On a type of Kenmotsu manifolds

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Abstract. The object of the present paper is to study Kenmotsu manifolds admitting a $W_2$-curvature tensor.


Key words: Kenmotsu manifold; $W_2$-curvature tensor; Einstein manifold; projective curvature tensor; concircular curvature tensor; quasi-conformal curvature tensor; conformal curvature tensor.

1 Introduction

In [12], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold $M$, the sectional curvature of plane sections containing $\xi$ is a constant, say $c$. If $c > 0$, $M$ is homogeneous Sasakian manifold of constant sectional curvature. If $c = 0$, $M$ is the product of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature. If $c < 0$, $M$ is a warped product space $\mathbb{R} \times f \mathbb{C}^n$. In 1971, Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions [7]. We call it Kenmotsu manifold. Recently, Kenmotsu manifolds have been studied by many authors such as De and Pathak [2], Jun, De and Pathak [6], Ozgür and De [9] and many others.

On the other hand, Pokhariyal and Mishra [11] have introduced new tensor fields, called $W_2$ and $E$-tensor fields, in a Riemannian manifold, and studied their properties. Then, Pokhariyal [10] has studied some properties of this tensor fields in a Sasakian manifold. Recently, Matsumoto, Ianus and Mihai [8] have studied P-Sasakian manifolds admitting $W_2$ and $E$-tensor fields and De and Sarkar [3] have studied P-Sasakian manifolds admitting $W_2$ tensor field.

The curvature tensor $W_2$ is defined by

$$W_2(X,Y,U,V) = R(X,Y,U,V) + \frac{1}{n-1}[g(X,U)S(Y,V) - g(Y,U)S(X,V)],$$

where $S$ is a Ricci tensor of type $(0,2)$.

In the present paper we have studied some curvature conditions on Kenmotsu manifolds. Firstly, we have studied their geometric and relativistic properties in...
Kenmotsu manifolds satisfying $W_2 = 0$. Then we have studied $W_2$-semisymmetric Kenmotsu manifolds. Also, we have classified Kenmotsu manifolds which satisfy $P \cdot W_2 = 0$, $\tilde{Z} \cdot W_2 = 0$, $C \cdot W_2 = 0$ and $\tilde{C} \cdot W_2 = 0$ where $P$ is the projective curvature tensor, $\tilde{Z}$ is the concircular curvature tensor, $\tilde{C}$ is the quasi-conformal curvature tensor and $C$ is the conformal curvature tensor.

2 Preliminaries

Let $(M^n, \phi, \xi, \eta, g)$ be an $n$-dimensional (where $n = 2m + 1$) almost contact metric manifold, where $\phi$ is a $(1, 1)$-tensor field, $\xi$ is the structure vector field, $\eta$ is a 1-form and $g$ is the Riemannian metric. It is well known that the $(\phi, \xi, \eta, g)$ structure satisfies the conditions [1]

\begin{align*}
\phi \xi &= 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\
\phi^2 X &= -X + \eta(X) \xi, \quad g(X, \xi) = \eta(X), \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y),
\end{align*}

for any vector fields $X$ and $Y$ on $M^n$.

If moreover

\begin{align*}
(\nabla_X \phi)Y &= -g(X, \phi Y)\xi - \eta(Y)\phi X, \\
\nabla_X \xi &= X - \eta(X)\xi,
\end{align*}

where $\nabla$ denotes the Riemannian connection of $g$ hold, then $(M^n, \phi, \xi, \eta, g)$ is called a Kenmotsu manifold. In this case, it is well known that [7]

\begin{align*}
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, \\
S(X, \xi) &= -(n - 1)\eta(X),
\end{align*}

where $S$ denotes the Ricci tensor. From (2.6), it easily follows that

\begin{align*}
R(X, \xi)Y &= g(X, Y)\xi - \eta(Y)X, \\
R(X, \xi)\xi &= \eta(X)\xi - X.
\end{align*}

The notion of the quasi-conformal curvature tensor was given by Yano and Sawaki [13]. According to them a quasi-conformal curvature tensor $\tilde{C}$ is defined by

\begin{align*}
\tilde{C}(X, Y)Z &= aR(X, Y)Z \\
&\quad + b[\tilde{S}(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\
&\quad - \frac{\tau}{n} \left[ a - 2b \right] [g(Y, Z)X - g(X, Z)Y],
\end{align*}

where $a$ and $b$ are constants and $R$, $S$, $Q$ and $\tau$ are the Riemannian curvature tensor type of (1, 3), the Ricci tensor of type (0, 2), the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and scalar curvature of the manifold respectively. If $a = 1$ and $b = -\frac{1}{n-2}$
then (2.10) takes the form

\[
\tilde{C}(X, Y)Z = R(X, Y)Z \\
- \frac{\tau}{n-1} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QY - g(X, Z)QY] \\
+ \frac{\tau}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y] = C(X, Y)Z,
\]

where \( C \) is the conformal curvature tensor [5].

We next define endomorphisms \( R(X, Y) \) and \( X \wedge_A Y \) of \( \chi(M) \) by

\[
R(X, Y)W = \nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_[X,Y]W, \\
(X \wedge_A Y)W = A(Y, W)X - A(X, W)Y,
\]

respectively, where \( X, Y, W \in \chi(M) \) and \( A \) is the symmetric \((0, 2)\)-tensor.

On the other hand, the projective curvature tensor \( P \) and the concircular curvature tensor \( \tilde{Z} \) in a Riemannian manifold \((M^n, g)\) are defined by

\[
P(X, Y)W = R(X, Y)W - \frac{1}{n-1} (X \wedge_S Y)W, \\
\tilde{Z}(X, Y)W = R(X, Y)W - \frac{\tau}{n(n-1)} (X \wedge_g Y)W,
\]

respectively.

An almost contact metric manifold is said to be an \( \eta \)-Einstein manifold if the Ricci tensor \( S \) satisfies the condition

\[
S(X, Y) = \lambda_1 g(X, Y) + \lambda_2 \eta(X)\eta(Y),
\]

where \( \lambda_1, \lambda_2 \) are certain scalars. A Riemannian or a semi-Riemannian manifold is said to semisymmetric if \( R(X, Y) \cdot R = 0 \), where \( R(X, Y) \) is considered as a derivation of the tensor algebra at each point of the manifold for the tangent vectors \( X, Y \).

In a Kenmotsu manifold, using (2.7) and (2.8), equations (2.12), (2.13), (2.11), and (2.10) reduce to

\[
\begin{align*}
P(\xi, X)Y &= -g(X, Y)\xi - \frac{1}{n-1} S(X, Y)\xi, \\
\tilde{Z}(\xi, X)Y &= (1 + \frac{\tau}{n-1}) (-g(X, Y)\xi + \eta(Y)X), \\
C(\xi, Y)W &= \frac{n-1+\tau}{(n-1)(n-2)} \{ g(Y, W)\xi - g(W)Y \} \\
&- \frac{1}{n-2} \{ S(Y, W)\xi - \eta(W)QY \}, \\
C(\xi, Y)W &= K \{ \eta(W)Y - g(Y, W)\xi \} + b \{ S(Y, W)\xi - \eta(W)QY \},
\end{align*}
\]

respectively, where \( K = a + (n-1)b + \frac{a}{n-1} + 2b \).

Also we have [1]

\[
(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y).
\]
A Kenmotsu manifold $M^n$ is said to be Einstein if its Ricci tensor $S$ is of the form
\begin{equation}
S(X, Y) = \lambda_1 g(X, Y),
\end{equation}
for any vector fields $X$, $Y$ and $\lambda_1$ is a certain scalar.

## 3 Kenmotsu manifolds satisfying $W_2 = 0$

In this section we consider a Kenmotsu manifold satisfying $W_2 = 0$. Then we have from (1.1)
\begin{equation}
R(X, Y, U, V) = \frac{1}{n-1} [g(Y, U)S(X, V) - g(X, U)S(Y, V)].
\end{equation}
Using $X = U = \xi$ in (3.1), we have
\begin{equation}
R(\xi, Y, \xi, V) = \frac{1}{n-1} [g(Y, \xi)S(\xi, V) - g(\xi, \xi)S(Y, V)].
\end{equation}
From (2.1), (2.7) and (2.9), we get
\begin{equation}
S(Y, V) = -(n-1)g(Y, V).
\end{equation}
Thus $M$ is an Einstein manifold.

**Theorem 1.** Let $M$ be an $n$-dimensional ($n > 1$) Kenmotsu manifold satisfying $W_2 = 0$. Then $M$ is an Einstein manifold.

Now using (3.3) in (3.1), we get
\begin{equation}
R(X, Y, U, V) = g(X, U)g(Y, V) - g(Y, U)g(X, V).
\end{equation}
Hence $M$ is of constant curvature $-1$. Then $M$ is locally isometric to the hyperbolic space $H^n(-1)$.

**Corollary 1.** Let $M$ be an $n$-dimensional ($n > 1$) Kenmotsu manifold satisfying $W_2 = 0$ and let $M$ be an Einstein manifold. Then $M^n$ is locally isometric to the hyperbolic space $H^n(-1)$.

## 4 $W_2$-semisymmetric Kenmotsu manifolds

**Definition 1.** An $n$-dimensional Kenmotsu manifold is called $W_2$-semisymmetric if it satisfies
\begin{equation}
R(X, Y) \cdot W_2 = 0,
\end{equation}
where $R(X, Y)$ is to be considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors $X, Y$.

**Proposition 1.** Let $M$ be an $n$-dimensional Kenmotsu manifold. Then the $W_2$-curvature tensor on $M$ satisfies the condition
\begin{equation}
W_2(X, Y, U, \xi) = 0.
\end{equation}
Theorem 2. A $W_2$-semisymmetric Kenmotsu manifold is an Einstein manifold.

Proof. From (4.1) we have

$$\begin{align*}
R(X, Y)W_2(Z, U)V - W_2(R(X, Y)Z, U)V \\
-W_2(Z, R(X, Y)U)V - W_2(Z, U)R(X, Y)V = 0.
\end{align*}$$

This equation implies

$$\begin{align*}
g(R(X, Y)W_2(Z, U)V, \xi) - g(W_2(R(X, Y)Z, U)V, \xi) \\
-g(W_2(Z, R(X, Y)U)V, \xi) - g(W_2(Z, U)R(X, Y)V, \xi) = 0.
\end{align*}$$

Putting $X = \xi$ in (4.4) we obtain

$$\begin{align*}
g(R(\xi, Y)W_2(Z, U)V, \xi) - g(W_2(R(\xi, Y)Z, U)V, \xi) \\
-g(W_2(Z, R(\xi, Y)U)V, \xi) - g(W_2(Z, U)R(\xi, Y)V, \xi) = 0.
\end{align*}$$

Using (4.4) in (4.5), we get

$$\begin{align*}
-g(Y, W_2(Z, U)V)V\xi + \eta(W_2(Z, U)V)Y + g(Y, Z)g(W_2(\xi, U)V, \xi) \\
-\eta(Z)g(W_2(Y, U)V, \xi) + g(Y, U)g(W_2(Z, \xi)V, \xi) - \eta(U)g(W_2(Z, Y)V, \xi) \\
+g(Y, V)g(W_2(Z, U)V, \xi) - \eta(V)g(W_2(Z, U)V, \xi) = 0.
\end{align*}$$

Taking the inner product with $\xi$ and using (4.2) in (4.6), we obtain

$$W_2(Z, U, V, Y) = 0.$$
5 Kenmotsu manifolds satisfying $P(X, Y) \cdot W_2 = 0$

In this section we consider a Kenmotsu manifold $M^n$ satisfying the condition

$$P(X, Y) \cdot W_2 = 0.$$  

This equation implies

$$P(X, Y)W_2(Z, U)V - W_2(P(X, Y)Z, U)V$$

$$-W_2(Z, P(X, Y)U)V - W_2(Z, U)P(X, Y)V = 0.$$  

(5.1)

Putting $X = \xi$ in (5.1), we obtain

$$g(P(\xi, Y)W_2(Z, U)V, \xi) - g(W_2(P(\xi, Y)Z, U)V, \xi)$$

$$-g(W_2(Z, P(\xi, Y)U)V, \xi) - g(W_2(Z, U)P(\xi, Y)V, \xi) = 0.$$  

(5.2)

Using (2.15) in (5.2), we have

$$g(Y, W_2(Z, U)V)\xi - g(Y, Z)W_2(\xi, U)V - g(Y, U)W_2(Z, \xi)V$$


(5.3)

Taking the inner product with $\xi$ and using (4.2) in (5.3), we get

$$g(Y, W_2(Z, U)V) + \frac{1}{n-1}S(Y, W_2(Z, U)V) = 0.$$  

(5.4)

Now using (1.1) in (5.4) we get

$$R(Z, U, V, Y) + \frac{1}{n-1}\{g(Z, V)S(U, Y) - g(U, V)S(Z, Y)\}$$

$$+ \frac{1}{n-1}R(Z, U, V, QY) + \frac{1}{(n-1)^2}\{g(Z, V)S(U, QY) - g(U, V)S(Z, QY)\} = 0,$$

where $S(QY, Z) = S^2(Z, Y)$.

Again using $Z = V = \xi$ in (5.5) and from (2.9) we get

$$S(QY, U) = -2(n-1)S(Y, U) - (n-1)^2g(Y, U).$$  

(5.6)

Hence we have the following:

**Theorem 3.** In an $n$-dimensional ($n > 1$) Kenmotsu manifold $M$ if the condition $P(X, Y) \cdot W_2 = 0$ holds on $M$, then the equation (5.6) is satisfied on $M$.

**Lemma 1.** [4] Let $A$ be a symmetric $(0, 2)$-tensor at point $x$ of a semi-Riemannian manifold $(M^n, g)$, $n > 1$, and let $T = g \wedge A$ be the Kulkarni-Nomizu product of $g$ and $A$. Then, the relation

$$T \cdot T = \alpha Q(g, T), \quad \alpha \in \mathbb{R}$$

is satisfied at $x$ if and only if the condition

$$A^2 = \alpha A + \lambda g, \quad \lambda \in \mathbb{R}$$

holds at $x$. 


From Theorem 3 and Lemma 1 we get the following:

**Corollary 3.** Let $M$ be an $n$–dimensional $(n > 1)$ Kenmotsu manifold satisfying the condition $P(X,Y)\cdot W_2 = 0$, then $T_T = \alpha Q(g,T)$, where $T = g\Lambda S$ and $\alpha = -2(n-1)$.

### 6 Kenmotsu manifolds satisfying $\tilde{Z}(X,Y) \cdot W_2 = 0$

In this section we consider a Kenmotsu manifold $M^n$ satisfying the condition $\tilde{Z}(X,Y) \cdot W_2 = 0$.

This equation implies

$$\tilde{Z}(X,Y)W_2(Z,U)V - W_2(\tilde{Z}(X,Y)Z,U)V - W_2(Z,\tilde{Z}(X,Y)U)V - W_2(Z,U)\tilde{Z}(X,Y)V = 0.\tag{6.1}$$

Now $X = \xi$ in (6.1), we have

$$\tilde{Z}(\xi,Y)W_2(Z,U)V - W_2(\tilde{Z}(\xi,Y)Z,U)V - W_2(Z,\tilde{Z}(\xi,Y)U)V - W_2(Z,U)\tilde{Z}(\xi,Y)V = 0.\tag{6.2}$$

Using (2.16) in (6.2), we get

$$\{1 + \frac{\tau}{n(n-1)}\}g(Y,W_2(Z,U)V)\xi - g(W_2(Z,U)V,\xi)Y - g(Y,Z)W_2(\xi,U)V + \eta(Z)W_2(Y,U)V - g(Y,V)W_2(Z,U)\xi + \eta(V)W_2(Z,U)Y = 0.\tag{6.3}$$

Taking the inner product with $\xi$ and using (4.2) in (6.3), we have

$$\{1 + \frac{\tau}{n(n-1)}\}g(Y,W_2(Z,U)V) = 0.$$

Again from (2.16) we have $1 + \frac{\tau}{n(n-1)} \neq 0$. Hence we have

$$W_2(Z,U,V,Y) = 0.$$

From the proof of Theorem 1, Theorem 2 and Corollary 1, we can say:

**Theorem 4.** An $n$–dimensional $(n > 1)$ Kenmotsu manifold $M$ satisfying the condition $\tilde{Z}(\xi,Y) \cdot W_2 = 0$ is an Einstein manifold and locally isometric to the hyperbolic space $H^n(-1)$.

### 7 Kenmotsu manifolds satisfying $C(X,Y) \cdot W_2 = 0$

In this section we consider a Kenmotsu manifold $M^n$ satisfying the condition $C(X,Y) \cdot W_2 = 0$.
This equation implies
\[(7.1)\]
\[
C(X,Y)W_2(Z,U)V - W_2(C(X,Y)Z,U)V
- W_2(Z,C(X,Y)U)V - W_2(Z,U)C(X,Y)V = 0.
\]
Putting \(X = \xi\) in (7.1), we have
\[(7.2)\]
\[
C(\xi,Y)W_2(Z,U)V - W_2(C(\xi,Y)Z,U)V
- W_2(Z,C(\xi,Y)U)V - W_2(Z,U)C(\xi,Y)V = 0.
\]
Using (2.17) in (7.1), we obtain
\[(7.3)\]
\[
\frac{n-1+\tau}{(n-1)(n-2)} \{ g(Y,W_2(Z,U)V)\xi - g(W_2(Z,U)V,\xi)Y
- g(Y,Z)W_2(\xi,U)V + \eta(Z)W_2(Y,U)V - g(Y,U)W_2(Z,\xi)V
+ \eta(U)W_2(Z,Y)V - g(Y,V)W_2(Z,U)V\xi + \eta(V)W_2(Z,U)V\xi \} = 0.
\]
Taking the inner product with \(\xi\) and using (4.2),
\[
\{ \frac{n-1+\tau}{(n-1)(n-2)} \} g(Y,W_2(Z,U)V) = 0.
\]
Let \(U_1\) and \(U_2\) be a part of \(M\) satisfying \(\tau + n - 1 = 0\) and
\[(7.4)\]
\[
W_2(Z,U,V,Y) = 0.
\]
This leads to the following:

**Theorem 5.** In \(n\)-dimensional \((n > 1)\) Kenmotsu manifold \(M\) satisfying the condition \(C(X,Y) \cdot W_2 = 0\). Then either \(\tau + n - 1 = 0\), or \(M\) is locally isometric to the hyperbolic space \(H^n(-1)\).

**8 Kenmotsu manifolds satisfying \(\tilde{C}(X,Y) \cdot W_2 = 0\)**

In this section we consider a Kenmotsu manifold \(M^n\) satisfying the condition
\[
\tilde{C}(X,Y) \cdot W_2 = 0.
\]
This equation implies
\[(8.1)\]
\[
\tilde{C}(X,Y)W_2(Z,U)V - W_2(\tilde{C}(X,Y)Z,U)V
- W_2(Z,\tilde{C}(X,Y)U)V - W_2(Z,U)\tilde{C}(X,Y)V = 0.
\]
Putting \(X = \xi\) in (8.1), we have
\[(8.2)\]
\[
\tilde{C}(\xi,Y)W_2(Z,U)V - W_2(\tilde{C}(\xi,Y)Z,U)V
- W_2(Z,\tilde{C}(\xi,Y)U)V - W_2(Z,U)\tilde{C}(\xi,Y)V = 0.
\]
Using (2.18) in (8.2), we get
\[
K \{ g(W_2(Z, U)V, \xi)Y - g(Y, W_2(Z, U)V)\xi - \eta(Z)W_2(Y, U)V \\
+ g(Y, Z)W_2(\xi, U)V - \eta(U)W_2(Z, Y)V + g(Y, U)W_2(Z, \xi)V \\
- \eta(V)W_2(Z, U)Y + g(Y, V)W_2(Z, U)\xi \}
\]
(8.3)
\[
+b\{ S(Y, W_2(Z, U)V)\xi - g(W_2(Z, U)V, \xi)QY + \eta(Z)W_2(QY, U)V \\
- S(Y, Z)W_2(\xi, U)V + \eta(U)W_2(Z, QY)V - S(Y, U)W_2(Z, \xi)V \\
+ \eta(V)W_2(Z, U)QY - S(Y, V)W_2(Z, U)\xi \} = 0,
\]
where \( K = a + (n - 1)b + \frac{n}{n - 1}\). Taking the inner product with \( \xi \) and using (4.2) in (8.3), we get
\[
Kg(Y, W_2(Z, U)V) = 0.
\]
Now using \( Z = V = \xi \) in (8.4). Then from (1.1) and (2.9), we have
\[
\frac{b}{n - 1} S(QY, U) - (\frac{K}{n - 1} - b)S(Y, U) - Kg(Y, U) = 0.
\]
If \( b = 0 \), then we get
\[
K\{ \frac{1}{n - 1}S(Y, U) - g(Y, U) \} = 0.
\]
Let \( U_1 \) and \( U_2 \) be a part of \( M \) satisfying \( K = 0 \) and
\[
S(Y, U) = -(n - 1)g(Y, U),
\]
respectively. If \( b \neq 0 \), then we get
\[
S(QY, W) = (\frac{K}{b} - n + 1)S(Y, W) + (n - 1)\frac{K}{b}g(Y, W).
\]
Therefore we have the following:

**Theorem 6.** Let \( M \) be an \( n \)-dimensional \((n > 1)\) Kenmotsu manifold satisfying the condition \( \tilde{C}(X, Y) \cdot W_2 = 0 \). Then we get

1. if \( b = 0 \), then either \( K = 0 \) on \( M \), or \( M \) is an Einstein manifold.
2. if \( b \neq 0 \), then the equation (8.6) holds on \( M \).

**Corollary 4.** Let \( M \) be an \( n \)-dimensional \((n > 1)\) Kenmotsu manifold satisfying the condition \( \tilde{C}(\xi, Y) \cdot \tilde{C} = 0 \), then \( T \cdot T = \alpha Q(g, T) \), where \( T = g \tilde{\nabla}^2 S \) and \( \alpha = \frac{K}{b} - n + 1 \).

**References**


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