On LP-Sasakian manifolds satisfying certain conditions on the concircular curvature tensor

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Abstract. In this paper we show that LP-Sasakian manifolds are Einstein manifolds if they satisfy the conditions $R(X, Y).S = 0$, $\tilde{C}(\xi, X).S = 0$ and $R.\tilde{C} = R.R$.

Key words: LP-Sasakian manifold; concircular curvature tensor; Einstein manifold.

1 Introduction

In 1989, K. Matsumoto [7] introduced the notion of Lorentzian Para Sasakian manifold. I. Mihai and R. Rosca [6] defined the same notion independently and thereafter many authors ([6], [14], [8]) studied LP-Sasakian manifolds. Cihan Özgür and U.C.De studied the same conditions on Para Sasakian manifolds. An LP-Sasakian manifold is called Ricci-semi-symmetric if $R(X, Y).S = 0$. In this paper we prove that an LP-Sasakian manifold is Ricci-semi symmetric if and only if it is an Einstein manifold. Also, we show that an LP-Sasakian manifold satisfying $\tilde{C}(\xi, X).S = 0$ is an Einstein manifold and manifold of constant scalar curvature $n(n - 1)$, were $\tilde{C}$ is a concircular curvature tensor. Finally, we show that $R.\tilde{C} = R.R$.

2 Preliminaries

An $2n + 1$- dimensional differentiable manifold $M$ is called an LP-Sasakian manifold [7], [8] if it admits a $(1, 1)$ tensor field $\varphi$, a contravariant vector field $\xi$, a $1$- form $\eta$ and Lorentzian metric $g$ which satisfy

\[
\begin{align*}
\varphi^2 &= I + \eta \otimes \xi, \quad \eta(\xi) = -1, \\
g(\varphi X, \varphi Y) &= g(X, Y) + \eta(X)\eta(Y), \\
(a) \quad \nabla_X \xi &= \varphi X, \quad (b) \quad g(X, \xi) = \eta(X), \\
\end{align*}
\]

and

\[
(\nabla_X \varphi)Y = g(X, Y)\xi + 2\eta(X)\eta(Y)\xi,
\]
On LP-Sasakian manifolds

where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$. It can be easily seen that in an LP-Sasakian manifold, the following relations hold:

\[
\varphi \xi = 0, \eta(\varphi X) = 0, \quad \text{rank} \varphi = 2n.
\]

Again if we put

\[
\Omega(X, Y) = g(X, \varphi Y),
\]

for any vector fields $X$ and $Y$, then the tensor field $\Omega(X, Y)$ is a symmetric $(0, 2)$ tensor field[7]. Also since the vector field $\eta$ is closed in an LP-Sasakian manifold we have ([7], [11]):

\[
(\nabla_X \eta)(Y) = \Omega(X, Y), \quad \Omega(X, \xi) = 0,
\]

for any vector fields $X$ and $Y$. Also, an LP-Sasakian manifold $M$ is said to be $\eta$-Einstein if its Ricci tensor $S$ is of the form

\[
S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)
\]

for any vector fields $X, Y$ where $a, b$ are functions on $M$. Further, on such an LP-Sasakian manifold the following relations hold ([8], [11]):

\[
g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = \eta(R(X, Z)Y) - g(X, Z)\eta(Y),
\]
\[
R(X, Y)\xi = \eta(Y)X - \eta(X)Y,
\]
\[
\begin{align*}
R(\xi, X)Y &= g(X, Y)\xi - \eta(Y)X, \\
R(\xi, X)\xi &= X + \eta(X)\xi,
\end{align*}
\]
\[
S(\varphi X, \varphi Y) = S(X, Y) + 2n\eta(X)\eta(Y),
\]

for any vector fields $X, Y, Z$, where $R(X, Y)Z$ is the curvature tensor, and $S$ is the Ricci tensor.

**Definition 1.** The concircular curvature tensor $\tilde{C}$ on LP-Sasakian manifold $M$ of dimensional $2n + 1$ is defined by

\[
\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y],
\]

for any vector fields $X, Y, Z$, where $R$ is the curvature tensor and $r$ is the scalar curvature.

**Definition 2.** An $2n + 1$-dimensional LP-Sasakian manifold is said to be Ricci-semi-symmetric if

\[
R(X, Y)S = 0,
\]

where $R$ is the curvature tensor and $S$ is the Ricci tensor.

**3 Main Results**

In this section, we prove the following theorems:
Theorem 3.1. Let $M$ be an $2n+1$-dimensional LP-Sasakian manifold. Then $M$ is Ricci-semi-symmetric if and only if it is an Einstein manifold.

Proof. We know that every Einstein manifold is Ricci-semi symmetric but the converse is not true in general. Here, we prove that in an LP-Sasakian manifold $R(X,Y).S = 0$ implies that the manifold is an Einstein manifold. It follows from (2.5) that

\[(3.1) \quad S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0.\]

Putting $X = \xi$ in (3.1) we get

\[(3.2) \quad S(R(\xi,Y)U,V) + S(U,R(\xi,Y)V) = 0.\]

Using (2.2), (2.3), from (3.2) we get

\[(3.3) \quad 2\eta(Y)S(Y,V) + 2\eta(S(Y,V)) = 0.\]

Now, putting $U = \xi$ in (3.3), we obtain

\[(3.4) \quad S(Y,V) = 2\eta(Y,V).\]

Therefore, $M$ is Einstein manifold. This the completes the proof of the theorem. □

Theorem 3.2. Let $M$ be an $2n+1$-dimensional LP-Sasakian manifold. Then $M$ satisfies in condition

\[\tilde{C}(\xi,X).S = 0,\]

if and only if either $M$ is Einstein manifold or $M$ has scalar curvature $r = 2n(2n+1)$.

Proof. Since $\tilde{C}(\xi,X).S = 0$, we have

\[\tilde{C}(\xi,X).S(Y,\xi) = 0.\]

This implies that

\[(3.4) \quad S(\tilde{C}(\xi,X)Y,\xi) + S(Y,\tilde{C}(\xi,X)\xi) = 0.\]

In view of (2.3), (2.4) in (3.4) we infer

\[(3.5) \quad (1 - \frac{r}{2n(2n+1)})[g(X,Y)\xi - \eta(Y)X,\xi] + S(Y,\xi) + S(Y,\xi) + S(X,Y)] = 0.\]

Using (2.3) in (3.5) we get

\[(3.6) \quad (1 - \frac{r}{2n(2n+1)})[2\eta(Y)\eta(\xi) + 2\eta(\xi)\eta(Y) + S(X,Y)] = 0.\]
In view of (2.1) in (3.6), it follows that

\[ \left( 1 - \frac{r}{2n(2n+1)} \right) [-2ng(X, Y) + S(X, Y)] = 0. \]

This implies \( S(X, Y) = 2ng(X, Y) \), or \( r = 2n(2n + 1) \). Therefore \( M \) is an Einstein manifold with the scalar curvature \( r = 2n(2n + 1) \). The converse is trivial. The proof is complete.

\[ \square \]

**Theorem 3.3.** Let \( M \) be an \( 2n + 1 \)-dimensional LP-Sasakian manifold. Then \( R.\tilde{C} = R.R \).

**Proof.** We have

\[
(R(X, Y).\tilde{C})(U, V, W) = R(X, Y)\tilde{C}(U, V)W - \tilde{C}(R(X, Y)U, V)W \\
- \tilde{C}(U, R(X, Y)V)W - \tilde{C}(U, V)R(X, Y)W.
\]

In view of (2.4) in (3.7) we have

\[
(R(X, Y).\tilde{C})(U, V, W) = R(X, Y)[R(U, V)W - \frac{r}{2n(2n+1)}(g(V, W)U - g(U, W)V)] \\
- R(R(X, Y)U, V)W + \frac{r}{2n(2n+1)}[g(V, W)R(X, Y)U - g(R(X, Y)U, W)V] \\
- R(U, R(X, Y)V)W + \frac{r}{2n(2n+1)}[g(R(X, Y)V, W)U - g(U, W)R(X, Y)V] \\
- R(U, V)R(X, Y)W + \frac{r}{2n(2n+1)}[g(V, R(X, Y)W)U - g(U, R(X, Y)W)V].
\]

We have

\[
- R(U, R(X, Y)V)W - R(U, V)R(X, Y)W \\
+ \frac{r}{2n(2n+1)}[g(R(X, Y)V, W)U + g(V, R(X, Y)W)U] \\
- g(R(X, Y)U, W)V - g(U, R(X, Y)W)V.
\]

Finally, we get

\[
\]

Therefore \( R.\tilde{C} = R.R \). This completes the proof of the theorem.

\[ \square \]

Now, in view of Theorem 2.1 of [1] and Theorem 3.3, if \( R(\xi, X).\tilde{C} = 0 \), then \( M \) is of constant curvature 1 and consequently we can state the following result

**Theorem 3.4.** An \( n \)-dimensional LP-Sasakian manifold \( M \) satisfies \( R(\xi, X).\tilde{C} = 0 \), if and only if \( M \) is locally isometric to the unit sphere \( S^n(1) \).
References


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