Vector optimization problems under d-invexity on Riemannian manifolds

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Abstract. The concept of $d-$invexity for functions defined on Riemannian manifolds is introduced. Then, this notion is used to obtain optimality conditions for vector optimization problems on Riemannian manifolds.

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1 Introduction

The concept of convexity on linear spaces plays an important role in many aspects of optimization theory. This concept is often not enjoyed by the real problems (see [6]). Therefore, various approaches have been proposed to relax the convexity assumption. One of the useful generalizations is invexity introduced by Hanson [9] who considered differentiable functions $f : X \to \mathbb{R}$, $X \subset \mathbb{R}^n$, for which there exists an $n-$dimensional vector function $\eta : X \times X \to \mathbb{R}^n$ such that for all $x, y \in X$,

$$f(x) - f(y) \geq \langle \bigtriangledown f(y), \eta(x, y)\rangle.$$

Since then numerous articles have been appeared in the literature reflecting further generalizations and applications in this category. Ben-Israel and Mond [4] introduced a new generalization of convex sets and convex functions and Craven [5] called them invex sets and preinvex functions, respectively. Ye [19] introduced the notion of $d-$invexity for functions which is another generalization of invex functions. Then, Antczak [1] and Mishra [13] used this notion for optimality conditions in multiobjective programming.

A manifold is not a linear space, Rapcsák [15] and Udriste [16] proposed a generalization of convexity which differs from the others (see also [17]). In this setting the linear space is replaced by a Riemannian manifold and the line segment by a geodesic.

On the other hand in the last few years, various concepts of nonsmooth analysis have been extended from Euclidean space to Riemannian manifold setting, in order to study optimization problems and related topics; see for instance [3, 8, 11]. In [2] invex sets, invex functions and preinvex functions on Riemannian manifolds are introduced (see also [14]).

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In the present paper the notion of d-invexity for the functions defined on Riemannian manifolds is introduced. Then, we apply this tool to solve vector optimization problems.

The organization of the paper is as follows: in Section 2 some results and facts from Riemannian geometry are collected.

Section 3 is devoted to the notions of invex sets, invex functions and preinvex functions on Riemannian manifolds.

In Section 4 the concept of $\eta-$directional derivative of functions on invex subsets of Riemannian manifolds is defined, the notion of d-invexity is introduced and some optimality conditions for d-invex functions are given.

In Section 5 giving the definition of d-invex functions, we obtain the necessary and sufficient optimality conditions for vector optimization problems on Riemannian manifolds.

2 Preliminaries

In this section, we recall some definitions and known results about Riemannian manifolds which will be used throughout the paper. We refer the reader to [10, 16] for the standard material of differential geometry.

Throughout this paper M is a C^∞ Riemannian manifold modelled on a Hilbert space H, with a Riemannian metric $\langle .,.\rangle_p$ on the T_pM and the corresponding norm $\| \, \|_p$. Let us recall that the length of a piecewise C^1 curve $\gamma : [a,b] \to M$ is defined by $L(\gamma) := \int_a^b \|\gamma'(t)\|_{\gamma(t)}dt$. The Riemannian distance $d(p,q)$, for every p, $q \in M$ is defined by $d(p,q) := \inf_\gamma\{L(\gamma)\}$, for every C^1 piecewise curve γ joining p to q.

On every Riemannian manifold there exists exactly one covariant derivation called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields X, Y on M. We also recall that a geodesic is a C^∞ smooth path γ such that $\nabla_{d\gamma(t)/dt}d\gamma(t)/dt = 0$. Any path γ joining p and q in M such that $L(\gamma) = d(p,q)$ is a geodesic, and it is called a minimal geodesic. For every $v \in TM$ there exists an open interval $J(v)$ containing 0 and exactly one geodesic $\gamma_v : J(v) \to M$ with $\frac{d}{dt}\gamma(t)|_{t=0} = v$. The exponential mapping $\exp : TM \to M$ is then defined as $\exp(v) = J_v(1)$ and the restriction of exp to is denoted by $\exp_p$ for every $p \in M$. We also recall that a simply connected complete Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold.

The following convention for equalities and inequalities will be used throughout this paper. If x, $y \in \mathbb{R}^m$ then,

$$
\begin{aligned}
&x \leq y \Leftrightarrow x_i \leq y_i, \quad i = 1,2,...,m \text{ with } \; x_j < y_j \text{ for at least one } j,\\
&x \leqq y \Leftrightarrow x_i \leq y_i, \quad i = 1,2,...,m,\\
&x = y \Leftrightarrow x_i = y_i, \quad i = 1,2,...,m,\\
&x < y \Leftrightarrow x_i < y_i, \quad i = 1,2,...,m.
\end{aligned}
$$

3 Invex sets and preinvex functions

In this section we recall some definitions and results from [2], which are used in the sequel.

Definition 3.1. Let M be a Riemannian manifold and $\eta : M \times M \to TM$ be a function such that for every $x,\, y \in M$, $\eta(x, y) \in T_yM$.
A nonempty subset S of M is said to be invex with respect to η or η-invex, if for every $x,\, y \in S$ there exists a unique geodesic $\alpha_{x,y} : [0, 1] \to M$ such that

$$\alpha_{x,y}(0) = y,\ \alpha'_{x,y}(0) = \eta(x, y),\ \alpha_{x,y}(t) \in S, \qquad for\ all\ t \in [0, 1].$$

Recall that a subset S of a Riemannian manifold is called convex if any two points $x, y \in S$ can be joined by exactly one geodesic of length $d(x, y)$ which belongs entirely to S (see [10]).

Example 3.1. Let M be a Hadamard manifold and $x_0,\, y_0 \in M,\ x_0 \neq y_0$. Let $B(x_0, r_1) \cap B(y_0, r_2) = \phi$ for some $0 < r_1, r_2 < \frac{1}{2}d(x_0, y_0)$, where $B(x, r) = \{y \in M : d(x, y) < r\}$ is an open ball with the center x and the radius r. Then, the set

$$S := B(x_0, r_1) \cup B(y_0, r_2),$$

is invex with respect to $\eta : M \times M \to M$ defined by

$$\eta(x, y) := \begin{cases} \exp_y^{-1} x & x, y \in B(x_0, r_1),\ \text{or}\ x, y \in B(y_0, r_2), \\ 0_y & \text{otherwise.} \end{cases}$$

Note that S is not a convex set, see [2] for more details.

Definition 3.2. Let M be a Riemannian manifold and $S \subseteq M$ be an invex set with respect to $\eta : M \times M \to TM$. Then, the function $f : S \to \mathbb{R}$ is said to be $\eta-$preinvex if for each $x,\, y \in S$,

$$f(\alpha_{x,y}(t)) \leq tf(x) + (1 - t)f(y), \qquad for\ all\ t \in [0, 1], \tag{3.1}$$

where $\alpha_{x,y}$ is the unique geodesic defined in Definition 3.1.

4 D-invexity

The definition and properties of differentiable invex functions and preinvex functions on invex subsets of Riemannian manifolds is studied in [2]. Now, we introduce the concept of d-invex functions defined on invex subsets of Riemannian manifolds.

Definition 4.1. Let $S \subseteq M$ be an open invex set with respect to $\eta : M \times M \to TM$. The function $f : S \to \mathbb{R}$ is said to be $\eta-$directional differentiable at $y \in S$ if

$$f'(y; \eta(x, y)) := \lim_{t \to 0^+} \frac{f(\alpha_{x,y}(t)) - f(y)}{t},$$

exists for each $x \in S$, where $\alpha_{x,y}$ is the unique geodesic defined in Definition 3.1. If f is $\eta-$directional differentiable at each $y \in S$ then, f is said to be $\eta-$directional differentiable on S.

Definition 4.2. Let M be a Riemannian manifold and $S \subseteq M$ be an open invex set with respect to $\eta : M \times M \to TM$. Let f be an $\eta-$directional differentiable function on S. Then, f is said to be d-invex if for every $x,\, y \in S$,

$$f(x) - f(y) \geq f'(y; \eta(x, y)). \tag{4.1}$$

If the inequality (4.1) is strict then, we say that f is a strictly d-invex function.

Note that if f is a locally Lipschitz function on an open $\eta-$invex set then, for every x, $y \in S$ the $\eta-$directional derivative of f exists, that is, $f'(y; \eta(x, y)) < +\infty$. Indeed, if K_y is a Lipschitz constant of f in a neighborhood of y then, for the points of the geodesic $\alpha_{x,y}(t)$ in that neighborhood we have

$$\frac{f(\alpha_{x,y}(t)) - f(y)}{t} \leq K_y \frac{d(\alpha_{x,y}(t), y)}{t}. \tag{4.2}$$

Since

$$\lim_{t \to 0^+} \frac{d(\alpha_{x,y}(t), y)}{t} = \|\eta(x, y)\|_y,$$

(see [8] Lemma 1) then, by taking limit in (4.2) we have $f'(y; \eta(x, y)) \leq K_y \|\eta(x, y)\|_y$.

In the following example the Riemannian manifold M is the $n-$sphere S^n, $\bar{p} := (0, ..., 1)$, $\bar{q} := (0, ..., -1)$ and S^+, S^- are open upper and open lower semispheres, respectively.

Example 4.1. Suppose that $S := S^+ \cup S^-$. Consider the function $\eta : M \times M \to TM$ defined by

$$\eta(x, y) := \begin{cases} \alpha'_{x,y}(0) & x, y \in S^+ \text{ or } x, y \in S^-, \\ 0_y & \text{otherwise}, \end{cases}$$

where $\alpha_{x,y}$ is the unique geodesic joining y to $x \in S$. It is obvious that S is an invex set with respect to η. Now, we define the function $\varphi : S \to \mathbb{R}$ as

$$\varphi(p) := \begin{cases} d(p, \bar{p}) & p \in S^+, \\ d(p, \bar{q}) & p \in S^-. \end{cases}$$

Then, the function φ is Lipschitz on S. Hence, it is $\eta-$directional differentiable.

In [2] we showed that on every Riemannian manifold M there exists a function $\eta : M \times M \to TM$ and an open subset S of M which is $\eta-$invex but it is not convex. Therefore, every locally Lipschitz function on the open set S is $\eta-$directional differentiable. In the following examples some locally Lipschitz functions on certain Riemannian manifolds are presented. Hence, these functions are $\eta-$directional differentiable on the set S.

Example 4.2. Let M be a connected, complete Riemannian manifold and $f : M \to (-\infty, +\infty]$ be a continuous function which is bounded below on M. Then, the inf-convolution of f defined by $f_\alpha(x) := \inf_{y \in M}\{f(y) + \alpha d^2(y, x)\}$, $\alpha > 0$ is locally Lipschitz on its domain.

Example 4.3. Let $\mathbb{S}(n)$ be the linear space of real $n \times n$ symmetric matrices. Endowing $\mathbb{S}(n)$ with the metric defined by $\langle A, B \rangle := \mathrm{tr}(A^t B)$, we obtain a Riemannian manifold. For $A \in \mathbb{S}(n)$, let $\lambda_1(A), \ldots \lambda_n(A)$ be the n (including repeated) real eigenvalues of A. Then, for each $k \in \{1, \ldots n\}$ the function $\lambda_k : \mathbb{S}(n) \to \mathbb{R}$ is locally Lipschitz, (see [11]).

Example 4.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(p) := \sum_{i=1}^n \ln(p_i)$. Consider the $n \times n$ matrix $g(p) := diag(p_1^{-2}, ..., p_n^{-2})$. Endowing $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n : x > 0\}$ with the Riemannian metric $g(u, v) = \langle g(p)u, v \rangle$ for u, $v \in \mathbb{R}^n_{++}$, we obtain a Riemannian manifold M_g. Considering $\mathbb{R}^n$ with the usual Euclidean metric, the map $\Phi : \mathbb{R}^n \to M_g$ defined by $\Phi(x) = (e^{x^1}, ..., e^{x^n})$ is an isometry. Suppose that the function $h : \mathbb{R}^n \to \mathbb{R}$ is defined by $h(x) := \sum_{i=1}^n x_i$. Then, h is Lipschitz on $\mathbb{R}^n$ of rank 1. Since $h(x) = f(\Phi(x))$ and isometries preserve the Riemannian distance, thus f is Lipschitz on M_g of rank 1, (see [7]).

Theorem 4.1. *Let M be a Riemannian manifold and $S \subseteq M$ be an open invex set with respect to $\eta : M \times M \to TM$. Suppose that $f : S \to \mathbb{R}$ is an $\eta-$directional differentiable and $\eta-$preinvex function on S. Then, f is a d-invex function.*

Proof. Let $x,\, y \in S$. Since S is an invex set with respect to η, there exists a unique geodesic $\alpha_{x,y} : [0,1] \to M$ such that

$$\alpha_{x,y}(0) = y,\ \alpha'_{x,y}(0) = \eta(x,y),\ \alpha_{x,y}(t) \in S, \quad \text{for all } t \in [0,1].$$

By the $\eta-$preinvexity of f we have

$$tf(x) + (1-t)f(y) \geq f(\alpha_{x,y}(t)), \qquad \text{for} \quad \text{all } t \in [0,1],$$

which implies

$$t(f(x) - f(y)) \geq f(\alpha_{x,y}(t)) - f(y).$$

Divide by t to obtain

$$f(x) - f(y) \geq \frac{1}{t}[f(\alpha_{x,y}(t)) - f(y)].$$

Taking the limit as $t \to 0$ we have

$$f(x) - f(y) \geq f'(y;\eta(x,y)).$$

□

Remark 4.2. Let M be a Hadamard manifold, $S \subseteq M$ be an open convex set and $f : S \to \mathbb{R}$ be a convex function. We define the function $\eta : M \times M \to TM$ by $\eta(x,y) := \exp_y^{-1}(x)$ for every $x,\, y \in M$ and set

$$\alpha_{x,y}(t) := \exp_y(t \exp_y^{-1}), \qquad \text{for all } t \in [0,1].$$

Then, we have

$$f'(y, \exp_y^{-1}(x)) \leq f(x) - f(y), \qquad \text{for all } x,\ y \in S.$$

Hence, every convex function defined on a convex subset S of a Hadamard manifold is $\eta-$directional differentiable on S. Moreover, f is d-invex and all results in this paper are valid for convex functions defined on convex subsets of Hadamard manifold, (see [2, Remark 3.1]).

Theorem 4.3. *Let M be a Reimannian manifold, $S \subseteq M$ be an open invex set with respect to $\eta : M \times M \to TM$. Suppose that $f : S \to \mathbb{R}$ is a d-invex function. If $\bar{x} \in S$ is a local optimal solution to the problem*

$$(P) \qquad \begin{aligned} &min\ f(x) \\ &s.t.\ x \in S, \end{aligned}$$

then, $\bar{x}$ is a global minimum in (P).

Proof. Suppose that $\bar{x} \in S$ is a local minimum of f. Then, there exits an open neighborhood $N_\varepsilon(\bar{x})$ such that

$$f(x) \geq f(\bar{x}), \qquad \text{for all } x \in S \cap N_\varepsilon(\bar{x}). \tag{4.3}$$

Fix $x \in S$. Since S is an invex set with respect to η, there exists a unique geodesic $\alpha_{x,\bar{x}}$ such that

$$\alpha_{x,\bar{x}}(0) = \bar{x},\ \alpha'_{x,\bar{x}}(0) = \eta(x,\ \bar{x}),\ \alpha_{x,\bar{x}}(t) \in S, \qquad \text{for all } \in [0,1].$$

By the continuity of the distance function d and the geodesic $\alpha_{x,\bar{x}}$, there exists a number $\delta > 0$ such that $d(\alpha_{x,\bar{x}}(t), \bar{x}) < \varepsilon$ for all $t \in (0,\delta)$. Hence, $f(\alpha_{x,\bar{x}}(t)) \geq f(\bar{x})$ for all $t \in (0,\delta)$. Since f is $d-$invex,

$$f(x) - f(\bar{x}) \geq f'(\bar{x}; \eta(x,\bar{x})) = \lim_{t\to 0^+} \frac{f(\alpha_{x,y}(t)) - f(\bar{x})}{t} \geq 0.$$

Thus, $f(x) \geq f(\bar{x})$ that is, $\bar{x}$ is a global minimum of f. □

It is obvious that if $f : S \to \mathbb{R}$ is a d-invex function then, any point $y \in S$ with $f'(y, \eta(x,y)) \geq 0$ for all $x \in S$, is a solution of (P).

The following corollary is an extended optimality condition for optimization problems introduced by Ferreira to invex setting, (see [8, Corollary 6]).

Corollary 4.4. *Let M be a Reimannian manifold, $S \subseteq M$ be an open invex set with respect to $\eta : M \times M \to TM$. Suppose that $f, g_i : S \to \mathbb{R}$, for $i = 1, ..., m$ are d-invex functions. Consider the following problem*

$$(\overline{P}) \qquad \begin{aligned} &min\ f(x) \\ &s.t.\ g_i \leq 0, \quad i = 1,...,m. \end{aligned}$$

Let $y \in S$ be a feasible point to $(\overline{P})$, that is, $g_i(y) \leq 0$ for $i = 1, ..., m$. Assume that for every feasible point x, there exists a vector $\mu = (\mu_1, ..., \mu_m) \in \mathbb{R}^m$ such that

$$f'(y, \eta(x,y)) + \sum_{i=1}^{m} \mu_i g_i'(y, \eta(x,y)) \geq 0, \quad \mu_i \geq 0, \quad \sum_{i=1}^{m} \mu_i g_i(y) = 0. \tag{4.4}$$

Then, y is a solution to $(\overline{P})$.

Proof. The functions $f, g_i : S \to \mathbb{R}$, for $i = 1, ..., m$ are d-invex and $\mu \geq 0$. Therefore the set $C := \{x \in S : g_i(x) \leq 0, \quad i = 1, ..., m\}$ is invex with respect to η and $h : S \to \mathbb{R}$ defined by $h(x) = f(x) + \sum_{i=1}^{m} \mu_i g_i(x)$ is a d-invex function. Moreover, $y \in C$ and $f(x) \geq h(x)$ for all $x \in C$. By (4.4) we have $h'(y, \eta(x,y)) \geq 0$. Hence, the d-invexity of h implies that

$$h(x) \geq h(y), \quad \text{for all } x \in C.$$

Thus, $f(x) \geq h(x) \geq h(y) = f(y)$ for all $x \in C$, and the proof is complete. □

The following theorem is an extension of Theorem 5.2 in [2]. By utilizing this theorem we give a sufficient optimality condition for d-invex functions defined on invex subsets of Hadamard manifolds. We recall that for a Riemannian manifold M the proximal subdifferential of a lower semicontinuous function $f : M \to (-\infty, +\infty]$ at $x \in M$ is the subset $\partial_p f(x)$ of T_xM consists of all ζ such that

$$f(y) \geq f(x) + \langle \zeta, \exp_x^{-1} y \rangle_x - \sigma d(x, y)^2, \tag{4.5}$$

for each y in a neighborhood of the point x (see [7]).

Theorem 4.5. *Let M be a Hadamard manifold, $S \subseteq M$ be an open invex set with respect to $\eta : M \times M \to TM$ with $\eta(x, y) \neq 0$ for $x \neq y$. Suppose that $f : M \to (-\infty, +\infty]$ is a lower semicontinuous function which is d-invex on S. Let $y \in S \cap dom(f)$ and $\zeta \in \partial_p f(y)$. Then, there exists a number $\delta > 0$ such that*

$$f(x) - f(y) \geq \langle \zeta, \eta(x, y) \rangle_y, \qquad for\ all\ x \in S \cap B(y, \delta).$$

The proof is similar to the proof of Theorem 5.2 in [2] and we omit it.

Now, we obtain a sufficient optimality condition which is an immediate consequence of Theorem 4.5.

Corollary 4.6. *Let M be a Hadamard manifold, $S \subseteq M$ be an open invex set with respect to $\eta : M \times M \to TM$ with $\eta(x, y) \neq 0$ for $x \neq y$. Suppose that $f : M \to (-\infty, +\infty]$ is a lower semicontinuous function which is d-invex on S and $0 \in \partial_p f(y)$. Then, y is a local minimum of f.*

Note that if M is a Hadamard manifold and we define $\eta : M \times M \to TM$ by $\eta(x, y) := 2 \exp_y^{-1}(x)$ for every $x, y \in M$ then, $\eta(x, y) \neq 0$ for $x \neq y$.

5 Vector optimization problems

Antczak in [1] used the notion of d-invexity to solve the vector optimization problems in $\mathbb{R}^n$. In this section we use the notion of d-invexity to solve the vector optimization problems on Riemannian manifolds.

Definition 5.1. Let M be a Riemannian manifold and $S \subseteq M$ be an open invex set with respect to $\eta : M \times M \to TM$. Suppose that $f : S \to \mathbb{R}^m$ is a vector valued functions. Then, f is said to be $\eta-$ preinvex on S if each of its components is $\eta-$preinvex on S.

It is worth noting that the concepts of d-invex and strictly d-invex vector valued functions can be defined similarly.

Now, we introduce some notions which are used in the sequel. Let M be a Riemannian manifold and $S \subseteq M$ be an open invex set with respect to $\eta : M \times M \to TM$. Consider the following vector optimization problem

$$(VOP) \qquad \begin{array}{l} min\ f(x) \\ s.t.\ g(x) \leq 0 \\ \quad x \in S, \end{array}$$

where, $f : S \to \mathbb{R}^m$ and $g : S \to \mathbb{R}^n$. Let $D := \{x \in S : g(x) \leq 0\}$ be the set of feasible solutions for (VOP). We denote $I := \{1, ..., m\}$, $J := \{1, ..., n\}$ and $J(x) := \{j \in J : g_j(x) = 0\}$, $\bar{J}(x) := \{j \in J : g_j(x) < 0\}$.

Motivated by [1, 13, 19] we utilize the $\eta-$directional derivative of functions defined on invex subsets of Riemannian manifolds to deduce the optimality conditions.

Definition 5.2. (a) The feasible point $\bar{x}$ is said to be a Pareto solution (a minimum) for (VOP) if there exists no $y \in D$ such that $f(y) \leq f(\bar{x})$.
$\bar{x}$ is said to be a local Pareto solution (a local minimum) for (VOP) if there is a neighborhood $N(\bar{x})$ such that there exists no $y \in D \cap N(\bar{x})$ such that $f(y) \leq f(\bar{x})$.
(b) The feasible point $\bar{x}$ is said to be a weak Pareto solution (a weak minimum) for (VOP) if there exists no $y \in D$ such that $f(y) < f(\bar{x})$.
$\bar{x}$ is said to be a local weak Pareto solution (a local weak minimum) for (VOP) if there is a neighborhood $N(\bar{x})$ such that there exists no $y \in D \cap N(\bar{x})$, $f(y) < f(\bar{x})$.

Lemma 5.1. *Let $\bar{x}$ be a local weak Pareto or a weak Pareto solution for (VOP) and f and g be $\eta-$directional differentiable functions on S. If g_j is continuous at $\bar{x}$ for all $j \in \bar{J}(\bar{x})$ then, the system*

$$f'(\bar{x}; \eta(x, \bar{x})) < 0, \tag{5.1}$$

$$g'_{J(\bar{x})}(\bar{x}; \eta(x, \bar{x})) < 0, \tag{5.2}$$

has no solution for $x \in S$.

Proof. Let $\bar{x}$ be a local weak Pareto solution for (VOP) and suppose that there exists a point $x^* \in S$ such that the inequalities (5.1) and (5.2) are true. Since S is an invex set with respect to η, there exists a unique geodesic $\alpha_{x^*,\bar{x}} : [0, 1] \to M$ such that

$$\alpha_{x^*,\bar{x}}(0) = \bar{x},\ \alpha'_{x^*,\bar{x}}(0) = \eta(x^*, \bar{x}),\ \alpha_{x^*,\bar{x}}(t) \in S, \quad \text{for all } t \in [0, 1].$$

If $\varphi_f(t) := f(\alpha_{x^*,\bar{x}}(t)) - f(\bar{x})$ then, $\varphi_f(0) = 0$. Now, using (5.1) we have

$$\begin{aligned} \lim_{t\to 0^+} \frac{\varphi_f(\alpha_{x^*,\bar{x}}(t)) - \varphi_f(0)}{t} &= \lim_{t\to 0^+} \frac{f(\alpha_{x^*,\bar{x}}(t)) - f(\bar{x})}{t} \\ &= f'(\bar{x}; \eta(x^*, \bar{x})) < 0. \end{aligned}$$

Hence, there is a number $\delta_f > 0$ such that $\varphi_f(t) < 0$ for all $t \in (0, \delta_f)$. It follows that

$$f(\alpha_{x^*,\bar{x}}(t)) < f(\bar{x}), \qquad \text{for all } t \in (0, \delta_f). \tag{5.3}$$

Similarly, by defining

$$\varphi_{g_{J(\bar{x})}}(t) := g_{J(\bar{x})}(\alpha_{x^*,\bar{x}}(t)) - g_{J(\bar{x})}(\bar{x}),$$

and using (5.2) one can prove that

$$g_{J(\bar{x})}(\alpha_{x^*,\bar{x}}(t)) < g_{J(\bar{x})}(\bar{x}), \qquad \text{for all } t \in (0, \delta_{g_{J(\bar{x})}}). \tag{5.4}$$

By the definition of $J(\bar{x})$, we obtain

$$g_{J(\bar{x})}(\alpha_{x^*,\bar{x}}(t)) < 0, \qquad \text{for all } t \in (0, \delta_{g_{J(\bar{x})}}).$$

Since g_j is continuous at $\bar{x}$ for $j \in \bar{J}(\bar{x})$, there is a number δ_j such that

$$(5.5) \qquad g_j(\alpha_{x^*,\bar{x}}(t)) < 0, \qquad \text{for all } t \in (0, \delta_j).$$

Let $\bar{\delta} := \min\{\delta_f, \delta_{g_{J(\bar{x})}}, \delta_j : j \in \bar{J}(\bar{x})\}$ and choose $\delta \leq \bar{\delta}$ small enough such that

$$(5.6) \qquad \alpha_{x^*,\bar{x}}(t) \in N_\delta(\bar{x}), \qquad \text{for all } t \in (0, \delta).$$

Hence, by (5.3)-(5.6) for all $t \in (0, \delta)$ we have

$$(5.7) \qquad \alpha_{x^*,\bar{x}}(t) \in N_\delta(\bar{x}) \cap D, \; f(\alpha_{x^*,\bar{x}}(t)) < f(\bar{x}).$$

This is a contradiction to the assumption that $\bar{x}$ is a local weak Pareto solution for (VOP). □

The following theorem is an extension of Theorem 2.1 in [18].

Theorem 5.2. *Let M be a Reimannian manifold, $S \subseteq M$ be an open invex set with respect to $\eta : M \times M \to TM$. If $f : S \to \mathbb{R}^m$ is an $\eta-$preinvex function on S. Then, either*

$$f(x) < 0 \qquad \textit{has a solution } x \in S$$

or

$$p^t f(x) \geq 0 \quad \textit{for all } x \in S, \quad \textit{for some } p \in \mathbb{R}^m_+,$$

but both alternatives are never true.

Proof. Following [12], the proof depends on establishing the convexity of the set $\Lambda := \bigcup\{\Lambda(x) : x \in S\}$, where

$$\Lambda(x) = \{u \in \mathbb{R}^m : u > f(x)\}, \qquad x \in S.$$

Let u_1, $u_2 \in \Lambda$ and $0 \leq \lambda \leq 1$. Then, there exist x_1, $x_2 \in S$ such that $u_1 \in \Lambda(x_1)$ and $u_2 \in \Lambda(x_2)$. By the invexity of S with respect to η, there exists a unique geodesic $\alpha_{x_1,x_2} : [0, 1] \to M$ such that

$$\alpha_{x_1,x_2}(0) = x_2, \alpha'_{x_1,x_2}(0) = \eta(x_1, x_2), \alpha_{x_1,x_2}(t) \in S, \quad \text{for all } t \in [0, 1].$$

Since f is an $\eta-$preinvex function on S,

$$f(\alpha_{x_1,x_2}(\lambda)) \leq \lambda f(x_1) + (1 - \lambda) f(x_2) < \lambda u_1 + (1 - \lambda) u_2,$$

hence, $\lambda u_1 + (1 - \lambda) u_2 \in \Lambda$. □

In [1] the necessary conditions for $\bar{x} \in D$ to be a weak Pareto solution for (VOP) are given. This is done by assuming the preinvexity of directional differentials of f and $g_{J(\bar{x})}$ at $\bar{x}$. Now we give necessary optimality criteria by extending the above condition on invex subsets of Riemannian manifolds.

Theorem 5.3. (Fritz John type necessary optimality conditions). *Suppose that $\bar{x}$ is a weak Pareto solution for (VOP) and g_j is continuous for $j \in \bar{J}(\bar{x})$. Let f and g be $\eta-$directional differentiable at $\bar{x}$ and $f'(\bar{x}; \eta(x, \bar{x}))$ and $g'_{J(\bar{x})}(\bar{x}; \eta(x, \bar{x}))$ be preinvex* *functions of x on S. Then, there exist $\bar{\lambda} \in \mathbb{R}^m_+$, $\bar{\mu} \in \mathbb{R}^n_+$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ satisfies the following conditions:*

$$\bar{\lambda}^t f'(\bar{x}; \eta(x, \bar{x})) + \bar{\mu}^t g'(\bar{x}; \eta(x, \bar{x})) \geq 0, \text{ for all } x \in S, \tag{5.8}$$

$$\bar{\mu}^t g(\bar{x}) = 0, \tag{5.9}$$

$$g(\bar{x}) \leq 0. \tag{5.10}$$

Proof. If $\bar{x}$ is a weak Pareto solution for (VOP) then, by Lemma 5.1 the system

$$f'(\bar{x}; \eta(x, \bar{x})) < 0, \quad g'_{J(\bar{x})}(\bar{x}; \eta(x, \bar{x})) < 0,$$

has no solution for $x \in S$. Since $f'(\bar{x}; \eta(x, \bar{x}))$ and $g'_{J(\bar{x})}(\bar{x}; \eta(x, \bar{x}))$ are preinvex functions on S then, by Theorem 5.2 there exist $\bar{\lambda} \in \mathbb{R}^m_+$, $\bar{\vartheta} \geq 0$ for $j \in J(\bar{x})$ such that

$$\sum_{i=l}^{m} \bar{\lambda}_i f'_i(\bar{x}; \eta(x, \bar{x})) + \sum_{j \in J(\bar{x})} \bar{\vartheta}_j g'_{J(\bar{x})}(\bar{x}; \eta(x, \bar{x})) \geq 0. \tag{5.11}$$

If we define $\bar{\mu}_j := \bar{\vartheta}_j$ for $j \in J(\bar{x})$ and $\bar{\mu}_j := 0$ for $j \in \bar{J}(\bar{x})$ then, (5.8) is satisfied. From feasibility of $\bar{x}$, we get (5.9). Now, (5.10) is immediate. □

Definition 5.3. The function g is said to satisfy the generalized Slater's constraint qualification at $\bar{x} \in D$ if g is d-invex at $\bar{x}$ and there exists a point $x^* \in D$ such that $g_j(x^*) < 0$, $j \in J(\bar{x})$.

The proofs of the following theorems are similar to the proofs of Theorems 12 and 13 in [1], respectively.

Theorem 5.4. (Karush-Kuhn-Tucker type necessary optimality conditions). *Suppose that $\bar{x}$ is a weak Pareto solution for (VOP) and g_j is continuous for $j \in \bar{J}(\bar{x})$. Let f and g be $\eta-$directional differentiable at $\bar{x}$ and $f'(\bar{x}; \eta(x, \bar{x}))$ and $g'_{J(\bar{x})}(\bar{x}; \eta(x, \bar{x}))$ be preinvex functions of x on S. Moreover, we assume that g satisfies the generalized Slater's constraint qualification at $\bar{x}$. Then, there exists $\bar{\mu} \in \mathbb{R}^n_+$ such that $(\bar{x}, \bar{\mu})$ satisfies the following conditions:*

$$\begin{cases} f'(\bar{x}; \eta(x, \bar{x})) + \bar{\mu}^t g'(\bar{x}; \eta(x, \bar{x})) \geq 0, \text{ for all } x \in S, \\ \bar{\mu}^t g(\bar{x}) = 0, \quad g(\bar{x}) \leq 0. \end{cases} \tag{5.12}$$

Theorem 5.5. (Sufficient optimality condition). *Let $\bar{x}$ be a feasible solution for (VOP) at which Karush-Kuhn-Tucker conditions (5.12) are satisfied. Moreover, if f and g are d-invex functions on D then, $\bar{x}$ is a weak Pareto solution for (VOP).*

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