Warped product semi-slant submanifolds of trans-Sasakian manifolds

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Abstract. In the present paper, we study warped product semi-slant submanifolds of trans-Sasakian manifolds. We have obtained results on the existence of warped product semi-slant submanifolds of trans-Sasakian manifolds in term of the canonical structure $F$.

Key words: Warped product; doubly warped product; slant submanifold; semi-slant submanifold; trans-Sasakian manifold; canonical structure.

1 Introduction

The notion of semi-slant submanifolds of almost Hermitian manifolds was introduced by N. Papaghuic [14]. In fact, semi-slant submanifolds in almost Hermitian manifolds are defined on the line of CR-submanifolds. In the setting of almost contact metric manifolds, semi-slant submanifolds are defined and investigated by J.L. Cabrerizo et. al [4].

In [13] J.A. Oubina studied a new class of almost contact metric structure, called trans-Sasakian which is, in some sense, an analogue of a locally conformal Kaehler structure on an almost Hermitian manifold.

On the other hand, in Gray-Hervella classification of almost Hermitian manifolds [7], there appears a class $W_4$, of Hermitian manifolds which are closely related to a locally conformal Kaehler manifolds. An almost contact metric structure on a manifold $M$ is called trans-Sasakian structure [13] if the product manifold $M \times \mathbb{R}$ belongs to class $W_4$. In [6], the authors introduced two subclasses of trans-Sasakian structure, the $C_5$ and $C_6$-structures, which contains the Kenmotsu and Sasakian structures, respectively. The class $C_6 \oplus C_5$ [11] coincides with the class of trans-Sasakian of type $(\alpha, \beta)$. We note that the trans-Sasakian structures of type $(0,0)$, $(0, \beta)$ and $(\alpha,0)$ are cosymplectic [3], $\beta$–Kenmotsu and $\alpha$–Sasakian [8], respectively.

Warped product manifolds were introduced by R.L. Bishop and B. O’Neill in [2]. The problem of existence or non-existence of warped product manifolds plays some important role in differential geometry and physics. The study of warped product semi-slant submanifolds of Kaehler manifolds was introduced by B. Sahin [15]. Later, K.A. Khan et.al studied warped product semi-slant submanifolds in cosymplectic
manifolds and showed that there exist no proper warped product semi-slant submanifolds in the forms $N_T \times fN_\theta$ and reversing the two factors in cosymplectic manifolds [9].

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In this paper, we have obtained some results for the existence of warped product semi-slant submanifolds of trans-Sasakian manifolds.

2 Preliminaries

Let $\tilde{M}$ be a $(2m + 1)$-dimensional manifold with almost contact structure $(\phi, \xi, \eta)$ is defined on $\tilde{M}$ by a $(1,1)$ tensor field $\phi$, a vector field $\xi$ and the dual 1-form $\eta$ of $\xi$, satisfying the following properties [3]

\begin{align*}
\phi^2 &= -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1.
\end{align*}

There always exists a Riemannian metric $g$ on an almost contact manifold $\tilde{M}$ satisfying the following compatibility condition

\begin{align*}
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y).
\end{align*}

An almost contact structure $(\phi, \xi, \eta)$ is said to be normal if the almost complex structure $J$ on the product manifold $\tilde{M} \times \mathbb{R}$ given by

\begin{align*}
J(X, f \frac{d}{dt}) &= (\phi X - f \xi, \ \eta(X) \frac{d}{dt}),
\end{align*}

has no torsion i.e., $J$ is integrable where $f$ is a $C^\infty$-function on $\tilde{M} \times \mathbb{R}$, in other words the tensor $[\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically on $\tilde{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$ [17].

An almost contact metric manifold $\tilde{M}$ is called trans-Sasakian [13] if

\begin{align*}
(\nabla_X \phi)Y &= \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X).
\end{align*}

for all $X, Y \in T\tilde{M}$, where $\alpha, \beta$ are smooth function on $\tilde{M}$ and $\nabla$ is the Levi-Civita connection of $g$ and in this case we say that the trans-Sasakian structure is of type $(\alpha, \beta)$.

If $\alpha = 0$ then $\tilde{M}$ is $\beta$-Kenmotsu manifold and if $\beta = 0$ then $\tilde{M}$ is $\alpha$-Sasakian manifold. Moreover, if $\alpha = 1$ and $\beta = 0$ then $\tilde{M}$ is a Sasakian manifold and if $\alpha = 0$ and $\beta = 1$ then $\tilde{M}$ is a Kenmotsu manifold. From (2.3), it follows that

\begin{align*}
\nabla_X \xi &= -\alpha \phi X + \beta(X - \eta(X)\xi).
\end{align*}

Let $\tilde{M}$ be submanifold of an almost contact metric manifold $\tilde{M}$ with induced metric $g$ and if $\nabla$ and $\nabla^\perp$ are the induced connections on the tangent bundle $TM$ and the normal bundle $T^\perp M$ of $\tilde{M}$, respectively then Gauss and Weingarten formulae are given by

\begin{align*}
\nabla_X Y &= \nabla_X Y + h(X, Y)
\end{align*}
\( \nabla_X N = -A_N X + \nabla^\perp_X N \),

for each \( X, Y \in TM \) and \( N \in T^\perp M \), where \( h \) and \( A_N \) are the second fundamental form and the shape operator (corresponding to the normal vector field \( N \)) respectively for the immersion of \( M \) into \( \bar{M} \). They are related as \([17]\)

\[ g(h(X,Y),N) = g(A_N X,Y), \]

where \( g \) denotes the Riemannian metric on \( \bar{M} \) as well as induced on \( M \).

For any \( X \in TM \), we write

\[ \phi X = PX + FX, \]

where \( PX \) is the tangential component and \( FX \) is the normal component of \( \phi X \).

Similarly, for any \( N \in T^\perp M \), we write

\[ \phi N = BN + CN, \]

where \( BN \) is the tangential component and \( CN \) is the normal component of \( \phi N \). The covariant derivatives of the tensor fields \( P \) and \( F \) are defined as

\[ (\nabla_X P)Y = \nabla_X PY - P\nabla_X Y, \]

\[ (\nabla_X F)Y = \nabla^\perp_X FY - F\nabla_X Y \]

for all \( X, Y \in TM \). The canonical structures \( P \) and \( F \) on a submanifold \( M \) are said to be parallel if \( \nabla P = 0 \) and \( \nabla F = 0 \), respectively.

We shall always consider \( \xi \) to be tangent to \( M \). The submanifold \( M \) is said to be invariant if \( \phi X \in TM \) for any \( X \in TM \). On the other hand \( M \) is said to be anti-invariant if \( P \) is identically zero, that is, \( \phi X \in T^\perp M \), for any \( X \in TM \).

For each non zero vector \( X \) tangent to \( M \) at \( x \), such that \( X \) is not proportional to \( \xi \), we denote by \( \theta(X) \), the angle between \( \phi X \) and \( PX \).

\( M \) is said to be slant \([5]\) if the angle \( \theta(X) \) is constant for all \( X \in TM - \{\xi\} \) and \( x \in M \). The angle \( \theta \) is called slant angle or Wirtinger angle. Obviously if \( \theta = 0 \), \( M \) is invariant and if \( \theta = \pi/2 \), \( M \) is an anti-invariant submanifold. If the slant angle of \( M \) is different from 0 and \( \pi/2 \) then it is called proper slant.

A characterization of slant submanifolds is given by the following

**Theorem 2.1** \([5]\]. Let \( M \) be a submanifold of an almost contact metric manifold \( \bar{M} \), such that \( \xi \in TM \). Then \( M \) is slant if and only if there exists a constant \( \lambda \in [0,1] \) such that

\[ P^2 = \lambda(-I + \eta \otimes \xi). \]

Furthermore, if \( \theta \) is slant angle, then \( \lambda = \cos^2 \theta \).

The following relations are straightforward consequences of equation (2.12):

\[ g(PX,PY) = \cos^2 \theta[g(X,Y) - \eta(X)\eta(Y)] \]
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\begin{equation}
\eta(X,Y) = \sin^2 \theta \{ g(X,Y) - \eta(X)\eta(Y) \}
\end{equation}

for any \(X, Y\) tangent to \(M\).

The study of semi-slant submanifolds of almost Hermitian manifolds was introduced by N. Papaghiuc [14], which were latter extended to almost contact metric manifold by J.L. Cabrerizo et.al [4]. We say \(M\) is a semi-slant submanifold of \(\bar{M}\) if there exist an orthogonal direct decomposition of \(TM\) as

\[TM = D_1 \oplus D_2 \oplus \{\xi\}\]

where \(D_1\) is an invariant distribution i.e., \(\phi(D_1) = D_1\) and \(D_2\) is slant with slant angle \(\theta \neq 0\). The orthogonal complement of \(FD_2\) in the normal bundle \(T^\perp M\), is an invariant subbundle of \(T^\perp M\) and is denoted by \(\mu\). Thus, we have

\[T^\perp M = FD_2 \oplus \mu.\]

Similarly, we say that \(M\) is anti-slant submanifold of \(\bar{M}\) if \(D_1\) is an anti-invariant distribution of \(M\) i.e., \(\phi D_1 \subseteq T^\perp M\) and \(D_2\) is slant with slant angle \(\theta \neq 0\).

3 Warped and doubly warped product submanifolds

Let \((N_1, g_1)\) and \((N_2, g_2)\) be two Riemannian manifolds and \(f\), a positive differentiable function on \(N_1\). The warped product of \(N_1\) and \(N_2\) is the Riemannian manifold \(N_1 \times fN_2 = (N_1 \times N_2, g)\), where

\begin{equation}
g = g_1 + f^2 g_2.
\end{equation}

A warped product manifold \(N_1 \times fN_2\) is said to be trivial if the warping function \(f\) is constant. We recall the following general formula on a warped product [2].

\begin{equation}
\nabla_X V = \nabla_V X = (X \ln f)V,
\end{equation}

where \(X\) is tangent to \(N_1\) and \(V\) is tangent to \(N_2\).

Let \(M = N_1 \times fN_2\) be a warped product manifold, this means that \(N_1\) is totally geodesic and \(N_2\) is totally umbilical submanifold of \(M\), respectively.

Doubly warped product manifolds were introduced as a generalization of warped product manifolds by B. Ünal [16]. A doubly warped product manifold of \(N_1\) and \(N_2\), denoted as \(f_2N_1 \times f_1N_2\) is endowed with a metric \(g\) defined as

\begin{equation}
g = f_2^2 g_1 + f_1^2 g_2
\end{equation}

where \(f_1\) and \(f_2\) are positive differentiable functions on \(N_1\) and \(N_2\) respectively.

In this case formula (3.2) is generalized as

\begin{equation}
\nabla_X V = (X \ln f_1)V + (V \ln f_2)X
\end{equation}

for each \(X \in TN_1\) and \(V \in TN_2\) [12].

If neither \(f_1\) nor \(f_2\) is constant we have a non trivial doubly warped product \(M = f_2N_1 \times f_1N_2\). Obviously in this case both \(N_1\) and \(N_2\) are totally umbilical submanifolds of \(M\).
**Theorem 3.1.** There do not exist proper doubly warped product submanifolds $M = f_2.N_1 \times f_1.N_2$ of a trans-Sasakian manifold $\bar{M}$ where $N_1$ and $N_2$ are any Riemannian submanifolds of $M$.

**Proof.** For any $X \in TN_1$ and $Z \in TN_2$ then by (3.4) we have
\[ \nabla_Z X = \nabla_X Z = (X \ln f_1)Z + (Z \ln f_2)X. \]
If $\xi \in TN_1$ then above equation gives
\[ (\xi \ln f_1)Z + (Z \ln f_2)\xi. \]
On the other hand, using (2.4) and the fact that $\xi$ is tangent to $N_1$ we have
\[ \nabla_Z \xi = -\alpha\phi Z + \beta Z. \]
By (2.5) and (2.8), we get
\[ \nabla_Z \xi + h(Z, \xi) = -\alpha PZ + \beta Z. \]
Using (3.5) and then comparing the tangential and normal components we obtain
\[ (\xi \ln f_1)Z + (Z \ln f_2)\xi = -\alpha PZ + \beta Z \]
and
\[ h(Z, \xi) = -\alpha FZ. \]
Taking product with $Z$ in (3.6) and using the fact that $\xi$, $Z$ and $PZ$ are orthogonal vector fields then
\[ \xi \ln f_1 = \beta, \quad Z \ln f_2 = 0. \]
This shows that $f_2$ is constant.
Similarly, if the structure vector field $\xi$ is tangent to $N_2$ and for any $X \in TN_1$ we obtain
\[ \xi \ln f_2 = \beta, \quad X \ln f_1 = 0, \]
showing that $f_1$ is constant. This completes the proof. \qed

The following corollary is an immediate consequence of the above theorem:

**Corollary 3.1.** There do not exist warped product submanifolds $M = N_1 \times f_2.N_2$ of a trans-Sasakian manifold $\bar{M}$ such that $\xi \in TN_2$, where $N_1$ and $N_2$ are any Riemannian submanifolds of $M$.

Thus, the only remaining case to study of warped product submanifolds $N_1 \times f_2.N_2$ of trans-Sasakian manifolds is that the structure vector field $\xi$ tangential to the first factor i.e., $\xi \in TN_1$. In this case, first we obtain some useful formulae for later use.

**Lemma 3.1.** Let $M = N_1 \times f_2.N_2$ be warped product submanifolds of a trans-Sasakian manifold $\bar{M}$ such that $\xi$ is tangent to $N_1$, where $N_1$ and $N_2$ are any Riemannian submanifolds of $M$. Then
(i) $\xi \ln f = \beta$,
(ii) $A_{FZ}X = -Bh(X, Z)$,
(iii) \( g(h(X, Y), FZ) = g(h(X, Z), FY) \),
(iv) \( g(h(X, Z), FW) = g(h(X, W), FZ) \)

for any \( X, Y \in \mathcal{TN}_1 \) and \( Z, W \in \mathcal{TN}_2 \) where \( \beta \) is a smooth function on \( \bar{M} \).

**Proof.** First part follows by Theorem 3.1. Now, for any \( X \in \mathcal{TN}_1 \) and \( Z \in \mathcal{TN}_2 \) we have
\[
(\nabla_X \phi)Z = \nabla_X \phi Z - \phi \nabla_X Z.
\]
Using (2.3) and the fact that \( \xi \in \mathcal{TN}_1 \), left hand side of the above equation is zero by orthogonality of two distributions, then
\[
\nabla_X \phi Z = \phi \nabla_X Z.
\]
By (2.5), (2.6), (2.8) and (2.9) we obtain
\[
\nabla_X PZ + h(X, PZ) - A_{FZ}X + \nabla^\perp_X FZ = P\nabla_X Z + F\nabla_X Z + Bh(X, Z) + Ch(X, Z).
\]
Equating the tangential and normal components and using (3.2), we get
\[
(3.7) \quad A_{FZ}X = - Bh(X, Z)
\]
and
\[
(3.8) \quad \nabla^\perp_X FZ = (X \ln f)FZ + Ch(X, Z) - h(X, PZ).
\]
Equation (3.7) follows part (ii). Parts (iii) and (iv) follow by taking the product in (ii) with \( Y \in \mathcal{TN}_1 \) and \( W \in \mathcal{TN}_2 \), respectively. \( \square \)

In the following section we shall investigate warped product semi-slant submanifolds of trans-Sasakian manifolds.

4 Warped product semi-slant submanifolds

We have seen that the warped products of the type \( N_1 \times fN_2 \) of trans-Sasakian manifolds do not exist if the structure vector field \( \xi \) is tangent to \( N_2 \). Thus, in this section we study warped product semi-slant submanifolds \( N_1 \times fN_2 \) of trans-Sasakian manifolds only when \( \xi \in \mathcal{TN}_1 \). If the manifolds \( N_0 \) and \( N_T \) (resp. \( N_\perp \)) are slant and invariant (resp. anti-invariant) submanifolds of a trans-Sasakian manifold \( \bar{M} \), then their warped product semi-slant submanifolds may given by one of the following forms:

(i) \( N_T \times fN_0 \),
(ii) \( N_\perp \times fN_0 \),
(iii) \( N_0 \times fN_T \),
(iv) \( N_0 \times fN_\perp \).

For the warped products of the type (i), we have

**Theorem 4.1.** Let \( M = N_T \times fN_0 \) be warped product semi-slant submanifolds of a trans-Sasakian manifold \( M \) such that \( \xi \) is tangent to \( N_T \). Then \( (\nabla_X F)Z \) lies in the invariant normal subbundle for all \( X \in \mathcal{TN}_T \) and \( Z \in \mathcal{TN}_0 \) where \( N_T \) and \( N_0 \) are invariant and proper slant submanifolds of \( M \).
Proof. For any $X \in TN_T$ and $Z \in TN_\theta$ we have
\[ g(\phi \bar{\nabla}_X Z, \phi Z) = g(\bar{\nabla}_X Z, Z). \]

Then from (3.2), we obtain
\[ g(\phi \bar{\nabla}_X Z, \phi Z) = (X \ln f)\|Z\|^2. \] (4.1)

On the other hand we have
\[ (\bar{\nabla}_X \phi) Z = \bar{\nabla}_X \phi Z - \phi \bar{\nabla}_X Z, \]
for any $X \in TN_T$ and $Z \in TN_\theta$. Using (2.3) and the fact that $\xi$ is in $TN_T$, left hand side of the above equation is zero, then
\[ \phi \bar{\nabla}_X Z = \bar{\nabla}_X \phi Z. \]

Taking the product with $\phi Z$ and then using (2.8), we get
\[ g(\phi \bar{\nabla}_X Z, \phi Z) = g(\bar{\nabla}_X (PZ + FZ), PZ + FZ). \]

Then on applying (2.5) and (2.6) we obtain
\[ g(\phi \bar{\nabla}_X Z, \phi Z) = g(\bar{\nabla}_X PZ, PZ) + g(h(X, PZ), FZ) \]
\[ -g(A_F Z X, PZ) + g(\bar{\nabla}_X FZ, FZ). \]

Thus from (2.7) and (3.2) we have
\[ g(\phi \bar{\nabla}_X Z, \phi Z) = (X \ln f)g(PZ, PZ) + g((\bar{\nabla}_X F) Z, FZ) + (X \ln f)g(FZ, FZ). \]

On using (2.13) and (2.14), we get
\[ g(\phi \bar{\nabla}_X Z, \phi Z) = (X \ln f) \cos^2 \theta\|Z\|^2 + g((\bar{\nabla}_X F) Z, FZ) + (X \ln f) \sin^2 \theta\|Z\|^2, \]

or
\[ g(\phi \bar{\nabla}_X Z, \phi Z) = (X \ln f)\|Z\|^2 + g((\bar{\nabla}_X F) Z, FZ). \] (4.2)

By equations (4.1) and (4.2), it follows that
\[ g((\bar{\nabla}_X F) Z, FZ) = 0. \] (4.3)

for any $X \in TN_T$ and $Z \in TN_\theta$. Since $N_\theta$ is a proper slant submanifold of $\tilde{M}$, then (4.3) implies $(\bar{\nabla}_X F) Z \in \mu$. The proof is complete. \qed

For the warped product of the type $N_\perp \times f N_\theta$ of a trans-Sasakian manifold $\tilde{M}$ such that $\xi \in TN_\perp$, we have the following theorem

**Theorem 4.2.** Let $M = N_\perp \times f N_\theta$ be warped product semi-slant submanifolds of a trans-Sasakian manifold $\tilde{M}$ such that $\xi$ is tangent to $N_\perp$. Then
\[ Z \ln f = \beta \eta(Z) \] (4.4)
for any $Z \in TN_\perp$ where $\beta$ is a smooth function on $\bar{M}$ and $N_\perp$ and $N_\theta$ are anti-invariant and proper slant submanifolds of $\bar{M}$, respectively.

**Proof.** For any $X \in TN_\theta$ and $Z \in TN_\perp$ we have

$$(\bar{\nabla}_X \phi)Z = \bar{\nabla}_X \phi Z - \phi \bar{\nabla}_X Z.$$  

On using (2.5), (2.6), (2.8), (2.9) and the structure equation of trans-Sasakian and the fact that $\xi \in TN_\perp$, we obtain

$$-\eta(Z)\alpha X + \beta \eta(Z)PX = A_{FZ}X + \nabla^N_{\frac{1}{2}}FZ - P\nabla_X Z$$

$$-F\nabla_X Z - Bh(X, Z) - Ch(X, Z).$$

Equating the tangential components and using (3.2), we get

(4.5) $$\eta(Z)\alpha X + \beta \eta(Z)PX = A_{FZ}X + (Z \ln f)PX + Bh(X, Z).$$

Taking the product with $PX$ in (4.5) and using the fact that $X$ and $PX$ are mutually orthogonal vector fields, then

$$\beta \eta(Z)g(PX, PX) = g(A_{FZ}X, PX) + (Z \ln f)g(PX, PX) + g(Bh(X, Z), PX).$$

Thus from (2.7) and (2.13) we get

(4.6) $$\{\beta \eta(Z) - (Z \ln f)\} \cos^2 \theta ||X||^2 = g(h(X, PX), FZ) - g(h(X, Z), FPX).$$

As $N_\theta$ is proper slant, interchanging $X$ by $PX$ in (4.6) and taking account of equation (2.12), we deduce that

(4.7) $$\{\beta \eta(Z) - (Z \ln f)\} \cos^2 \theta ||X||^2 = -g(h(X, PX), FZ) + g(h(PX, Z), FX).$$

On adding (4.6) and (4.7), we obtain

(4.8) $$2\{\beta \eta(Z) - (Z \ln f)\} \cos^2 \theta ||X||^2 = g(h(PX, Z), FX) - g(h(X, Z), FPX).$$

Thus by Lemma 3.1 (iv), the right hand side of the above equation is zero then we have

$$\{\beta \eta(Z) - (Z \ln f)\} \cos^2 \theta ||X||^2 = 0.$$  

Since $N_\theta$ is proper slant and $X$ is non-null, then $Z \ln f = \beta \eta(Z)$. Hence the theorem is proved. \[\square\]

Now, the case (iii) is dealt with the following

**Proposition 4.1.** Let $\bar{M}$ be a $(2n+1)-$dimensional trans-Sasakian manifold and $M = N_\theta \times fN_T$ be warped product semi-slant submanifolds of $\bar{M}$ such that $\xi$ is tangent to $N_\theta$. Then

(4.9) $$g((\bar{\nabla}_X F)Z, FZ) = \sec^2 \theta g((\bar{\nabla}_X F)PZ, FPZ)$$

for any $X \in TN_T$ and $Z \in TN_\theta$ where $N_T$ and $N_\theta$ are invariant and proper slant submanifolds of $\bar{M}$, respectively.
Proof. Assume $M = N_{\theta} \times f N_{T}$ be warped product semi-slant submanifolds of a trans-Sasakian manifold $M$ such that $\xi$ is tangent to $N_{\theta}$. Then for any $X \in TN_{T}$ and $Z \in TN_{\theta}$ we have

$$(\bar{\nabla}_X \phi)Z = \nabla_X \phi Z - \phi \nabla_X Z.$$  

Since $\xi \in TN_{\theta}$ then from (2.3), (2.5), (2.6), (2.8) and (2.9), we get

$$-\eta(Z)\alpha X - \beta \eta(Z)\phi X = \nabla_X PZ + h(X, PZ) - AFZX + \nabla_X FZ - P\nabla_X Z - F\nabla_X Z - Bh(X, Z) - Ch(X, Z).$$

On comparing the tangential and normal parts we have

$$(4.10) -\eta(Z)\alpha X - \beta \eta(Z)\phi X = \nabla_X PZ - AFZX - P\nabla_X Z - Bh(X, Z)$$

and

$$(4.11) \quad (\bar{\nabla}_X F)Z = Ch(X, Z) - h(X, PZ).$$

Taking the product with $FZ$ in (4.11) we have

$$g((\bar{\nabla}_X F)Z, FZ) = g(Ch(X, Z), FZ) - g(h(X, PZ), FZ) = g(\phi h(X, Z), \phi Z) - g(Bh(X, Z), PZ) - g(h(X, PZ), \phi Z).$$

That is,

$$(4.12) \quad g((\bar{\nabla}_X F)Z, FZ) = -g(Bh(X, Z), PZ) + g(Bh(X, PZ), Z).$$

As $\theta \neq \frac{\pi}{2}$, then substituting $Z$ by $PZ$ in (4.12) and using (2.12) we obtain

$$g((\bar{\nabla}_X F)PZ, FPZ) = \cos^2 \theta \{ -g(Bh(X, Z), PZ) + g(Bh(X, PZ), Z) \}.$$

Using (4.12), we get

$$g((\bar{\nabla}_X F)PZ, FPZ) = \cos^2 \theta g((\bar{\nabla}_X F)Z, FZ).$$

This proves our assertion. \hfill \Box

The case (iv) is dealt in the following Theorem.

**Theorem 4.3.** Let $M = N_{T} \times \perp N_{\perp}$ be warped product submanifold of a trans-Sasakian manifold $M$ such that $N_{T}$ an invariant submanifold tangent to $\xi$ and $N_{\perp}$ is an anti-invariant submanifold of $M$. Then $(\bar{\nabla}_X F)Z$ lies in the invariant normal subbundle for each $X \in TN_{T}$ and $Z \in TN_{\perp}$.

**Proof.** As $M = N_{T} \times \perp N_{\perp}$ be warped product submanifold with $\xi$ is tangent to $N_{T}$, then for any $X \in TN_{T}$ and $Z \in TN_{\perp}$ we have

$$(\bar{\nabla}_X \phi)Z = \nabla_X \phi Z - \phi \nabla_X Z.$$  

On using (2.3) and the fact that $\xi$ tangential to $N_{T}$ the left hand side of the above equation is zero. Thus, we have

$$\nabla_X \phi Z = \phi \nabla_X Z.$$
Then from (2.5) and (2.6) we obtain

\[-A_{FZ}X + \nabla_X^\perp FZ = \phi(\nabla_X Z + h(X, Z)).\]

Which on using (2.8) and (2.9) yields

\[-A_{FZ}X + \nabla_X^\perp FZ = P\nabla_X Z + F\nabla_X Z + Bh(X, Z) + Ch(X, Z).\]

From the normal components of the above equation and formula (3.2) gives

\[(4.13) \quad \nabla_X^\perp FZ = (X \ln f)FZ + Ch(X, Z).\]

Taking the product in (4.13) with FW_1 for any W_1 \in TN_\perp, we get

\[g(\nabla_X^\perp FZ, FW_1) = (X \ln f)g(FZ, FW_1) + g(Ch(X, Z), FW_1)\]

or,

\[g(\nabla_X^\perp FZ, FW_1) = (X \ln f)g(\phi Z, \phi W_1) + g(\phi h(X, Z), \phi W_1).\]

Then from (2.2) we have

\[(4.14) \quad g(\nabla_X^\perp FZ, FW_1) = (X \ln f)g(Z, W_1).\]

On the other hand for any X \in TN_T and Z \in TN_\perp we have

\[(\nabla_X F)Z = \nabla_X^\perp FZ - F\nabla_X Z.\]

Taking the product with FW_1, for any W_1 \in TN_\perp and using (3.2), we get

\[(4.15) \quad g((\nabla_X F)Z, FW_1) = g(\nabla_X^\perp FZ, FW_1) - (X \ln f)g(Z, W_1).\]

Equations (4.14) and (4.15), it follows that

\[(4.16) \quad g((\nabla_X F)Z, FW_1) = 0,\]

for any X \in TN and Z, W_1 \in TN_\perp. Now, if W_2 \in TN_T then using the formula (2.11), we get

\[g((\nabla_X F)Z, \phi W_2) = g(\nabla_X^\perp FZ, \phi W_2) - g(F\nabla_X Z, \phi W_2).\]

As N_T is an invariant submanifold then \phi W_2 \in TN_T for any W_2 \in TN_T, thus using the fact that the product of tangential component with normal is zero, we obtain that

\[(4.17) \quad g((\nabla_X F)Z, \phi W_2) = 0,\]

for any X, W_2 \in TN_T and Z \in TN_\perp. Thus from equations (4.16) and (4.17) it follows that (\nabla_X F)Z \in \mu. This proves the theorem completely.

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