Harmonic maps on Riemannian manifolds

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Abstract. The aim of this article is to underline several properties of harmonic maps on Riemannian manifolds. First we define and give some examples of harmonic maps on compact Riemannian manifolds. Then we prove the first and the second variation formula, we write the Euler-Lagrange equation and we discuss the stability of harmonic maps.

Key words: Euler-Lagrange equation; Riemannian manifolds; harmonic maps; first and second variational formulas; the instability of a harmonic map.

1 Definition and examples

Let us consider two compact Riemannian manifolds \((M, g)\), \((N, h)\), \(\dim M = m\), \(\dim N = n\) and \(\Phi : M \rightarrow N\) a \(C^\infty\)-morfism. We define the integral of energy

\[
E(\Phi) = \frac{1}{2} \int_M \sum_{i=1}^m (\Phi^* h)(e_i, e_i) v_g,
\]

where \(\Phi^*: T_x M \rightarrow T_{\Phi(x)} N\) is the differential of \(\Phi\), \((e_i)_{i=1}^m\) is an orthonormal frame in \(T_x M\). A critical point for the energy is called a harmonic map. We write

\[
\text{Har}(M, N) = \{ \Phi : M \rightarrow N | \Phi \text{ harmonic map} \}
\]

(in general, this is not a manifold).

The \(C^\infty\)-application \(F : (-\varepsilon, \varepsilon) \times M \rightarrow N\), \(F(0, x) = \Phi(x)\), \(F(t, x) = \Phi_t(x)\), \(\forall x \in M\), \(\forall t \in (-\varepsilon, \varepsilon)\) induces a variational vector field

\[
V(x) = \frac{d}{dt} \bigg|_{t=0} \Phi_t(x) = F_* \left( \frac{\partial}{\partial t} \right)_{(0,x)} \in T_{\Phi(x)} N
\]

and a covariant derivative \(\tilde{\nabla}_X \in \Gamma(\Phi_*^{-1} TN)\) with respect to the Riemannian metric \(h\), satisfying

\[
\tilde{\nabla}_X (\Phi_* Y) - \tilde{\nabla}_Y (\Phi_* X) - \Phi_* ([X,Y]) = 0, \quad \forall X, Y \in \chi(M), \quad \tilde{\nabla}_X V = N \nabla_{\Phi_* X} V.
\]

Theorem 1.1. (the first variational formula). The function \(\Phi\) is a harmonic map if and only if satisfies the Euler-Lagrange equation \(\tau(\Phi) = 0\), where

\[
\tau(\Phi)(x) = \sum_{i=1}^m (\tilde{\nabla}_{e_i} \Phi_* e_i - \Phi_* \nabla e_i)(x), \quad \forall x \in M
\]
or
\[ \tau(\Phi)(x) = \sum_{i=1}^{m} (N \nabla_{\Phi e_i} \Phi e_i - \Phi e_i \nabla e_i)(x), \quad \forall x \in M \]

and \( \frac{d}{dt} \bigg|_{t=0} E(\Phi_t) = - \int_M h(V, \tau(\Phi)) v_g. \)

Proof. We can write
\[
\frac{d}{dt} \bigg|_{t=0} E(\Phi_t) = \frac{1}{2} \int_M \sum_{i=1}^{m} \frac{d}{dt} h(\Phi_t e_i, \Phi_t e_i) v_g = \frac{1}{2} \int_M \sum_{i=1}^{m} \frac{d}{dt} h(F_t e_i, F_t e_i)(t, x)v_g =
\]
\[
= \frac{1}{2} \int_M \sum_{i=1}^{m} \left( \frac{\partial}{\partial t} \right) h(F_t e_i, F_t e_i) v_g = \int_M \sum_{i=1}^{m} h \left( \nabla_{\nabla e_i} F_t e_i, F_t e_i \right) v_g =
\]
\[
= \int_M \sum_{i=1}^{m} \left( \nabla_{e_i} F_t \left( \frac{\partial}{\partial t} \right), F_t e_i \right) = \int_M \sum_{i=1}^{m} \left( e_i h \left( F_t \frac{\partial}{\partial t}, F_t e_i \right) \right) - h \left( F_t \frac{\partial}{\partial t}, \nabla_{e_i} F_t e_i \right) v_g.
\]

We have
\[
\sum_{i=1}^{m} \left( e_i h \left( F_t \frac{\partial}{\partial t}, F_t e_i \right) \right) = \sum_{i=1}^{m} e_i g(X_t, e_i) = \sum_{i=1}^{m} \left( g(\nabla_{e_i} X_t, e_i) + g(X_t, \nabla_{e_i} e_i) \right) =
\]
\[
\leq \text{div} X_t + \sum_{i=1}^{m} g(X_t, \nabla_{e_i} e_i) = \text{div} X_t + \sum_{i=1}^{m} h \left( F_t \frac{\partial}{\partial t}, F_t \nabla_{e_i} e_i \right),
\]

so
\[
\frac{d}{dt} \bigg|_{t=0} E(\Phi_t) = - \int_M h(\nabla, \tau(\Phi)) v_g \quad \text{and} \quad \tau(\Phi) = \sum_{i=1}^{m} \left( \nabla_{e_i} \Phi e_i - \Phi e_i \nabla e_i \right). \quad \text{Locally,}
\]
\[
\tau(\Phi) = \Delta \Phi + \sum_{i,j=1}^{m} g^{ij} \left( \sum_{\alpha,\beta=1}^{n} N^{\alpha}_{\beta} \frac{\partial \Phi^\alpha}{\partial x^i} \frac{\partial \Phi^\beta}{\partial x^j} \right),
\]
\[
\tau(\Phi) = \Delta \Phi + \sum_{i,j=1}^{m} g^{ij} \left( \sum_{\alpha,\beta=1}^{n} N^{\alpha}_{\beta} \frac{\partial \Phi^\alpha}{\partial x^i} \frac{\partial \Phi^\beta}{\partial x^j} \right).
\]

The Euler-Lagrange equation is a non-linear partial differential equation.

**Examples.**

1. \( \Phi : (M, g) \to (N, h) \), \( \Phi(x) = q, \forall x \in M \) is a constant harmonic map \( (\tau(\Phi) = 0) \).
2. \( \Phi : (M, g) \to (R^n, g_0) \), \( \Phi = (\Phi^1, \ldots, \Phi^n) \in C^\infty(M, R^n) \) is a harmonic linear map if and only if \( \Phi^i \) is a harmonic function, \( \forall i = 1, n \).
3. \( \Phi : (S^1, g) \to (N, h) \), where \( e_1 = \frac{\partial}{\partial x} \), \( x \in R \), \( \nabla_{e_1} e_1 = 0 \). The function \( \Phi \) is a harmonic map if and only if \( \nabla_{x} \Phi^i = 0 \) (i.e. the equation of geodesics for \( \Phi_1 = \Phi^e_1 \)).
4. \( \Phi : (M, g) \to (N, h) \), an isometric harmonic immersion is equivalent to minimality.
5. \( \Phi : (M, g) \rightarrow (N, h) \), a harmonic Riemannian submersion is equivalent to the minimality of the submanifolds \( \Phi^{-1}(\Phi(x)) \) in \( M \).

6. \( \Phi : (M, g) \rightarrow (N, h) \) is a holomorphic harmonic map between two Kahler manifolds.

7. \( \pi : (\mathbb{C}^n, g_0) \rightarrow (\mathbb{C}^n|\Lambda, g_\Lambda) \) is the projection between two Kähler manifolds with zero sectional curvature (\( \mathbb{C}^n|\Lambda \) is the complex torus, \( \Lambda = \mathbb{C}^n \)).

8. \( \pi : (S^{n+1}, g_{S^{n+1}}) \rightarrow (P^n(\mathbb{C}), h) \) is the Hopf map, i.e. a Riemannian harmonic submersion between the sphere \( S^{n+1} = SU(n+1)/SU(n) \) (with sectional curvature equal to 1) and the projective complex space \( P^n(\mathbb{C}) = SU(n+1)/S(U(1) \times U(n)) \) (with sectional curvature in \([1,4]\)).

## 2 The second variational formula

Let us consider a new continuous variation with respect of two parameters \( s, t \),

\[ F : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times M \rightarrow N, \quad F(s, t, x) = \Phi_{s,t}(x), \quad F(0, 0, x) = \Phi(x), \quad \forall x \in M. \]

The variational vector fields are as follows:

\[ V(x) = \frac{\partial}{\partial s} \bigg|_{s=0} \Phi_s(x) = F_s \left( \frac{\partial}{\partial s} \right) \bigg|_{(s, t) = (0, 0)}, \]

\[ W(x) = \frac{\partial}{\partial t} \bigg|_{t=0} \Phi_t(x) = F_s \left( \frac{\partial}{\partial t} \right) \bigg|_{(s, t) = (0, 0)} \in \Gamma(\Phi^{-1}_*TN). \]

The integral of energy is

\[ E(\Phi_{s,t}) = \frac{1}{2} \int_M \sum_{i=1}^m h(\phi_{s,t} e_i, \phi_{s,t} e_i) v_g, \]

and the Hessian of energy is

\[ \text{Hess} \left( E_{\Phi} \right)(V, W) = \frac{\partial^2}{\partial s \partial t} \bigg|_{(s, t) = (0, 0)} E(\Phi_{s,t}). \]

**Theorem 2.1.** (the second variational formula). If \( \Phi \) is a harmonic map between \( M \) and \( N \), then the Hessian of the energy is

\[ \text{Hess} \left( E_{\Phi} \right)(V, W) = \int_M h(J_\Phi(V), W) v_g, \]

where the elliptic differential Jacobi operator has \( J_\Phi(V) \) of the form

\[ J_\Phi(V) = -\sum_{i=1}^m (\nabla_{e_i} \nabla_{e_i} - \nabla_{\nabla_{e_i} e_i}) V - \sum_{i=1}^m R(V, e_i e_i) e_i, \quad J_\Phi = \Delta_\Phi - R_\Phi, \]

the classical Laplacian \( \Delta_\Phi \) being defined by

\[ \int_M h(\Delta_\Phi V, W) v_g = \int_M h(V, \nabla_\Phi W) v_g = \int_M h(V, \nabla_\Phi W) v_g, \quad \forall V, W \in \Gamma(\Phi^{-1}_*TN). \]

**Proof.** Indeed, by straightforward calculation, we obtain:

\[ \frac{\partial}{\partial t} E(\Phi_{s,t}) = -\int_M h \left( F_s \frac{\partial}{\partial t} \sum_{i=1}^m (\nabla_{e_i} F_s e_i - F_s \nabla_{e_i} e_i) \right) v_g. \]
We differentiate with respect to \( t \):
\[
\frac{\partial^2}{\partial s \partial t} E(\Phi, t) = - \int_M h \left( \nabla_{\partial_t} F_s \frac{\partial}{\partial t}, \sum_{i=1}^m \left( \nabla_{e_i} F_s e_i - F_s \nabla_{e_i} e_i \right) \right) v_g - \int_M h \left( F_s \frac{\partial}{\partial t}, \sum_{i=1}^m \nabla_{\partial_t} \left( \nabla_{e_i} F_s e_i - F_s \nabla_{e_i} e_i \right) \right) v_g.
\]

We find
\[
\nabla_{\partial_t} \nabla_{e_i} F_s e_i = \nabla_{e_i} \nabla_{\partial_t} F_s e_i + N R \left( F_s \frac{\partial}{\partial s}, F_s e_i \right) F_s e_i + \nabla_{\left[ \overset{\partial_t}{\nabla}_{e_i} \right]} F_s e_i
\]
and
\[
\nabla_{\partial_t} \nabla_{e_i} e_i = \nabla_{\nabla_{e_i} e_i} F_s \frac{\partial}{\partial s}, \text{ because } \left[ \frac{\partial}{\partial s}, e_i \right] = 0, \forall i = 1, m.
\]

So, we have \( \text{Hess}(E)_\Phi(V, W) = \int_M h(J_\Phi V, W), \) where the Jacobi’s operator is \( J_\Phi(V) = \Delta_\Phi V - R_\Phi V. \) Moreover,
\[
h(\Delta_\Phi V, W) = - \sum_{i=1}^m (e_i h(\nabla_{e_i} V, W) - h(\nabla_{e_i} V, \nabla_{e_i} W)) + \sum_{i=1}^m h(\nabla_{\nabla_{e_i} e_i} V, W) =
\]
\[
= -\text{div} X + \sum_{i=1}^m h(\nabla_{e_i} V, \nabla_{e_i} W), g(X, Y) = h(\nabla_Y V, W).
\]

By integration, we obtain the final conclusion. \( \square \)

3 The instability theorem

For a harmonic map \( \Phi : (M, g) \to (N, h), \) we define the following notions:

\( \bullet \) \( \text{index}(\Phi) = \sup \{ \text{dim} F | F \subset \Gamma(\Phi_s^{-1} T N) \text{ a subspace with Hess } (E)_\Phi \text{ non-positive definite } \}, \)
\( \bullet \) \( \text{null } (\Phi) = \text{dim} \{ V \in \Gamma(\Phi_s^{-1} T N) | \text{Hess } (E)_\Phi(V, W) = 0, \forall W \in \Gamma(\Phi_s^{-1} T N) \}, \)
\( \bullet \) \( V_\lambda(\Phi) = \{ V \in \Gamma(\Phi_s^{-1} T N) | J_\Phi V = \lambda V \}, \) \( \text{dim} V_\lambda(\Phi) = \) the multiplicity of \( \lambda, \)
\( \bullet \) \( \text{Spec}(J_\Phi) = \{ \lambda | J_\Phi V = \lambda V, V \in \Gamma(\Phi_s^{-1} T N) \} \) \( (\text{with increasing elements}) \).

We say that \( \Phi \) is \textit{low - stable} if and only if the index vanishes \( (\Phi) = 0 \) (with positive eigenvalues for the Hessian) and if it isn’t, \( \Phi \) is called \textit{instable}. We can write the index \( (\Phi) = \sum_{\lambda \leq 0} \text{dim} V_\lambda(\phi), \) null \( (\Phi) = \text{dim} V_0(\Phi) = \text{dim}(\ker J_\Phi). \)

**Theorem 3.1.** 1. If \( (N, h) \) has a non-positive sectional curvature, then \( \Phi \) is low-stable.

2. We have \( T_\Phi \text{Har}(M, N) \subset \ker J_\Phi \) \( (\text{in general, the equality is false}) \) and \( \text{dim} T_\Phi \text{Har}(M, N) \leq \text{null}(\Phi) \) \( (\text{as a vector space}). \)

3. (Xin, 1980). For \( (S^m, g_{S^m}^0) \) (the sectional curvature is equal with 1), \( m \geq 3, \) \( (N, h) \) a compact Riemannian manifold, any non-constant harmonic map \( \Phi : S^m \to N \) is unstable.
4. By generalization, for a compact Riemannian manifold \((M, g)\), \(\pi_1(M) = \pi_2(M) = 0\) and \((N, h), (M', g')\) two arbitrary compact Riemannian manifolds, the harmonic maps \(\Phi : M \to N\) and \(\Psi : M \to M'\) are unstable (index \((\Phi) > 0\), index \((\Psi) > 0\)).

Proof. 1. The condition \(N k(u, v) \leq 0\) is equivalent to \(h(N R(u, v), u) \leq 0\), which is equivalent to \(h(R\phi V, V) \leq 0, \forall V \in (\phi^{-1} T N)\). So,

\[
\int h(\bar{\Delta} \phi V, V) v_g = \int h(\nabla \phi V, \phi V) v_g \geq 0
\]

and \(\int h(J \phi V, V) v_g \geq 0, \forall V \in (\phi^{-1} T N)\).

2. If \((\phi_s)_{s} \in \text{Har}(M, N)\) is a 1-parameter family of harmonic maps, \(\phi_0 = \phi, \phi_{s,t} \in C^\infty(M, N)\) a variation of \(\phi_s, \phi_{s,0} = \phi_s\), then \(\frac{\partial}{\partial t} \bigg|_{t=0} E(\phi_{s,t}) = 0 \) (\(\phi_s\) is harmonic, \(\forall s\)) and

\[
\int h(\phi_s V, W) v_g = \frac{\partial^2}{\partial s \partial t} \bigg|_{(s, t) = (0, 0)} E(\phi_{s,t}) = 0, \forall W \in \Gamma(\phi^{-1} T N).
\]

So, \(V \in \ker (J_\phi)\) and \(\dim T_\phi \text{Har}(M, N) \leq \text{null}(\Phi)\).

3. Step 1. We decompose the tangent vector space in \(x \in M, \) in a direct sum \(T_x \mathbb{R}^{m+1} = T_x S^m \oplus T_x S^m^\perp\) s.t., for any vector field \(V = \sum_{i=1}^{m+1} a^i \frac{\partial}{\partial x_i} = V^T + V^\perp,\)

\[
V^T = \sum_{i=1}^{m+1} (a_i - x_i(a,x)) \frac{\partial}{\partial x_i}, \quad V^\perp = (a, x) \sum_{i=1}^{m+1} x_i \frac{\partial}{\partial x_i} = W.
\]

We obtain \(\nabla_X W = -(a, x) X, \forall X \in T_x S^m\).

Step 2. We consider an orthonormal local frame on \((S^m, g^S_{\mathbb{R}}), (e_i)_{i=1,m},\) and we prove \(\Delta W = W = \text{grad} S^m(a, x).\) Indeed,

\[
\Delta W = -\sum_{i=1}^{m} (\nabla_{e_i} \nabla_{e_i} W - \nabla_{e_i} e_i W) = \\
= -\sum_{i=1}^{m} (\nabla e_i (- (a, x) e_i) - (a, x) \nabla e_i e_i) = \sum_{i=1}^{m} (e_i(a, x)) e_i = W.
\]

Step 3. We have \(\nabla_Y e_i = 0, \forall Y \in \chi(S^m)\) and

\[
\Delta \phi_s W = -\sum_{i=1}^{m} (\nabla e_i \nabla e_i - \nabla_{e_i} e_i) \phi_s W = -\sum_{i=1}^{m} \nabla e_i \nabla e_i \phi_s W.
\]

But \(\phi_s W = d\phi(W), \nabla_X W = -(a, x) W,\) so

\[
\nabla_{e_i} \phi_s W = (\nabla_{e_i} d\phi)(W) + d\phi (\nabla_{e_i} W) = (\nabla_{e_i} d\phi)(W) - d\phi((a, x) e_i).
\]

By analogy,

\[
\Delta \phi_s W = -\sum_{i=1}^{m} (\nabla e_i \nabla_{e_i} d\phi)(W) - \sum_{i=1}^{m} (\nabla e_i d\phi)(V e_i W) + \sum_{i=1}^{m} \nabla e_i (d\phi((a, x) e_i))
\]
and because $d\Phi(a, x)e_i = (a, x)\Phi_*e_i$, $\Phi$ is harmonic, $\nabla e_i = 0$,

$$\Phi_*W = \sum_{i=1}^m \nabla_{e_i}(d\Phi(a, x)e_i),$$

we obtain the relation: $(\Delta d\Phi)(W) = \sum_{i=1}^m N R(\Phi_*W, \Phi_*e_i)\phi_i e_i - \Phi_*\rho(W)$, where

$$\rho(W) = \sum_{i=1}^m R(W, e_i)e_i = (m-1)W.$$

So,

$$\Delta \Phi_*W = \sum_{i=1}^m N R(\Phi_*W, \Phi_*e_i)\Phi_*e_i + (2-m)\Phi_*W.$$

**Step 4.**

$$0 \leq \int_M h(J\Phi(\Phi_*W), \Phi_*W)v_g = \int_M h(\Delta \Phi_*W -$$

$$- \sum_{i=1}^m N R(\Phi_*W, \Phi_*e_i)\phi_i e_i, \Phi_*W)v_g = (2-m)\int_M h(\Phi_*W, \Phi_*W)v_g.$$

For $m \geq 3$ and index $(\Phi) = 0$, we have $\int_M h(\Phi_*W, \Phi_*W)v_g = 0$, i.e. $\Phi_*W = 0$, $\forall W \in T_xS^m$, so $\Phi_* = 0$ and $\Phi$ is constant. $\square$

For related problems, see [1]-[10].

**References**


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