag{2.10}$$

$$\mathcal{Q}_X Y = (\bar{\nabla}_X N)Y + h(X, TY) - nh(X, Y) \tag{2.11}$$

for any $X, Y \in TM$.

Similarly, for any $V \in T^{\perp}M$, denoting tangential and normal parts of $(\bar{\nabla}_X \phi)V$ by $\mathcal{P}_X V$ and $\mathcal{Q}_X V$ respectively, we obtain

$$\mathcal{P}_X V = (\bar{\nabla}_X t)V + TA_V X - A_{nV} X \tag{2.12}$$

$$\mathcal{Q}_X V = (\bar{\nabla}_X n)V + h(tV, X) + NA_V X \tag{2.13}$$

where the covariant derivatives of T, N, t and n are defined by

$$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y \tag{2.14}$$

$$(\bar{\nabla}_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y \tag{2.15}$$

$$(\bar{\nabla}_X t)V = \nabla_X tV - t\nabla^{\perp}_X V \tag{2.16}$$

$$(\bar{\nabla}_X n)V = \nabla^{\perp}_X nV - n\nabla^{\perp}_X V, \ \forall X, Y \in TM, V \in T^{\perp}M. \tag{2.17}$$

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be *totally umbilical* if

$$h(X, Y) = g(X, Y)H, \tag{2.18}$$

where H is the mean curvature vector. A submanifold M is said to be *totally geodesic* if $h(X, Y) = 0$, for each $X, Y \in TM$ and M is *minimal* if $H = 0$.

3 Slant submanifolds of a nearly trans-Sasakian manifold

Slant immersions in complex geometry were defined by B.Y. Chen [5]. Moreover, A. Lotta has introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold [9, 10] and slant submanifolds in Sasakian manifolds have been studied by J.L. Cabrerizo et al. [3].

For any $x \in M$ and $X \in T_xM$ if the vectors X and ξ are linearly independent, the angle denoted by $\theta(X) \in [0, \frac{\pi}{2}]$ between ϕX and T_xM is well defined. If $\theta(X)$ does not depend on the choice of $x \in M$ and $X \in T_xM$, we say that M is *slant* in $\bar{M}$. The constant angle θ is then called the *slant angle* of M in $\bar{M}$. The *anti-invariant submanifold* of an almost contact metric manifold is a slant submanifold with slant angle $\theta = \frac{\pi}{2}$ and an *invariant submanifold* is a slant submanifold with the slant angle $\theta = 0$. If the slant angle θ of M is different from 0 and $\frac{\pi}{2}$, then it is called a *proper slant* submanifold. If M is a slant submanifold of an almost contact manifold then the tangent bundle TM of M is decomposed as

$$TM = D \oplus < \xi > \tag{3.1}$$

where $< \xi >$ denotes the distribution spanned by the structure vector field ξ and D is a complementary distribution of $< \xi >$ in TM, known as the *slant distribution.* For a proper slant submanifold M of an almost contact manifold $\bar{M}$ with a slant angle θ, Lotta [10] proved that

$$QX = -\cos^2\theta(X - \eta(X)\xi), \ \forall X \in TM. \tag{3.2}$$

Recently, Cabrerizo et al. [3] extended the above result into a characterization for a slant submanifold in a contact metric manifold. In fact, they obtained the following crucial theorem.

Theorem 3.1 [3]. *Let M be a submanifold of an almost contact metric manifold $\bar{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = -\lambda(I - \eta \otimes \xi). \tag{3.3}$$

Furthermore, in such case, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Hence, for a slant submanifold we have

$$g(TX, TX) = \cos^2\theta(g(X, Y) - \eta(X)\eta(Y)) \tag{3.4}$$

$$g(NX, NY) = \sin^2\theta(g(X, Y) - \eta(X)\eta(Y)), \ \forall X, Y \in TM. \tag{3.5}$$

In the following theorems we assume M as a totally umbilical proper slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$.

Theorem 3.2. *Let M be a totally umbilical proper slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$, then following conditions are equivalent*

(i) $H \in \mu$,

(ii) $\alpha = \frac{g(\nabla_{TX}\xi, X)}{2(\|X\|^2 - \eta^2(X))}$

for any $X \in TM$.

Proof. For any $X \in TM$, we have $h(X, TX) = g(X, TX)H = 0$. Then from (2.5) and (2.6) and the structure equation of nearly trans-Sasakian manifold for one vector field $X \in TM$, above equation reduced to

$$0 = \phi(\nabla_X X + h(X, X)) + A_{NX}X - \nabla^{\perp}_X NX - \nabla_X TX$$
$$+2\alpha\{g(X, X)\xi - \eta(X)X\} - 2\beta\eta(X)\phi X.$$

After using (2.8), equating the tangential components of above equation, we obtain

$$0 = T\nabla_X X - \nabla_X TX + th(X, X) + A_{NX}X + 2\alpha\{g(X, X)\xi - \eta(X)X\} - 2\beta\eta(X)TX).$$

As M is a totally umbilical submanifold then the term $A_{NX}X$ becomes $Xg(H, NX)$, then above equation becomes

$$0 = T\nabla_X X - \nabla_X TX + g(X, X)tH + Xg(H, NX)$$
$$+2\alpha g(X, X)\xi - 2\alpha\eta(X)X - 2\beta\eta(X)TX. \tag{3.6}$$

If $H \in \mu$, then from (3.6) we obtain

$$T\nabla_X X - \nabla_X TX = -2\alpha\|X\|^2\xi + 2\alpha\eta(X)X + 2\beta\eta(X)TX. \tag{3.7}$$

Taking the inner product in (3.7) with ξ, we get

$$g(\nabla_X TX, \xi) = 2\alpha\{\|X\|^2 - \eta^2(X)\}$$

Interchanging X by TX, we derive

$$g(\nabla_{TX}T^2X, \xi) = 2\alpha\{\|TX\|^2 - \eta^2(TX)\}$$

or,

$$g(T^2X, \nabla_{TX}\xi)\} = -2\alpha\{\|TX\|^2 - \eta^2(TX).$$

Then from equations (3.3) and (3.4), we obtain

$$\cos^2\theta g(X, \nabla_{TX}\xi) - \cos^2\theta\eta(X)g(\xi, \nabla_{TX}\xi) = 2\alpha\cos^2\theta\{\|X\|^2 - \eta^2(X)\}.$$

Therefore, we conclude that

$$\alpha = \frac{g(\nabla_{TX}\xi, X) - \eta(X)g(\nabla_{TX}\xi, \xi)}{2(\|X\|^2 - \eta^2(X))}. \tag{3.8}$$

Now, we have $g(\xi, \xi) = 1$. Taking the covariant derivative of above equation with respect to TX for any $X \in TM$, we obtain $g(\bar{\nabla}_{TX}\xi, \xi) + g(\xi, \bar{\nabla}_{TX}\xi) = 0$, which implies $g(\nabla_{TX}\xi, \xi) = 0$ and then (3.8), gives

$$\alpha = \frac{g(\nabla_{TX}\xi, X)}{2(\|X\|^2 - \eta^2(X))}. \tag{3.9}$$

This is part *(ii)* of the theorem. If (3.9) holds then from equation (3.6) we get $H \in \mu$. □

Now, if $\alpha = 0$ on M then from (3.9), we obtain $g(\nabla_{TX}\xi, X) = 0$. Interchanging X by TX and then using (3.3), we get

$$g(\nabla_{T^2X}\xi, TX) = g(\nabla_{\cos^2\theta(-X+\eta(X)\xi)}\xi, TX) = 0.$$

Thus the above equation can be written as

$$-\cos^2\theta g(\nabla_X\xi, TX) + \cos^2\theta\eta(X)g(\nabla_\xi\xi, TX) = 0.$$

From the structure equation (2.4), we get $\nabla_\xi\xi = 0$, thus we obtain

$$\cos^2\theta g(\nabla_X\xi, TX) = 0. \tag{3.10}$$

Thus, from equation (3.10) we have if either M is an anti-invariant submanifold or $\nabla_X\xi = 0$ i.e., ξ is a killing vector field on M or M is trivial. If ξ is not killing then we can take atleast two linearly independent vectors X and TX to span D_θ i.e., the $dimM \geq 3$.

From the above conclusion we obtain the following theorem.

Theorem 3.3. *Let M be a totally umbilical slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$ such that $\alpha = 0$ on M then one of the following statements is true*

(i) $H \in \mu$

(ii) *M is an anti-invariant submanifold*

(iii) *If M is a proper slant submanifold then $dimM \geq 3$*

(iv) *M is trivial.*

(v) *ξ is a Killing vector field on M.*

Theorem 3.4. *A totally umbilical proper slant submanifold M of a nearly trans-Sasakian manifold $\bar{M}$ is totally geodesic if $H \in \mu$.*

Proof. For any $X, Y \in TM$, we have

$$\bar{\nabla}_X\phi Y = (\bar{\nabla}_X\phi)Y + \phi\bar{\nabla}_XY. \tag{3.11}$$

Then from (2.5), (2.6), (2.8) and (2.18), the above equation takes the form

$$\nabla_XTY + g(X, TY)H - A_{NY}X + \nabla^\perp_XNY = (\bar{\nabla}_X\phi)Y + T\nabla_XY + N\nabla_XY + g(X, Y)\phi H.$$

Taking the product with ϕH, we obtain

$$g(\nabla^\perp_XNY, \phi H) = g((\bar{\nabla}_X\phi)Y, \phi H) + g(N\nabla_XY, \phi H) + g(X, Y)g(H, H).$$

Using (2.6), we get

$$g(\bar{\nabla}_XNY, \phi H) = g((\bar{\nabla}_X\phi)Y, \phi H) + g(X, Y)\|H\|^2. \tag{3.12}$$

Similarly, we have

$$g(\bar{\nabla}_YNX, \phi H) = g((\bar{\nabla}_Y\phi)X, \phi H) + g(X, Y)\|H\|^2 \tag{3.13}$$

Then from (3.12) and (3.13), we derive

$$g(\bar{\nabla}_X NY + \bar{\nabla}_Y NX, \phi H) = 2g(X,Y)\|H\|^2 + g((\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X, \phi H).$$

Thus from the structure equation (2.4), we obtain

$$g(\bar{\nabla}_X NY + \bar{\nabla}_Y NX, \phi H) = 2g(X,Y)\|H\|^2. \tag{3.14}$$

Now, for any $X \in TM$, we have

$$\bar{\nabla}_X \phi H = (\bar{\nabla}_X \phi)H + \phi\bar{\nabla}_X H.$$

Then from (2.5), (2.6), (2.8) and (2.9), we obtain

$$-A_{\phi H}X + \nabla^{\perp}_X \phi H = P_X H + Q_X H - TA_H X - NA_H X + t\nabla^{\perp}_X H + \nabla^{\perp}_X H.$$

Taking the inner product with NY, the above equation reduces to

$$zcabla^{\perp}_X \phi H, NY) = -g(NA_H X, NY) + g(Q_X H, NY). \tag{3.15}$$

Thus from (3.10) and (3.15), we obtain

$$g(\nabla^{\perp}_X \phi H, NY) = -\sin^2\theta\{g(X,Y)\|H\|^2 - \eta(A_H X)\eta(Y)\} + g(Q_X H, NY).$$

Since $\bar{\nabla}$ is a metric connection and NY and ϕH are orthogonal, then

$$g(\bar{\nabla}_X NY, \phi H) = \sin^2\theta\{g(X,Y)\|H\|^2 - \eta(X)\eta(Y)\|H\|^2\} - g(Q_X H, NY). \tag{3.16}$$

Similarly, we have

$$g(\bar{\nabla}_Y NX, \phi H) = \sin^2\theta\{g(X,Y)\|H\|^2 - \eta(X)\eta(Y)\|H\|^2\} - g(Q_Y H, NX). \tag{3.17}$$

Then from (3.16) and (3.17), we obtain

$$\begin{aligned} g(\bar{\nabla}_X NY + \nabla_Y NX, \phi H) = \; & 2\sin^2\theta \cdot g(X,Y)\|H\|^2 - g(Q_X H, NY) \\ & -g(Q_Y H, NX) - 2\sin^2\theta \cdot \eta(X)\eta(Y)\|H\|^2. \end{aligned} \tag{3.18}$$

Thus equations (3.14) and (3.18), imply

$$\begin{aligned} 2g(X,Y)\|H\|^2 = \; & 2\sin^2\theta \cdot g(X,Y)\|H\|^2 - g(Q_X H, NY) \\ & -g(Q_Y H, NX) - 2\sin^2\theta \cdot \eta(X)\eta(Y)\|H\|^2. \end{aligned}$$

The above equation can be written as

$$\{\cos^2\theta \cdot g(X,Y) + \sin^2\theta \cdot \eta(X)\eta(Y)\}\|H\|^2 = \frac{-1}{2}\{g(Q_X H, NY) + g(Q_Y H, NX)\}.$$

In view of equations (2.13), (2.17) and the fact that $H \in \mu$, then the above equation takes the form

$$\begin{aligned} & \cos^2\theta \cdot g(X,Y)\|H\|^2 + \sin^2\theta \cdot \eta(X)\eta(Y)\|H\|^2 \\ & \quad = -\sin^2\theta \cdot g(X,Y)\|H\|^2 + \sin^2\theta \cdot \eta(X)\eta(Y)\|H\|^2. \end{aligned} \tag{3.19}$$

Then from (3.19) we conclude that $g(X,Y)\|H\|^2 = 0$, for any $X, Y \in TM$. Since M is proper slant then we can not take $g(X,Y) = 0$, for any non-zero vector $X, Y \in TM$, thus, we obtain $H = 0$. Hence, M is totally geodesic. □

4 Hemi-slant submanifolds of a nearly trans-Sasakian manifold

Hemi-slant submanifolds were introduced by A. Carriazo [4] as a special case of bi-slant submanifolds and he called them pseudo-slant submanifolds. In this section, we study a special class of hemi-slant submanifolds which are totally umbilical.

Definition 4.1. *A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a hemi-slant submanifold if there exist two orthogonal complementary distributions D_1 and D_2 satisfying:*

(i) $TM = D_1 \oplus D_2 \oplus \langle\xi\rangle$

(ii) *D_1 is a slant distribution with slant angle $\theta \neq \pi/2$*

(iii) *D_2 is totally real i.e., $\phi D_2 \subseteq T^{\perp}M$.*

If μ is an invariant subspace of the normal bundle $T^{\perp}M$, then for pseudo-slant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as $T^{\perp}M = \mu \oplus ND_1 \oplus ND_2$.

In the following propositions we work out the integrability conditions of the involved distributions in the definition of a hemi-slant submanifold, which play an important role from geometric viewpoint.

Proposition 4.1. *Let M be a hemi-slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$, then the anti-invariant distribution $D_2 \oplus \langle\xi\rangle$ is integrable if and only if*

$$A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z = 2(\bar{\nabla}_Z\phi)W, \quad \forall\ Z,\ W \in D_2 \oplus \langle\xi\rangle. \tag{4.1}$$

Proof. For any $Z, W \in D_2 \oplus \langle\xi\rangle$, we have

$$\bar{\nabla}_Z\phi W = (\bar{\nabla}_Z\phi)W + \phi\bar{\nabla}_Z W = (\bar{\nabla}_Z\phi)W + \phi\nabla_Z W + \phi h(Z, W).$$

Using (2.6), we obtain

$$-A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W = (\bar{\nabla}_Z\phi)W + \phi\nabla_Z W + \phi h(Z, W). \tag{4.2}$$

Interchanging Z and W, we get

$$-A_{\phi Z}W + \nabla^{\perp}_{W}\phi Z = (\bar{\nabla}_W\phi)Z + \phi\nabla_W Z + \phi h(Z, W). \tag{4.3}$$

Then from (4.2) and (4.3), we obtain

$$A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z = (\bar{\nabla}_Z\phi)W - (\bar{\nabla}_W\phi)Z + \phi[Z, W].$$

Using (2.4), we derive

$$\begin{aligned} A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z &= \phi[Z, W] + 2(\bar{\nabla}_Z\phi)W \\ -\alpha\{2g(Z, W)\xi - \eta(W)Z - \eta(Z)W\} &+ \beta\{\eta(W)\phi Z + \eta(Z)\phi W\}. \end{aligned} \tag{4.4}$$

Taking the inner product in (4.4) with ϕX, for any $X \in D_1$

$$A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z, \phi X) = g(\phi[Z, W], \phi X) + 2g((\bar{\nabla}_Z\phi)W, \phi X).$$

Thus from (2.2), the above equation takes the form

$$g([Z,W],X) = g(A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z, \phi X)) - 2g((\bar{\nabla}_{Z}\phi)W, \phi X). \tag{4.5}$$

The distribution $D_2 \oplus \langle\xi\rangle$ is integrable if and only if the right hand side of the above equation is zero. Hence the result follows from (4.5). □

Proposition 4.2. *Let M be a hemi-slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$, then the slant distribution $D_1 \oplus \langle\xi\rangle$ is integrable if and only if*

$$2(\bar{\nabla}_X\phi)Y + g(h(Y,TX) - h(X,TY) + \nabla^{\perp}_{Y}NX - \nabla^{\perp}_{X}NY \in \mu, \quad \forall\ X, Y \in D_1 \oplus \langle\xi\rangle.$$

Proof. For any $X, Y \in D_1 \oplus \langle\xi\rangle$, we have

$$\phi[X,Y] = \phi\bar{\nabla}_Y X - \phi\bar{\nabla}_X Y = \bar{\nabla}_Y\phi X - (\bar{\nabla}_Y\phi)X - \bar{\nabla}_X\phi Y + (\bar{\nabla}_X\phi)Y.$$

Then from (2.4) and (2.8), we obtain

$$\begin{aligned}\phi[X,Y] &= 2(\bar{\nabla}_X\phi)Y + \bar{\nabla}_Y TX + \bar{\nabla}_Y NX - \bar{\nabla}_X TY - \bar{\nabla}_X NY \\ &\quad -\alpha\{2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\} + \beta\{\eta(Y)\phi X + \eta(X)\phi Y\}.\end{aligned}$$

Using (2.5) and (2.6), we get

$$\begin{aligned}\phi[X,Y] &= 2(\bar{\nabla}_X\phi)Y + \nabla_Y TX + h(Y,TX) - A_{NX}Y \\ &\quad +\nabla^{\perp}_{Y}NX - \nabla_X TY - h(X,TY) + A_{NY}X - \nabla^{\perp}_{X}NY \\ &\quad -\alpha\{2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\} + \beta\{\eta(Y)\phi X + \eta(X)\phi Y\}.\end{aligned} \tag{4.6}$$

Taking the product in (4.6) with ϕZ, for any $Z \in D_2$, we derive

$$\begin{aligned}g(\phi[X,Y],\phi Z) &= 2g((\bar{\nabla}_X\phi)Y,\phi Z) + g(h(Y,TX),\phi Z) + g(\nabla^{\perp}_{Y}NX,\phi Z) \\ &\quad -g(-h(X,TY),\phi Z) - g(\nabla^{\perp}_{X}NY,\phi Z) + \beta\eta(Y)g(\phi X,\phi Z) + \beta\eta(X)g(\phi Y,\phi Z).\end{aligned}$$

Thus from (2.2), we obtain

$$\begin{aligned}g([X,Y],Z) &= g(2(\bar{\nabla}_X\phi)Y + h(Y,TX) + \nabla^{\perp}_{Y}NX - h(X,TY) \\ &\quad -\nabla^{\perp}_{X}NY,\phi Z) + \beta\eta(Y)g(X,Z) + \beta\eta(X)g(Y,Z).\end{aligned} \tag{4.7}$$

Then by orthogonality of two distributions the last two terms of right hand side in (4.7) are identically zero, then

$$g([X,Y],Z) = g(2(\bar{\nabla}_X\phi)Y + h(Y,TX) + \nabla^{\perp}_{Y}NX - h(X,TY) - \nabla^{\perp}_{X}NY,\phi Z). \tag{4.8}$$

Thus the assertion follows from (4.8). □

Theorem 4.1. *Let M be a hemi-slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$, then at least one of the following statement is true*

(i) $dimD_2 = 1$

(ii) $H \in \mu$

(iii) *M is proper slant.*

Proof. For any $U, V \in TM$, we have

$$(\bar{\nabla}_U \phi)V + (\bar{\nabla}_V \phi)U = \alpha(2g(U,V)\xi - \eta(V)U - \eta(U)V) - \beta(\eta(V)\phi U - \eta(U)\phi V).$$

If we take the vector fields $Z, W \in D_2$, then the above equation will be

$$(\bar{\nabla}_Z \phi)W + (\bar{\nabla}_Z \phi)W - 2\alpha g(Z,W)\xi = 0.$$

In particular, if we take the above equation for one vector $Z \in D_2$, i.e.,

$$(\bar{\nabla}_Z \phi)Z - \alpha\|Z\|^2\xi = 0. \tag{4.9}$$

Therefore the tangential and normal parts of the above equation are $\mathcal{P}_Z Z = \alpha\|Z\|^2\xi$ and $\mathcal{Q}_Z Z = 0$, respectively. From (2.10) and tangential component of (4.9), we obtain

$$(\bar{\nabla}_Z T)Z = T\nabla_Z Z = A_{NZ}Z + th(Z,Z) - \alpha\|Z\|^2\xi.$$

Taking the product with $W \in D_2$, we obtain

$$g(T\nabla_Z Z, W) = g(h(Z,W), NZ) + g(th(Z,Z), W).$$

Using the fact that M is a totally umbilical submanifold and $TW = 0$ for any $W \in D_2$, then the above equation takes the form

$$0 = g(H, NZ)g(Z,W) + \|X\|^2 g(tH, W). \tag{4.10}$$

Thus the equation (4.10) has a solution if either $dimD_2 = 1$ or $H \in \mu$ or $D_2 = \{0\}$, i.e., M is proper slant. □

Now, we have the following theorem.

Theorem 4.2. *Let M be a totally umbilical hemi-slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$. Then at least one of the following statements is true*

(*i*) *M is an anti-invariant submanifold,*

(*ii*) $\alpha = \frac{g(\nabla_Z\xi, TZ)}{\|Z\|^2}$ *for any $Z \in TM$,*

(*iii*) *M is a totally geodesic submanifold,*

(*iv*) $dimD_2 = 1$,

(*v*) *M is a proper slant submanifold.*

Proof. If $H \neq 0$ then from equation (3.19) only possibility is that the slant distribution $D_1 = \{0\}$ i.e., M is anti-invariant submanifold which is case (*i*). If $D_1 \neq \{0\}$ and $H \in \mu$, then by Theorem 3.2, we have $\alpha = \frac{g(\nabla_Z\xi, TZ)}{\|Z\|^2}$ for any $Z \in TM$, which is case (*ii*). Moreover, if $H \in \mu$ then by Theorem 3.4, M is totally geodesic. Finally, if $H \notin \mu$, then the equation (4.10) has a solution if either $dimD_2 = 1$ or $D_2 = \{0\}$ which are cases (*iv*) and (*v*), respectively. Hence, the theorem is proved completely. □

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