The geometry of $(2,0)-$jet bundles

Violeta Zalutchi

Abstract. In this note we make an introduction to higher order holomorphic geometry. Two types of holomorphic jet bundles are compared. The $(2,0)$-jet bundles present some resemblances with the well known real osculator bundle. For this we study the main topics of this geometry: the fibre structure, its complexified tangent bundle, the decomposition by a nonlinear complex connection, $N-$linear complex connection.


Key words: Jet bundle; holomorphic bundle; complex nonlinear connection.

1 Holomorphic jet bundles

The complex jet bundles notion was firstly introduced by Green and Griffiths, [9], with the aimed to get a natural filtration of the sheaf of $k-$jet differentials.

There are known two different structures of jet bundles. According to Stoll and Wong, [20], one of the structures is called the full jet bundle (or holomorphic tangent bundle of order $k$) and the second is called the restricted jet bundle, which has a natural geometric meaning which we shall emphasize in the next section.

In a set of papers, [20, 8, 21], are studied the problems of algebraic geometry for a holomorphic jet bundle. The holomorphic jet bundle has a natural structure of complex manifold, whose total space will be further studied by the same schedule as for the real case for the known $k-$ osculator bundle, topic intensively investigated in the last decade, especially by the Romanian geometers, [12, 13, 7, 18, 6], etc.

In this introductory section we briefly revise the building of both the mentioned above holomorphic structures and will point out their differences.

Let $M$ be a complex manifold, $\dim_C M = n$, and let $(z^i)_{i=1}^n$ be the complex coordinates in a local chart $(U, \varphi)$ near $z \in U$. Consider the differential operator $D_i = \frac{\partial}{\partial z^i}$ and its conjugate $\overline{D_i} = \frac{\partial}{\partial \bar{z}^i}$. The set of germ of holomorphic $k-$jets, denoted $T^k M$ is the subsheaf of the sheaf of homomorphisms $\text{Hom}_C(O_M, O_M)$ built on the operators $\sum_{I=1}^{k} D_I \circ D_J$, where $D_I = D_{i_1} \circ D_{i_2} \circ \ldots \circ D_{i_p}$, $I = \{1 \leq i_1 \leq i_2 \leq \ldots \leq i_p \leq k\}$ and $D_J = D_{j_1} \circ D_{j_2} \circ \ldots \circ D_{j_q}$, $J = \{1 \leq j_1 \leq j_2 \leq \ldots \leq j_q \leq k\}$.

Of course, this means that any element of $T^k M$ is locally of the form
\[
\sum_{I+J=1}^{k} a_{IJ} D_I \circ D_J,
\]
where $a_{IJ}$ are symmetric functions in all indices of $I$, respectively of $J$. The operators $D_I \circ D_J$ will be called to be of $(I, J)$--type. $T^k M$ is locally free and $T^{k-1} M$ injects into $T^k M$. There exists an isomorphism $\mu : D_I \circ D_J \to (\partial I) \circ (\partial J)$, and the symmetric product $\circ^k TM = \circ^p TM \circ \circ^q TM$. Generally, $T^k M$ is not a holomorphic vector bundle, but $\circ^p TM$ is a holomorphic one, and via this isomorphism $\mathcal{T}((k, o)) \equiv \circ^k TM$ is a holomorphic vector bundle as well, called the $k$-jet bundle. For the changes of holomorphic coordinates on $\mathcal{T}((k, o)) M$, we refer to in \cite[formula (1)]{20}.

The second kind of holomorphic bundle is related to $k$--th order jets differentials, being introduced by Green and Griffiths in \cite{9}. We consider $\mathcal{H}_z$, $z_0 \in M$, the sheaf of germs of holomorphic curves $\{ f : \Delta_r \to M, f \in \mathcal{H}_z, f(0) = z_0 \}$. Let $(z^i)_{i=1}^{\overline{mn}}$ be the complex coordinates in a local chart $(U, \varphi)$ and $(z^i)_{i=1}^{\overline{mn}}$ complex coordinates in $(U', \psi')$ from $z_0 \in M$, with the changes
\[
\left. \frac{\partial z^i}{\partial z^j} \right|_0 \neq 0.
\]
Since $M$ is a complex manifold, we have $\frac{\partial z^i}{\partial z^j} = 0$, $\forall i, j = 1, \overline{m}n$.

We denote by $f' = z' \circ f$, $\forall i = 1, \overline{m}n$ and consider a complex parameter $\theta \in \Delta_r \subset C$. According to \cite{20}, $f, g \in \mathcal{H}_z$ are said to be $k$--equivalent, and denoted as $f \overset{k}{\sim} g$, iff $f'(0) = g'(0)$, and $\frac{\partial^p f^j}{\partial^q \theta^p}(0) = \frac{\partial^p g^j}{\partial^q \theta^p}(0), \forall i = 1, \overline{m}, p = 1, \overline{k}$.

The class of $f$ is $[f]_k$ and the set of all classes is $\mathcal{J}((k, 0)) M = \cup_{z_0 \in M} \mathcal{H}_{z_0}//\mathcal{J}$. By $j^k f(0) = \left( f(0), \frac{df}{d\theta}(0), ..., \frac{d^k f}{d\theta^k}(0) \right)$ we denote the $k$-jet of $f \in [f]_k$.

Let $\pi((k, 0)) : \mathcal{J}((k, 0)) M \to M$ be the canonical projection. Then we immediately check that $(\mathcal{J}((k, 0)) M, \pi((k, 0)))$ has a fibre bundle structure, named in \cite{20} the restricted $k$--jet bundle, and in \cite{8} the parametrized $k$--jet bundle. Actually, $\mathcal{J}((k, 0)) M$ has the following geometric meaning. Let $c : \Delta_r \to M$ be a holomorphic curve parametrically given by $z^i = z^i(\theta)$, $i = 1, \overline{m}n$, with $z^i(0) = z_0$ and $\frac{dz^i}{d\theta}(0) = 0$. Then $c$ given by
\[
\left. \frac{dz^i}{d\theta}(0) \right|_0 = 0.
\]
span the class $[f]_k$ if we set $f' = z'^* \circ f$. Therefore the terminology of parametrized $k$--jet bundle is justifiable.

$\mathcal{J}((k, 0)) M$ has indeed a structure of a differentiable manifold, which we subsequently mark out. If $(z^i)_{i=1}^{\overline{mn}}$ are complex coordinates in $(U, \varphi)$ near $z \in U$, then
\[
Z = \begin{pmatrix}
\left. \frac{dz^1}{d\theta}(0) \right|_0 & \left. \frac{dz^2}{d\theta}(0) \right|_0 & \left. \frac{dz^3}{d\theta}(0) \right|_0 & \left. \frac{dz^4}{d\theta}(0) \right|_0 & \left. \frac{dz^5}{d\theta}(0) \right|_0 \\
\left. \frac{d^2 z^1}{d\theta^2}(0) \right|_0 & \left. \frac{d^2 z^2}{d\theta^2}(0) \right|_0 & \left. \frac{d^2 z^3}{d\theta^2}(0) \right|_0 & \left. \frac{d^2 z^4}{d\theta^2}(0) \right|_0 & \left. \frac{d^2 z^5}{d\theta^2}(0) \right|_0 \\
\left. \frac{d^3 z^1}{d\theta^3}(0) \right|_0 & \left. \frac{d^3 z^2}{d\theta^3}(0) \right|_0 & \left. \frac{d^3 z^3}{d\theta^3}(0) \right|_0 & \left. \frac{d^3 z^4}{d\theta^3}(0) \right|_0 & \left. \frac{d^3 z^5}{d\theta^3}(0) \right|_0 \\
... & \left. \frac{d^k z^1}{d\theta^k}(0) \right|_0 & \left. \frac{d^k z^2}{d\theta^k}(0) \right|_0 & \left. \frac{d^k z^3}{d\theta^k}(0) \right|_0 & \left. \frac{d^k z^5}{d\theta^k}(0) \right|_0 \\
\end{pmatrix}
\]
will be complex coordinates in $(U, \Phi)$ from $\mathcal{J}((k, 0)) M$. Let us note that $(Z^i)$ are holomorphic functions, since:
\[
\left. \frac{\partial z^i}{\partial z^j}(0) \right|_0 = 0 ; \left. \frac{\partial z^i}{\partial \theta}(0) \right|_0 = \left. \frac{d}{d\theta} \left( \frac{\partial z^i}{\partial z^j} \right)(0) \right|_0 = 0 ;
\]
\[
\frac{\partial^{(2)}}{\partial \eta^1} = \frac{1}{2} \frac{\partial}{\partial \theta} \left( \frac{\partial^{(1)}}{\partial \eta^1} \right) (0) = \frac{1}{2} \frac{\partial}{\partial \theta} \left( \frac{\partial^{(1)}}{\partial \eta^1} \right) (0) = 0; \text{Analogue, } \frac{\partial^{(2)}}{\partial \eta^1} (0) = 0, \text{ etc.}
\]

Moreover, due to the holomorphy requirement, we have:

\[
\begin{align*}
(1.3) & \quad \eta^i = \frac{dz^i}{d\theta} \\
(2) & \quad \eta^i = \frac{d}{d\theta} \eta^i = \frac{1}{2} \frac{\partial}{\partial z^j} \eta^j \\
(k) & \quad \eta^i = \frac{1}{k!} \frac{d}{d\theta} \eta^i = \frac{1}{k!} \frac{\partial}{\partial z^j} \eta^j.
\end{align*}
\]

If \( g \in J^k f(0) \), we easily infer that

\[
\sum_{\alpha_1, \ldots, \alpha_p = 1}^{n} (D_{(\alpha_1)} f) \eta^{i_1} \ldots \eta^{i_n} = \sum_{\alpha_1, \ldots, \alpha_p = 1}^{n} (D_{(\alpha_1)} g) \eta^{i_1} \ldots \eta^{i_n}, \forall I \subset \{1 \leq i_1 \leq \ldots \leq i_p \leq n\} \text{ and } D_{Jf} = D_{Jg} = 0.
\]

Let \((U', \Phi')\) be another chart over \( Z \) from \( J^{(k,0)} M \), with \( Z = (z^i, \eta^1, \ldots, \eta^k) \).

Then the changes on the complex manifold \( J^{(k,0)} M \) are given by:

\[
\begin{align*}
(1.4) & \quad z'^i = z'^i(z^i) \\
(1) & \quad \eta'^i = \frac{\partial z'^i}{\partial z^j} \eta^j \\
(2) & \quad 2 \eta'^i = \frac{\partial \eta'^i}{\partial z^j} \eta^j + 2 \frac{\partial \eta'^i}{\partial \eta^j} \eta^j \\
(k) & \quad k \eta'^i = \frac{\partial \eta'^i}{\partial z^j} \eta^j + 2 \frac{\partial \eta'^i}{\partial \eta^j} \eta^j + \ldots + k \frac{\partial \eta'^i}{\partial \eta^j} \eta^j.
\end{align*}
\]

Some remarks are appropriate at this place.

Firstly, \( J^{(k,0)} M \) has not a vector bundle structure aside from \( k = 1 \), and then it is identified with \( T^* M \), the holomorphic tangent bundle.

Further, the described structure for \( J^{(k,0)} M \) depends on the choice of the parameter \( \theta \) on the curves \( j^k f \). If we consider the \( \mathbb{C}^* \)-action \( f_\lambda(\theta) = f(\lambda \theta), \lambda \in \mathbb{C}^* \), then \( j^k f_\lambda(0) = \left( f(0), \lambda \frac{df}{d\theta}(0), \ldots, \lambda^k \frac{d^k f}{d\theta^k}(0) \right) \) and hence, \( J^{(k,0)} M \) we assume that it is well defined up to this \( \mathbb{C}^* \)-action.

We note that the rank of the fibre bundle \( J^{(k,0)} M \) is \( kn \), while the dimension of the complex manifold structure is \( (k + 1)n \).

As proved in [20], the map locally given by

\[
j^k f(0) \rightarrow \frac{d^k f}{d\theta^k}(0) D_{J1} + \frac{d^{k-1} f}{d\theta^{k-1}}(0) \frac{d^k f}{d\theta^k}(0) D_{J2} + \ldots + \frac{d^k f}{d\theta^k}(0) D_{Jk} \circ \ldots \circ D_{Jk}
\]
is a holomorphic (but not biholomorphic) map from \( J^{(k,0)}M \) onto \( T^kM \). The bundle \( J^{(k-1,0)}M \) is not injected into \( J^{(k,0)}M \), like is the case for \( T^kM \). There are some other differences between these bundles, such as their ranks. In exchange, \( J^{(k,0)}M \) is a fibre bundle over \( M \) but also over \( J^{(h,0)}M \), for \( 1 \leq b < k \), \( \pi^{(k,h)} : J^{(k,0)}M \to J^{(h,0)}M \), \((1) \) \((h) \) locally by restricting to the coordinates \( (z^i, \eta^j, \zeta^j) \).

A significant remark is that \((1.2) \) provides a particular jet, let us say of \( (k,0) \)-type for \( f \). In general, a \( (p,q) \)-jet on \( M \) can be spanned by \( \frac{\partial f}{\partial \theta_0}(0), \frac{\partial f}{\partial \theta_0}(0), \frac{\partial^2 f}{\partial \theta_0^2}(0), \frac{\partial^3 f}{\partial \theta_0^3}(0), \frac{\partial^4 f}{\partial \theta_0^4}(0), \ldots \), where \( f \in \mathcal{F}(M) \), is not necessary holomorphic in \( z_0 = f(0) \). In this position \( J^{(p,q)}M \) is not always holomorphic. Certainly, if \( f \) is in \( \mathcal{H}_{z_0} \) then \( \frac{\partial f}{\partial \theta_0}(0) = 0 \), and it shows that \( J^{(p,0)}M \) is a holomorphic subbundle of \( J^{(p,q)}M \).

Finally, we specify that the corresponding real \( k \)-jet bundle has a special geometric name, that of \( k \)-osculator bundle, \( \text{Osc}^kM \) bundle (e.g., \([12, 13, 7, 18]\)). The real and imaginary parts of \( J^{(k,0)}M \) define both of them \( \text{Osc}^kM \) bundles, while this does not generally hold for \( J^{(p,q)}M \).

2 The geometry of holomorphic \( J^{(2,0)}M \) bundles

In this section for the sake of simplicity we limit ourselves to the case of the \( J^{(2,0)}M \) space of \((2,0)\)-holomorphic functions, while the general case \( J^{(k,0)}M \) can be immediately derived.

There exits in our approach a strongly similitude to the study of the real case for the \( \text{Osc}^kM \) bundle.

Let \( M \) be a complex manifold, \( T_C M = T'M \oplus T''M \), the complexified tangent bundle of \((1,0)\-) and of \((0,1)\-)type vectors, respectively. If \((z^i)_{i=1}^m\) are complex coordinates, then \( T'_M \) is spanned by \( \{ \frac{\partial}{\partial z^i} \}_{i=1}^m \) and \( T''_M \) is spanned by \( \{ \frac{\partial}{\partial \bar{z}^i} \}_{i=1}^m \), moreover \( T'M \) is a holomorphic vector bundle.

A similar construction can be made for the \( T_C(J^{(2,0)}M) \) bundle, still mentioning that the corresponding tangent subspaces do not admit a vector bundle structure.

Indeed, let \( Z = (z^i, \eta^j, \zeta^j) \) be local coordinates in the chart \((U, \Phi)\) from \( J^{(2,0)}M \), with the \((1.4) \) changes for \( k = 2 \). Further, for simplifying the formalism, we shall use the following notations \( Z = (z^i, \eta^i =: \eta^i, \zeta^i =: \zeta^i) \).

A local basis in \( T'_Z(J^{(2,0)}M) \) is \( \{ \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \zeta^i} \}_{i=1}^m \) and in \( T''_Z(J^{(2,0)}M) \) theirs conjugates \( \{ \frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{\eta}^i}, \frac{\partial}{\partial \bar{\zeta}^i} \}_{i=1}^m \). Due to holomorphic changes on \( J^{(2,0)}M \), that is all of \( \frac{\partial z^i}{\partial \bar{z}^j}, \frac{\partial \eta^i}{\partial \bar{\eta}^j}, \frac{\partial \zeta^i}{\partial \bar{\zeta}^j}, \frac{\partial \bar{z}^i}{\partial z^j}, \frac{\partial \bar{\eta}^i}{\partial \eta^j}, \frac{\partial \bar{\zeta}^i}{\partial \zeta^j} \) are vanishing, and also theirs conjugates, it follows that local bases from \( T'_Z(J^{(2,0)}M) \) change w.r.t. the transformations \((1.4) \) by the rules:

\[
\frac{\partial}{\partial z^j} = \frac{\partial z^i}{\partial z^j} \frac{\partial}{\partial z^i} + \frac{\partial \eta^i}{\partial z^j} \frac{\partial}{\partial \eta^i} + \frac{\partial \zeta^i}{\partial z^j} \frac{\partial}{\partial \zeta^i}.
\]
In view of (1.4) we immediately infer that \( \frac{\partial z^n}{\partial y^p} = \frac{\partial y^p}{\partial y^p} = \frac{\partial z^n}{\partial y^p} \), but in change \( \frac{\partial q_i}{\partial z^j} = \frac{\partial q_i}{\partial z^j} \) contain the second order derivatives of \( z^n \), while \( \frac{\partial q_i^j}{\partial z^j} \) contains even the 3-th derivatives of \( z^n \). Thus the nonlinear type of the changes (2.1) is obvious. By conjugation everywhere in (2.1) we obtain the changes of local bases \( \{ \frac{\partial}{\partial z^j}, \frac{\partial}{\partial y^p}, \frac{\partial}{\partial y^p} \} \) from \( T_{2}(J^{(2,0)}M) \).

On \( T_C(J^{(2,0)}M) \) the natural complex structure \( J^2 = -I \) acts as follows:

\[
(2.2) \quad J \left( \frac{\partial}{\partial z^j} \right) = \imath \frac{\partial}{\partial q_i} \quad ; \quad J \left( \frac{\partial}{\partial q_i} \right) = \imath \frac{\partial}{\partial q_i} \quad ; \quad J \left( \frac{\partial}{\partial q_i} \right) = \imath \frac{\partial}{\partial q_i}
\]

but also the almost second order tangent structure \( F^3 = 0 \),

\[
(2.3) \quad F \left( \frac{\partial}{\partial z^j} \right) = \frac{\partial}{\partial q_i} \quad ; \quad F \left( \frac{\partial}{\partial q_i} \right) = \frac{\partial}{\partial q_i} \quad ; \quad F \left( \frac{\partial}{\partial q_i} \right) = 0
\]

and analogously for the conjugates.

The structure \( F \) is globally defined - as follows from the previous remarks concerning the (2.1) changes. Moreover, \( F \) commutes with \( J \) and \( \text{rank } F = 2n, \text{rank } J = 3n \).

Consider \( W = \text{Ker } F^2 \) and \( V = \text{Ker } F \). We infer that \( V \subset W \) and \( F(W) = V \) are locally spanned by \( W = < \frac{\partial}{\partial q_i}, \frac{\partial}{\partial y^p} > \) and \( V = < \frac{\partial}{\partial q_i} > \).

The distribution \( V \) will be called vertical. Like in real case, we can consider the following well defined Liouville fields on \( T_C(J^{(2,0)}M) \),

\[
(2.4) \quad (1) C = \eta^i \frac{\partial}{\partial q_i} \quad \text{and} \quad (2) C = \eta^i \frac{\partial}{\partial q_i} + 2 \xi^i \frac{\partial}{\partial q_i}
\]

and their conjugates.

We say that a complex field \( S \in T^r(J^{(2,0)}M) \) is a complex 2-spray iff \( F \circ S = (2) \).

**Proposition 2.1.** Any complex 2-spray is locally written as \( S = \eta^i \frac{\partial}{\partial z^j} + 2 \xi^i \frac{\partial}{\partial y^p} = 3G^i(z, \eta, \zeta) \frac{\partial}{\partial z^j} \), where \( G^i \) are the coefficients of the spray and they change by the rule

\[
3G^i = 3 \frac{\partial z^n}{\partial z^j} G^j - \left( \eta^i \frac{\partial z^n}{\partial z^j} + 2 \xi^i \frac{\partial z^n}{\partial y^p} \right).
\]

Let \( W(J^{(2,0)}M) \) be the fibre bundle defined by the distribution \( W \) at any point \( Z \in J^{(2,0)}M \).

**Definition 2.2.** A complex nonlinear connection on \( J^{(2,0)}M \) (in brief, c.n.c.), is a complex subbundle \( H(J^{(2,0)}M) \), supplementary to \( W(J^{(2,0)}M) \) in \( T'(J^{(2,0)}M) \).

This notion has a special role in the "linearizing" of the geometry of \( T_C(J^{(2,0)}M) \). Obviously, we have \( \text{dim } H(J^{(2,0)}M) = n \) and by conjugation we get

\[
(2.5) \quad T_C(J^{(2,0)}M) = H(J^{(2,0)}M) \oplus W(J^{(2,0)}M) \oplus \overline{H}(J^{(2,0)}M) \oplus \overline{W}(J^{(2,0)}M).
\]
It has been pointed out that \( \pi^{(2,0)} : J^{(2,0)} M \to M \) admits a fibre bundle structure over \( J^{(1,0)} M \), identified with \( TM \). Let \( \pi^{* (2,0)} M \) be the pull-back bundle of \( T' M \) onto \( T' (J^{(2,0)} M) \). The sections of \( \pi^{* (2,0)} M \) are identified with the vectors of \( T' (J^{(2,0)} M) \) of the form \( \{ \frac{\partial}{\partial \eta^j} \} \). Denoting by \( \iota^h : T' M \to H(J^{(2,0)} M) \subset T'(J^{(2,0)} M) \) the isomorphism which provides \( \pi^{* (2,0)} \circ \iota^h = \text{Id}_{T'M} \).

**Proposition 2.3.** The following sequence is exact:

\[
0 \to W(J^{(2,0)} M) \xrightarrow{i} T'(J^{(2,0)} M) \xrightarrow{\pi^{* (2,0)} M} 0.
\]

A splitting \( \mathcal{C} \) of this sequence determines a (c.n.c.) on \( J^{(2,0)} M \) and \( \text{Ker} \mathcal{C} = H(J^{(2,0)} M) \). Let be \( \frac{\delta}{\delta z^j} =: \iota^h (\frac{\partial}{\partial \eta^j}) \), and then the vectors \( \{ \frac{\delta}{\delta z^j} \}_{i=1}^n \) determine a local basis in the horizontal bundle \( H(J^{(2,0)} M) \), called adapted base to the (c.n.c.). Let be

\[
(2.6) \quad \frac{\delta}{\delta z^j} = \frac{\partial z^{\eta_i}}{\partial \delta z^j} \frac{\delta}{\partial \eta^i} = N^i_j \frac{\partial}{\partial \eta^i} - N^i_j \frac{\partial}{\partial \xi^i}.
\]

Since the horizontal lift \( \iota^h \) is an isomorphism of fibres, it follows that the adapted basis \( \{ \frac{\delta}{\delta z^j} \} \), which spans the horizontal distribution, satisfies:

\[
(2.7) \quad \frac{\delta}{\delta z^j} = \frac{\partial z^{\eta_i}}{\partial \delta z^j} \frac{\delta}{\partial \eta^i}.
\]

**Theorem 2.4.** The functions \( N^i_j, N^j_i \), called the coefficients of the (c.n.c.), change by the rules:

\[
(2.8) \quad N^i_k \frac{\partial z^{l \eta_k}}{\partial z^j} = \frac{\partial z^{\eta_i}}{\partial \delta z^j} N^i_j - \frac{\partial \eta^i}{\partial \delta z^j},
\]

\[
(2.9) \quad N^j_k \frac{\partial z^{l \eta_k}}{\partial z^j} = \frac{\partial z^{\eta_i}}{\partial \delta z^j} N^j_k + \frac{\partial \eta^i}{\partial \delta z^j} N^j_k - \frac{\partial \eta^i}{\partial \delta z^j}.
\]

The proof relies on replacing the changes (2.1) in (2.7).

Summarizing, \( T'(J^{(2,0)} M) \) is a direct sum of the \( H(J^{(2,0)} M) \) and \( W(J^{(2,0)} M) \) bundles, of rank \( n \) and respectively \( 2n \), locally spanned by \( \{ \frac{\delta}{\delta z^j} \} \) and respectively \( \{ \frac{\partial}{\partial \eta^j} \} \). Moreover, the bundle \( V(J^{(2,0)} M) \) defined by the vertical distribution and called the vertical bundle of \( (2,0) \)-jets, is a holomorphic subbundle of \( W(J^{(2,0)} M) \) and hence of \( T'(J^{(2,0)} M) \). Now we can split \( W(J^{(2,0)} M) \) by means of the almost tangent structure \( F \) into \( V(J^{(2,0)} M) \) and some supplement \( H_1(J^{(2,0)} M) =: F(H(J^{(2,0)} M)) \).

From (2.6) and the definition (2.3) of the \( F \) structure, it results that the distributions of \( H_1(J^{(2,0)} M) \) are spanned by a set of vectors of the form:

\[
(2.9) \quad \frac{\delta}{\delta \eta^j} =: F(\frac{\delta}{\delta z^j}) = \frac{\partial}{\partial \eta^j} - N^i_j \frac{\partial}{\partial \xi^i},
\]

where \( N^i_j \) satisfies the same change rule as in (2.8) and \( \frac{\delta}{\delta \eta^j} = \frac{\partial z^{\eta_i}}{\partial \delta z^j} \frac{\delta}{\partial \eta^i} - \frac{\partial z^{\eta_i}}{\partial \delta z^j} \frac{\delta}{\partial \xi^i} \). Hence the following decomposition is obtained, \( T'(J^{(2,0)} M) = H(J^{(2,0)} M) \oplus H_1(J^{(2,0)} M) \oplus \).
V(J^{(2,0)}M), each of these of rank \( n \), and by conjugation is obtained a decomposition for \( T^n(J^{(2,0)}M) \). Finally a decomposition as a sum of six subbundles for \( T_C(J^{(2,0)}M) \) is obtained.

Let \( \{dz^i, \delta\eta^i, \delta\xi^i\}_{i=1}^{n} \) be the dual of the obtained adapted basis \( \{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \xi^i}\}_{i=1}^{n} \) by using a (c.n.c.). If \( \delta\eta^i = d\eta^i + M_j^i dz^j \) and \( \delta\xi^i = d\xi^i + M_j^i d\eta^i + M_j^i dz^j \), then

**Proposition 2.5.** \( M_j^i = N_j^i \) and \( M_j^i = N_j^i + N_k^i N_j^k \).

The proof follows from the definition of the dual basis: \( dz^i(\frac{\partial}{\partial x^j}) = \delta\eta^i(\frac{\partial}{\partial \eta^j}) = d\xi^i(\frac{\partial}{\partial \xi^j}) = \delta^i_j \), the Kronecker symbol, and zero for all the other choices.

We shall further use the following abbreviations: \( \delta_{0j} := \frac{\partial}{\partial \eta^j} ; \delta_{1j} := \frac{\partial}{\partial \xi^j} ; \delta_{2j} = \frac{\partial}{\partial \tau^j} \) and \( \delta_{0j}, \delta_{1j}, \delta_{2j} \) for their conjugates.

**Proposition 2.6.** The brackets of the adapted basis of a (c.n.c) are given by:

\[
\begin{align*}
[\delta_{0j}, \delta_{0k}] &= (1) A_{(jk)} (1) \delta_{11} + (1) A_{(jk)} (1) \delta_{21} \& \text{where } A_{(jk)} : A_j^k - A_k^j - \delta_{0k} N_j^i - \delta_{0j} N_k^i \\
[\delta_{0j}, \delta_{0k}] &= (1) A_{j^k} (1) \delta_{11} + (1) A_{k^j} (1) \delta_{21} \& \text{where } A_{j^k} := \delta_{0k} N_j^i + A_j^k - A_k^j \\
[\delta_{0j}, \delta_{1k}] &= (1) A^k_{j^1} (1) \delta_{21} + (1) A_{j^1} (1) \delta_{21} \& \text{where } B_{j^1} := \delta_{1k} N_j^i \& \alpha = 1, 2 \\
[\delta_{0j}, \delta_{1k}] &= (1) A_{j^2} (1) \delta_{21} + (1) A_{j^2} (1) \delta_{21} \& \text{where } B_{j^2} := \delta_{1k} N_j^i \\
[\delta_{0j}, \delta_{2k}] &= (1) C_{j^1} (1) \delta_{21} + (1) C_{j^1} (1) \delta_{21} \& \text{where } C_{j^1} := \delta_{2k} N_j^i \\
[\delta_{0j}, \delta_{2k}] &= (1) C_{j^2} (1) \delta_{21} + (1) C_{j^2} (1) \delta_{21} \& \text{where } C_{j^2} := \delta_{2k} N_j^i \\
[\delta_{1j}, \delta_{1k}] &= (1) B_{j^1} (1) \delta_{21} := (B_{j^1} - B_{k^1}) \delta_{21} \\
[\delta_{1j}, \delta_{1k}] &= (1) B_{j^1} (1) \delta_{21} - (1) C_{j^1} (1) \delta_{21} \\
[\delta_{1j}, \delta_{2k}] &= (1) C_{j^1} (1) \delta_{21} \& [\delta_{1j}, \delta_{2k}] = (1) C_{j^2} (1) \delta_{21} \\
[\delta_{2j}, \delta_{2k}] &= 0 ; [\delta_{2j}, \delta_{2k}] = 0.
\end{align*}
\]

We note that the horizontal distribution \( H \) is integrable if \( A_{j^k} = A_{j^k} = 0 \) for all \( \alpha = 1, 2 \), and with respect to the adapted bases of a (c.n.c) we have:

\[
J(\delta_{\beta j}) = i\delta_{\beta j} ; J(\delta_{\beta j}) = -i\delta_{\beta j} ; \& \beta = 0, 1, 2
\]

and

\[
F(\delta_{0j}) = \delta_{1j} ; F(\delta_{1j}) = \delta_{2j} ; F(\delta_{2j}) = 0 ; \\
F(\delta_{0j}) = \delta_{1j} ; F(\delta_{1j}) = \delta_{2j} ; F(\delta_{2j}) = 0.
\]
Hence we may consider the following structure:

\[(2.11) \quad F^*(\delta_{2j}) = \delta_{1j} ; \quad F^*(\delta_{1j}) = \delta_{0j} ; \quad F^*(\delta_{0j}) = 0 ; \quad F^*(\delta_{2j}) = \delta_{1j} ; \quad F^*(\delta_{1j}) = \delta_{0j} ; \quad F^*(\delta_{0j}) = 0.\]

Let \(h, h_1, v, \bar{h}, h_1, \bar{v}\) be the projectors on the six distributions determined by a \((c.n.c.)\) on \(J^{(2,0)}M\) and the almost second order tangent structure \(F\).

**Proposition 2.7.** \(F^*\) is a well defined almost second order tangent structure, \(F^*\circ h = 0\), called adjoint to \(F\), and its restriction to \(T'(J^{(2,0)}M)\) satisfies:

\(i) \quad F^*h = 0, \quad FF^* = v + h_1 ; \quad FF^* = h + h_1 ; \quad ii) \quad vF^* = 0 ; \quad F^*h = 0 ; \quad (h_1 + v)F^* = 0 ; \quad iii) \quad F^2F^* = vF ; \quad F^*F^2 = hF ; \quad F^*F^2 = v ; \quad F^*F^2 = h.\)

Similar formulas are fulfilled on \(T''(J^{(2,0)}M)\).

We shall further introduce a special type of derivation law acting on \(TC(J^{(2,0)}M)\), named \(N\)–complex linear connection of \(J^{(2,0)}M\), in brief \(N\)–\((c.l.c.)\).

Any derivation law \(D\) on \(TC(J^{(2,0)}M)\) admits the decomposition \(D = D' + D''\). Moreover, \(D\) is called distinguished, \(d-(c.l.c.)\), if it preserves the six distributions of \(TC(J^{(2,0)}M)\) defined by a \((c.n.c.)\).

**Proposition 2.8.** \(D\) is a \(d-(c.l.c.)\) if and only if \(DJ = 0\) and \(Dh = Dh_1 = Dv = Dh = Dh_1 = D\bar{v} = 0\).

**Proof.** \(DJ = 0\) implies \(D'\) is of \((1,0)\) type and \(D''\) is of \((0,1)\) type. Further, \(Dh = 0\) means that \(D_X(hY) = hD_XY\), and hence \(D\) preserves the horizontal distribution. Similarly, \(Dh_1 = 0\) means that \(D\) preserves \(H_1\), etc. \(\square\)

A \(d-(c.l.c.)\) admits the decomposition \(D = D^h + D^{h_1} + D^v + D^h + D^{h_1} + D^v\) as follows:

\[D^h_{\delta_{0j}} \delta_{0j} = L^h_{jk} \delta_{0i} ; \quad D^h_{\delta_{1j}} \delta_{1j} = F^h_{jk} \delta_{0i} ; \quad D^h_{\delta_{0j}} \delta_{2j} = L^h_{jk} \delta_{2i} ; \quad D^h_{\delta_{1j}} \delta_{2j} = F^h_{jk} \delta_{2i} ; \quad D^h_{\delta_{2j}} \delta_{0j} = 0 ; \quad D^h_{\delta_{2j}} \delta_{1j} = C^h_{jk} \delta_{1i} ; \quad D^h_{\delta_{2j}} \delta_{2j} = C^h_{jk} \delta_{2i} ; \quad D^v_{\delta_{0j}} \delta_{0j} = 0 ; \quad D^v_{\delta_{1j}} \delta_{1j} = L^v_{jk} \delta_{1i} ; \quad D^v_{\delta_{2j}} \delta_{2j} = L^v_{jk} \delta_{2i} ; \quad \text{and} \]

\[D^{h_1}_{\delta_{0j}} \delta_{0j} = 0 ; \quad D^{h_1}_{\delta_{1j}} \delta_{1j} = L^{h_1}_{jk} \delta_{1i} ; \quad D^{h_1}_{\delta_{2j}} \delta_{2j} = L^{h_1}_{jk} \delta_{2i} ; \quad \text{with} \quad D_{X\bar{Y}} = D^h_{X\bar{Y}}.\]

The coefficients of the \(d-(c.l.c.)\) \(D\) chang by the rules:

\[L^h_{jk} = \frac{\partial z^i}{\partial \bar{z}^j} \frac{\partial z^p}{\partial \bar{z}^q} \frac{\partial z^q}{\partial \bar{z}^k} L^p_{rq} + \frac{\partial z^i}{\partial \bar{z}^j} \frac{\partial^2 z^p}{\partial \bar{z}^q \partial \bar{z}^k} ; \quad \alpha = 0, 1, 2\]
and all the other coefficients are \(d\)– complex tensors, i.e.,

\[
L^{\alpha}_{jk} = \frac{\partial z^i}{\partial z^p} \frac{\partial z^q}{\partial z^k} L^{\alpha}_{pq} \quad \alpha = 0, 1, 2
\]

\[
F^{\alpha}_{jk} = \frac{\partial z^i}{\partial z^p} \frac{\partial z^q}{\partial z^k} F^{\alpha}_{pq} \quad \alpha = 0, 1, 2
\]

\[
C^{\alpha}_{jk} = \frac{\partial z^i}{\partial z^p} \frac{\partial z^q}{\partial z^k} C^{\alpha}_{pq} \quad \alpha = 0, 1, 2
\]

Such connection generally infer heavy computation.

We say that \(D\) is a \(N\)--(c.l.c) if and only if

\[
0 = \alpha 1^2 \quad \beta 1^2, \quad \gamma 1^2 \quad \delta 1^2, \quad \epsilon 1^2
\]

\[
F^{\alpha}_{jk} = F^{\alpha}_{jk}; \quad F^{\alpha}_{jk} = F^{\alpha}_{jk}; \quad C^{\alpha}_{jk} = C^{\alpha}_{jk}; \quad C^{\alpha}_{jk} = C^{\alpha}_{jk}
\]

**Theorem 2.9.** \(D\) is a \(N\)--(c.l.c) if and only if \(DJ = DF = DF^* = 0\).

*Proof.* \(DJ = 0\) implies that \(D\) preserves both distributions \(T^*(J^{(2,0)}M)\) and \(T^*(J^{(2,0)}M)\).

If \(DF = DF^* = 0\), then in view of Proposition 2.5 it follows \(Dh = Dh_1 = Du = D\bar{h} = Dh_1 = D\bar{v} = 0\), and hence \(D\) is a \(d\)--(c.l.c). Expressing \(DX(FY) = FD_XY\) for \(X = \delta_{ik}\) and \(Y = \delta_{ij}\) we infer \(L^{\alpha}_{jk} = L^{\alpha}_{jk}\). Taking \(X\) and \(Y\) as choices in the set \(\{\delta_{ik}, \delta_{ij}\}\), one can prove the remaining claims. \(\square\)

The torsions and the curvatures of a \(N\)--(c.l.c.) are deduced by direct computation, replacing \(X, Y, Z\) with the fields of the adapted frame \(\{\delta_{ik}, \delta_{ij}\}\) \(\alpha = 0, 1, 2\).

**References**


Author’s address:
Violeta Zalutchi
University “Transilvania” of Brasov,
Faculty of Mathematics and Informatics,
50 Iuliu Maniu Str., Brasov 500091, Romania.
E-mail: zalvio@yahoo.com