On weakly pseudo-projectively symmetric manifolds

J. P. Jaiswal and R. H. Ojha

Abstract. The object of present paper is to study weakly pseudo-projectively symmetric manifolds and pseudo-projectively flat weakly Ricci-symmetric manifolds.


Key words: Weakly symmetric manifold; weakly pseudo-projectively symmetric manifold; weakly Ricci-symmetric manifold; quasi-Einstein manifold.

1 Introduction

The notion of weakly symmetric Riemannian manifolds was introduced by L. Tamassy and T. Q. Binh [10] and also studied in [1].

A non-flat Riemannian manifold \((M^n, g)\) \((n > 2)\) is called weakly symmetric manifold if its curvature tensor \(K\) of type \((0, 4)\) satisfies the condition

\[
\]

(1.1)

for all vector fields \(X, Y, Z, U, V \in \chi(M^n)\), where \(A, B, C, D\) and \(E\) are 1-forms (not simultaneously zero) and \(\nabla\) the operator of covariant differentiation with respect to the Riemannian metric \(g\). The 1-forms are called the associated 1-forms of the manifold and an \(n\) dimensional manifold of this kind is denoted by \((WS)_n\).

Tamassy and Binh [11] further studied weakly symmetric Sasakian manifolds and proved that such a manifold does not always exist. In [3] the authors established the existence of \((WS)_n\) by an example and proved that in \((WS)_n\), the associated 1-forms \(B = C\) and \(D = E\). So (1.1) reduces to the following form

\[
\]

(1.2)

Some authors like De and Bandyopadhyay [4], Shaikh and Baishya [9] extended this notion for conformal curvature tensor, quasi-conformal curvature tensor respectively. Recently Malik and Samavaki [7] have also studied weakly symmetric Riemannian manifolds.
In 2002 Prasad [8] defined and studied a tensor field $\bar{P}$ on a Riemannian manifold of dimension $n$ ($n > 2$) which includes the projective curvature tensor $P$. This tensor field $\bar{P}$ is known as pseudo-projective curvature tensor and given by

$$\bar{P}(X, Y, Z) = aK(X, Y, Z) + b[R(Y, Z)X - R(X, Z)Y] - \frac{r}{n} \left[ \frac{a}{n - 1} + b \right] [g(Y, Z)X - g(X, Z)Y],$$  \tag{1.3}$$

where $a$ and $b$ are constants such that $a, b \neq 0$, $K$ is the curvature tensor, $R$ is the Ricci tensor and $r$ is the scalar curvature.

A non-pseudo projectively flat Riemannian manifold $(M^n, g)$ ($n > 2$) is said to be weakly pseudo-projectively symmetric manifold if the pseudo-projective curvature tensor $\bar{P}$ of type $(0, 4)$ satisfies the condition

$$\nabla_X \bar{P}(Y, Z, U, V) = A(X) \bar{P}(Y, Z, U, V) + B(Y) \bar{P}(X, Z, U, V) + C(Z) \bar{P}(Y, X, U, V) + D(U) \bar{P}(Y, Z, X, V) + E(V) \bar{P}(Y, Z, U, X),$$  \tag{1.4}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where $A, B, C, D$ and $E$ are defined as before. Such an n-dimensional manifold is denoted by $(WPPS)_n$.

Section 2 is concerned with preliminaries. It is shown that like $(WS)_n$, in a $(WPPS)_n$ we have always $B = C$ and $D = E$ and hence (1.4) reduces to the form

$$\nabla_X \bar{P}(Y, Z, U, V) = A(X) \bar{P}(Y, Z, U, V) + B(Y) \bar{P}(X, Z, U, V) + C(Z) \bar{P}(Y, X, U, V) + D(U) \bar{P}(Y, Z, X, V) + E(V) \bar{P}(Y, Z, U, X),$$  \tag{1.5}$$

where $A, B, D$ are non-zero 1-forms.

In section 3 we have investigated the nature of scalar curvature of a $(WPPS)_n$. It is proved that if in a $(WPPS)_n$ the Ricci tensor is of Codazzi type or constant scalar curvature then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor $R$ corresponding to the eigenvector $Q$ defined by $g(X, Q) = \lambda(X)$, for all $X$. Further we have studied some other properties.

Section 4 is devoted in the study of pseudo-projectively flat $(WRS)_n$. At first we have proved that in a pseudo-projectively flat $(WRS)_n$ the vector field $\rho$ defined by $g(X, \rho) = H(X)$ is a proper concircular vector field. Finally we have shown that a pseudo-projectively flat $(WRS)_n$ ($n > 2$) is a quasi-Einstein manifold.

2 Preliminaries

In this section we derive some formulae which will be required to the study of a $(WPPS)_n$. Let $\{e_i, i = 1, 2, ..., n\}$ be an orthonormal basis of the tangent at any point of the manifold. Then from (1.3), we have the following

$$\sum_{i=1}^{n} \bar{P}(e_i, Y, Z, e_i) = [a + (n - 1)b]P(Y, Z),$$  \tag{2.1}$$
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where

\[ P(Y, Z) = R(Y, Z) - \frac{r}{n} g(Y, Z), \]

\[ \sum_{i=1}^{n} \overline{P}(X, Y, e_i, e_i) = 0, \]

\[ \sum_{i=1}^{n} \overline{P}(e_i, e_i) = 0. \]

It can be easily seen that the pseudo-projective curvature tensor \( \overline{P} \) is skew symmetric with respect to first two indices but neither symmetric nor skew-symmetric with respect to last two indices. Also neither symmetric nor skew-symmetric in first and last two indices.

**Proposition 2.1.** In a Riemannian manifold \( (M^n, g) \) \( (n > 2) \) the pseudo-projective curvature tensor satisfies the following relations

\[ \begin{aligned}
(I) \overline{P}(X, Y, Z, U) + \overline{P}(Y, Z, X, U) + \overline{P}(Z, X, Y, U) &= 0, \\
(II) \overline{P}(X, Y, U, Z) + \overline{P}(Y, Z, U, X) + \overline{P}(Z, X, U, Y) &= 0.
\end{aligned} \]

**Proposition 2.2.** The defining condition of \( (\text{WP}P \text{S})_n \) can always be expressed in the form (1.5).

**Proof.** Interchanging \( Y \) and \( Z \) in (1.4), we get

\[ \overline{(\nabla X \overline{P})}(Z, Y, U, V) = A(X) \overline{P}(Z, Y, U, V) + B(Z) \overline{P}(X, Y, U, V) + C(Y) \overline{P}(Z, X, U, V) + D(U) \overline{P}(Z, Y, X, V) + E(V) \overline{P}(Z, Y, U, X). \]

Adding (1.4) and (2.5) then using skew-symmetric property of \( \overline{P} \), we obtain

\[ \mu(Y) \overline{P}(X, Z, U, V) + \mu(Z) \overline{P}(X, Y, U, V) = 0, \]

where \( \mu(X) = B(X) - C(X), \forall X \).

Now we choose a particular vector field \( \rho \) such that \( \mu(\rho) \neq 0 \). Substituting \( Y = Z = \rho \) in (2.6) we get \( \overline{P}(X, \rho, U, V) = 0 \). Again putting \( Z = \rho \) in (2.6) we get \( \overline{P}(X, Y, U, V) = 0 \) for all vector fields \( X, Y, U \) and \( V \) which contradicts our assumption that the manifold in not pseudo projectively flat. Hence we must have \( \mu(X) = 0 \) for all \( X \), and \( B = C \). Similarly, by interchanging \( U \) and \( V \) in (1.4) and proceeding as above, it can be easily seen that \( D = E \). Thus all the associated 1-forms \( A, B, C, D \) and \( E \) coincide, since \( B = C \) and \( D = E \). Therefore (1.4) can be written as (1.5). \( \square \)

3 The nature of scalar curvature

Let \( L \) be the symmetric endomorphism of the tangent space at any point of the manifold corresponding to the Ricci tensor \( R \) i.e.

\[ g(LX, Y) = R(X, Y). \]
Theorem 3.1. If in a Riemannian manifold \((M^n, g)\) \((n > 2)\), the Ricci tensor if of Codazzi type then relation (3.4) holds. Converse also holds if the scalar curvature is constant.

Proof. From (1.3) it follows by virtue of Bianchi identity that

\[
(\nabla_X \bar{P})(Y, Z, U, V) + (\nabla_Y \bar{P})(Z, X, U, V) + (\nabla_Z \bar{P})(X, Y, U, V) \nonumber
\]

\[
\]

\[
= \frac{1}{n} \left[ \frac{a}{n - 1} + b \right] [dr(X)\{g(Z, U)g(Y, V) - g(Y, U)g(Z, V)\} + dr(Y)\{g(X, U)g(Z, V) - g(Z, U)g(X, V)\} + dr(Z)\{g(Y, U)g(X, V) - g(X, U)g(Y, V)\}]. 
\]

If the Ricci tensor is of Codazzi type [5] i.e. if

\[
(\nabla_X R)(Z, U) = (\nabla_Z R)(X, U), 
\]

which implies

\[
dr(X) = 0, \forall X. 
\]

By virtue of (3.2) and (3.3), (3.1) becomes

\[
(\nabla_X \bar{P})(Y, Z, U, V) + (\nabla_Y \bar{P})(Z, X, U, V) + (\nabla_Z \bar{P})(X, Y, U, V) = 0. 
\]

Conversely suppose that in a Riemannian manifold (3.4) holds, then (3.1) becomes

\[
\]

\[
= \frac{1}{n} \left[ \frac{a}{n - 1} + b \right] [dr(X)\{g(Z, U)g(Y, V) - g(Y, U)g(Z, V)\} + dr(Y)\{g(X, U)g(Z, V) - g(Z, U)g(X, V)\} + dr(Z)\{g(Y, U)g(X, V) - g(X, U)g(Y, V)\}]. 
\]

Putting \(Y = V = e_i\) in (3.5) and then taking summation over \(i, 1 \leq i \leq n\), we obtain

\[
b[(\nabla_X R)(Z, U) - (\nabla_Z R)(X, U)] 
\]

\[
= \frac{1}{n} \left[ \frac{a}{n - 1} + b \right] \{dr(X)g(Z, U) - dr(Z)g(X, U)\}. 
\]

Since \(r\) is constant, then (3.6) shows that Ricci tensor is of Codazzi type. \(\Box\)

Theorem 3.2. If in a \((WPPS)\) the Ricci tensor is of Codazzi type or the scalar curvature is constant then \(\frac{r}{n}\) is an eigenvalue of the Ricci tensor \(R\) corresponding to the eigenvector \(Q\) defined by \(g(X, Q) = \lambda(X)\), for all \(X\) provided that \(a + \frac{n-2b}{2} \neq 0\).
Proof. First we Suppose that the Ricci tensor is of Codazzi type so by virtue of (3.4) and (1.5), we have

\[ \lambda(X) \hat{P}(Y, Z, U, V) + \lambda(Y) \hat{P}(Z, X, U, V) + \lambda(Z) \hat{P}(X, Y, U, V) = 0, \]  

where

\[ \lambda(X) = A(X) - 2B(X), \forall X. \]

Putting \( Y = V = e_i \) in (3.7) and then taking summation over \( i, 1 \leq i \leq n \), we obtain

\[ \{a + (n - 1)b\}[\lambda(X)P(Z, U) - \lambda(Z)P(X, U)] + \lambda(\hat{P}(Z, X, U)) = 0. \]

Again putting \( X = U = e_i \) in (3.8) and then taking summation over \( i, 1 \leq i \leq n \) and then using (2.3) we obtain

\[ \lambda(LZ) = \frac{r}{n} \lambda(Z), \]

provided that \( a + \frac{(n - 2)b}{2} \neq 0 \), i.e.

\[ R(Z, Q) = \frac{r}{n} g(Z, Q). \]

Next suppose that scalar curvature is constant. By virtue of (3.1) and (1.5), we obtain

\[ \lambda(X) \hat{P}(Y, Z, U, V) + \lambda(Y) \hat{P}(Z, X, U, V) + \lambda(Z) \hat{P}(X, Y, U, V) \]

\[ = b[g(Y, V)\{(\nabla_X R)(Z, U) - (\nabla_Z R)(X, U)\} \]

\[ + g(Z, V)\{(\nabla_Y R)(X, U) - (\nabla_X R)(Y, U)\} \]

\[ + g(X, V)\{(\nabla_Z R)(Y, U) - (\nabla_Y R)(Z, U)\}] \]

\[ - \frac{1}{n} \left[ \frac{a}{n - 1} + b \right] [dr(X)[g(Z, U)g(Y, V) - g(Y, U)g(Z, V)] \]

\[ + dr(Y)[g(X, U)g(Z, V) - g(Z, U)g(X, V)] \]

\[ + dr(Z)[g(Y, U)g(X, V) - g(X, U)g(Y, V)]]]. \]

Setting \( Y = V = e_i \) in (3.10) and then taking summation over \( i, 1 \leq i \leq n \), we get

\[ \{a + (n - 1)b\}[\lambda(X)P(Z, U) - \lambda(Z)P(X, U)] + \lambda(\hat{P}(Z, X, U)) \]

\[ = (n - 2) \left\{ b[(\nabla_X R)(Z, U) - (\nabla_Z R)(X, U)] \right. \]

\[ - \frac{1}{n} \left( \frac{a}{n - 1} + b \right) [dr(X)g(Z, U) - dr(Z)g(X, U)] \right\}. \]

Again contracting over \( X \) and \( U \), we obtain

\[ \frac{(n - 2)}{2n} dr(Z) = \lambda(LZ) - \frac{r}{n} \lambda(Z), \]

provided \( a + \frac{(n - 2)b}{2} \neq 0 \). Since the manifold is of constant scalar curvature then (3.12) reduces to (3.9) provided that \( a + \frac{(n - 2)b}{2} \neq 0. \)
Theorem 3.3. If the scalar curvature of \((WPPS)_n\) vanishes then relation (3.17) holds provided that \(a + \frac{(n-2)b}{2} \neq 0\). Converse also holds if \(T(X) \neq 0\) for all \(X\).

Proof. Putting \(Y = V = e_i\) in (1.5) and then taking summation over \(i, 1 \leq i \leq n\), we get by virtue of (2.1) that

\[
\{a + (n-1)b\} (\nabla X P)(Z, U) = [a + (n-1)b][A(X)P(Z, U) + B(Z)P(X, U) + D(U)P(Z, X)] + B(\bar{P}(X, Z, U)) + D(\bar{P}(X, U, Z)).
\]

(3.13)

Let \(\rho_1, \rho_2, \rho_3\) be the vector fields associated to the 1-forms \(A, B\) and \(D\) respectively. Therefore we have \(A(X) = g(X, \rho_1), B(X) = g(X, \rho_2), D(X) = g(X, \rho_3)\). Substituting \(Z = U = e_i\) in (3.13) and then taking summation over \(i, 1 \leq i \leq n\), we get

\[
P(X, \rho_2) + P(X, \rho_3) = 0,
\]

(3.14)

provided \(a + \frac{(n-2)b}{2} \neq 0\).

Which implies that

\[
R(X, \rho_2) + R(X, \rho_3) = \frac{r}{n}[g(X, \rho_2) + g(X, \rho_3)].
\]

(3.15)

In view of (3.15), we have

\[
R(X, \tilde{\rho}) = \frac{r}{n}g(X, \tilde{\rho}),
\]

(3.16)

where \(g(X, \tilde{\rho}) = T(X) = B(X) + D(X), \tilde{\rho} = \rho_2 + \rho_3\). Since the scalar curvature \(r\) of \((WPPS)_n\) is zero then from (3.16) \(R(X, \tilde{\rho}) = 0\) and so by virtue of (1.3), we have

\[
\bar{P}(X, Y, \tilde{\rho}, U) = aR(X, Y, \tilde{\rho}, U).
\]

(3.17)

Also if (3.17) holds in \((WPPS)_n\), then by virtue of (3.16) it follows from (1.3) that \(r = 0\) for \(T(X) \neq 0\) for all \(X\). \(\square\)

4 Pseudo projectively flat weakly Ricci-symmetric manifolds

The notion of weakly Ricci symmetric manifolds was introduced by Tamassy and Binh ([11]).

A non-flat Riemannian manifold \((M^n, g)\) \((n > 2)\) is called weakly Ricci symmetric if its Ricci tensor of type \((0,2)\) is not identically zero and satisfies the condition

\[
(\nabla X R)(Y, Z) = A(X)R(Y, Z) + B(Y)R(X, Z) + D(Z)R(Y, X),
\]

(4.1)

where \(A, B, D\) and \(\nabla\) are as before. Such an \(n\)-dimensional manifold is denoted by \((WRS)_n\). In [6] Jana and Shaikh have studied quasi-conformally flat weakly Ricci symmetric manifolds.

Proposition 4.1. In a \((WRS)_n\) with \(\sigma(X) \neq 0\) the scalar curvature can not be zero and the Ricci tensor will be of the form \(R(X, Y) = rH(X)H(Y)\) where the vector field associated with the 1-form \(H\) is a unit vector field.
Proof. From (4.1) it follows that

\[(\nabla x R)(Y, Z) - (\nabla x R)(Z, Y) = [B(Y) - D(Y)]R(X, Z) + [D(Z) - B(Z)]R(X, Y).\]

Since \(R\) is symmetric, (4.2) can be written as

\[[B(Y) - D(Y)]R(X, Z) = [B(Z) - D(Z)]R(X, Y).\]

Let \(\sigma(X) = B(X) - D(X)\) for any vector field \(X\). Then (4.3) becomes

\[\sigma(Y)R(X, Z) = \sigma(Z)R(X, Y).\]

Let \(\{e_i, i = 1, 2, \ldots, n\}\) be an orthonormal basis of the tangent space at any point of the manifold. Putting \(X = Z = e_i\) in (4.4) and then taking summation over \(i, 1 \leq i \leq n\), we get

\[r \sigma(Y) = \sigma(RY),\]

where \(\sigma(X) = g(X, \delta)\) for any vector field \(X\) and \(r\) is the scalar curvature. From (4.4) we have

\[\sigma(\delta)R(X, Z) = \sigma(Z)R(X, \delta) = \sigma(Z)\sigma(RX).\]

Using (4.5) in (4.6), we get

\[R(X, Z) = \frac{r}{\sigma(\delta)}\sigma(Z)\sigma(X) = rH(X)H(Z),\]

where \(H(X) = \frac{\sigma(X)}{\sqrt{\sigma(\delta)}}\) and \(g(X, \rho) = H(X), \rho\) is a unit vector field. Now from (4.7) it follows that if \(r = 0\), then \(R(X, Z) = 0\) which is inadmissible by the definition of \((WRS)_n\). So \(r \neq 0\).

\[\square\]

**Proposition 4.2.** In a \((WRS)_n\), with \(\sigma(X) \neq 0\), \(r\) is an eigenvalue of the Ricci tensor corresponding to the eigenvector \(\delta\).

Proof. From (4.5) it follows that \(r g(Y, \delta) = R(Y, \delta)\), which shows that \(r\) is an eigenvalue of the Ricci tensor corresponding to the eigenvector \(\delta\).

\[\square\]

**Theorem 4.3.** In a pseudo projectively flat \((WRS)_n\), \((n > 2)\) with \(\sigma(X) \neq 0, a + (n - 1)b \neq 0, a + b \neq 0\), the vector field \(\rho\) defined by \(g(X, \rho) = H(X)\) is a proper concircular vector field.

Proof. Differentiating (1.3) covariantly we have

\[\left(\nabla W \hat{P}\right)(X, Y, Z) = a(\nabla W K)(X, Y, Z) + b(\nabla W R)(X, Y, Z) - (\nabla W R)(X, Y) \cdot Y\]

\[\left(\frac{a}{n} + b\right) dr(W) [g(Y, Z) X - g(X, Z) Y].\]

Contracting above we get

\[\left(\text{div} \hat{P}\right)(X, Y, Z) = a(\text{div} K)(X, Y, Z) + b(\nabla X R)(Y, Z) - (\nabla Y R)(X, Z)\]

\[\left(\frac{a}{n} + b\right) [dr(X) g(Y, Z) - dr(Y) g(X, Z)].\]
We know that in a Riemannian manifold

\[(\nabla_X R)(Y, Z) = \nabla_Y R, (X, Z).\]

Using (4.10) in (4.9), we have

\[(\text{div}\bar P)(X, Y, Z) = \left(\frac{a}{n-1} + b\right) \left[\text{dr}(X)g(Y, Z) - \text{dr}(Y)g(X, Z)\right],\]

since \(\text{div}\bar P = 0\), so (4.11) becomes

\[(a + b)\left[\nabla_Y R, (X, Z) = \frac{1}{n} \left[\frac{a}{n-1} + b\right] \left[\text{dr}(X)g(Y, Z) - \text{dr}(Y)g(X, Z)\right].\]

Now (4.7) implies

\[(\nabla_Y R)(X, Z) = \text{dr}(Y)H(X)H(Z) + (\nabla_Y H)(X)H(Z)\]

In view (4.13), (4.12) becomes

\[(a + b)\left[\nabla_Y R, (X, Z) - \nabla_Y R, (X, Z)\right] = \left(\frac{a}{n-1} + b\right) \left[\text{dr}(X)g(Y, Z) - \text{dr}(Y)g(X, Z)\right].\]

Putting \(Y = Z = e_i\) in the above expression and then taking summation over \(i, 1 \leq i \leq n\), we get

\[(a + b)\left[\text{dr}(\rho)H(X) + r\left[\nabla_\rho H, (X) + H(X)\sum_{i=1}^n (\nabla_{e_i} H)(e_i)\right]\right] = \left(\frac{a}{n-1} + b\right) \left[\text{dr}(X)g(Y, Z) - \text{dr}(Y)g(X, Z)\right].\]

Now substituting \(Y = Z = \rho\) in (4.14) gives

\[r(a + b)(\nabla_\rho H)(X) = \left[\frac{a(n^2 - n - 1) + b(n - 1)^2}{n(n-1)}\right] \left[\text{dr}(X) - \text{dr}(\rho)H(X)\right].\]

By virtue of (4.16), (4.15) assumes the form

\[\left(\frac{a + (n - 1)b}{n(n-1)}\right) \left[\text{dr}(X) + \text{dr}(\rho)H(X)\right] + r(a + b)H(X)\sum_{i=1}^n (\nabla_{e_i} H)(e_i) = 0.\]
Now putting $X = \rho$ in (4.17), we have

\[
\left\{ \frac{a + (n - 1)b}{n} \right\} dr(\rho) = -r(a + b) \sum_{i=1}^{n} (\nabla_{e_i} H)(e_i).
\]

From (4.17) and (4.18) it follows that

\[
dr(X) = dr(\rho) H(X),
\]

if $a + (n - 1)b \neq 0$.

Again setting $Z = \rho$ in (4.14) and then using (4.19) we get

\[
r(a + b)\{(\nabla_X H)(Y) - (\nabla_Y H)(X)\} = 0,
\]

which shows that

\[
(\nabla_X H)(Y) - (\nabla_Y H)(X) = 0,
\]

if $a + b \neq 0$ (since $r \neq 0$).

By virtue of (4.19), it follows from (4.16) that

\[
(\nabla H)(X) = 0,
\]

provided that $a + b \neq 0$.

Again substituting $Y = \rho$ in (4.14) and then using (4.19) and (4.21) we get

\[
(\nabla X f)(Z) = \left\{ \frac{a + (n - 1)b}{n(n - 1)} \right\} \left\{ \frac{dr(\rho)}{r(a + b)} \right\} [H(X)H(Z) - g(X, Z)].
\]

Now we consider the scalar function

\[
f = \left\{ \frac{a + (n - 1)b}{n(n - 1)} \right\} \left\{ \frac{dr(\rho)}{r(a + b)} \right\},
\]

then we have

\[
(\nabla X f) = -\left\{ \frac{a + (n - 1)b}{n(n - 1)} \right\} \left\{ \frac{dr(\rho)}{r(a + b)} \right\} dr(X)
\]

\[
+ \left\{ \frac{a + (n - 1)b}{n(n - 1)} \right\} \left\{ \frac{1}{r(a + b)} \right\} d^2r(\rho, X).
\]

By virtue of (4.19) we get

\[
d^2r(X, Y) = d^2r(\rho, Y) H(X) + dr(\rho)(\nabla_Y H)(X).
\]

Now in a Riemannian manifold the second covariant differentiation of any function $h \in C^\infty(M)$ is defined by

\[
d^2h(X, Y) = X(Y h) - (\nabla_X Y)h,
\]

for all $X, Y \in \chi(M)$, which shows that

\[
d^2h(X, Y) = d^2h(Y, X),
\]
for all \( X, Y \in \chi(M) \).
And hence by virtue of (2.24) and (2.20), we have
\[
d^2r(\rho,Y)H(X) = d^2r(\rho,X)H(Y).
\]
Putting \( Y = \rho \) in above, we get
\[
d^2r(\rho,X) = d^2r(\rho,\rho)H(X) = \phi H(X),
\]
where \( \phi = d^2r(\rho,\rho) \) is a scalar function.
Now in consequence of (4.26) and (4.19), (4.23) assumes the form
\[
\nabla_X f = \nu H(X),
\]
where
\[
\nu = \left\{ \frac{a + (n - 1)b}{n(n - 1)} \right\} \left\{ \frac{1}{r^2(a + b)} \right\} \left[ r\phi - (d^2r(\rho))^2 \right].
\]
Now we consider a 1-form \( \alpha \) given by
\[
\alpha(X) = \left\{ \frac{a + (n - 1)b}{n(n - 1)} \right\} \left\{ \frac{dr(\rho)}{r(a + b)} \right\} H(X) = fH(X).
\]
In view of (4.28), (4.27) and (4.20) we have
\[
d\alpha(X,Y) = 0,
\]
i.e. the 1-form \( \alpha \) is closed. So (4.22) can be written as follows
\[
(\nabla_X H)(Y) = \alpha(X)H(Y) - fg(X,Y),
\]
where \( \alpha \) is closed. But this means that the vector field \( \rho \) corresponding to the 1-form \( H \) defined by \( g(X,\rho) = H(X) \) is a proper concircular vector field [12].

**Theorem 4.4.** A pseudo projectively flat \((WRS)_{\rho,n} \) \((n > 2)\) with \( \sigma(X) \neq 0 \), \( a + (n - 1)b \neq 0 \), \( a + b = 0 \) is of constant scalar curvature and Ricci tensor is symmetric along the direction of the unit vector field \( \rho \) defined by \( g(X,\rho) = H(X) \) for all \( X \).

**Proof.** Suppose \( a + b = 0 \) and \( a + (n - 1)b \neq 0 \), then (4.12) becomes
\[
dr(X)g(Y,Z) - dr(Y)g(X,Z) = 0,
\]
which gives
\[
dr(X) = 0,
\]
for all \( X \). Now setting \( Y = Z = e_i \) in (4.13) and then taking summation over \( i, 1 \leq i \leq n \), we get by virtue of (4.32) that
\[
(\nabla_{\rho}H)(X) + H(X)\sum_{i=1}^{n} (\nabla_{e_i}H)(e_i) = 0.
\]
Putting \( X = \rho \) in above we get
\[
\sum_{i=1}^{n} (\nabla_{e_i} H)(e_i) = 0.
\]

Using above relation in (4.33) we obtain
(4.34)
\[
(\nabla_{\rho} H)(X) = 0.
\]

Again setting \( Y = \rho \) in (4.13) and then using (4.34) and (4.32) we get
\[
(\nabla_{\rho} R)(X, Z) = 0,
\]
for all \( X, Z \). The proof is complete. \( \square \)

**Theorem 4.5.** In a pseudo projectively flat \((\text{WRS})_n (n > 2)\) with \( \sigma(X) \neq 0 \) and \( a + b \neq 0 \) the vector field \( \rho \) defined by \( g(X, \rho) = H(X) \) is a unit parallel vector field.

**Proof.** Suppose \( a + b \neq 0 \) and \( a + (n - 1)b = 0 \) then (4.12) becomes
(4.35)
\[
(\nabla_X R)(Y, Z) = (\nabla_Y R)(X, Z),
\]
for all \( X, Y \) and \( Z \), which gives
\[
d\epsilon(X) = 0,
\]
for all \( X \).

Hence (4.14) assumes the form
(4.36)
\[
\{(\nabla_X H)(Y)H(Z) + (\nabla_X H)(Z)H(Y)
\]
\[
- (\nabla_Y H)(X)H(Z) - (\nabla_Y H)(Z)H(X)\} = 0.
\]

Putting \( Y = Z = \rho \) in above, we get
(4.37)
\[
(\nabla_{\rho} H)(X) = 0,
\]
for all \( X \). Again putting \( Y = \rho \) in (4.36) and using (4.37) we obtain \((\nabla_X H)(Z) = 0\), for all \( X, Z \); this implies \( g(Z, \nabla_X \rho) = 0 \), for all \( X, Z \). Since \( g \) is non-degenerate, it follows that \( \nabla_X \rho = 0 \), for all \( X \), which shows the result. \( \square \)

**Remark 4.6.** In a pseudo projectively flat \((\text{WRS})_n (n > 2)\) the case \( a + (n - 1)b = 0 \) and \( a + b = 0 \) can not occur simultaneously. Suppose if possible they occur simultaneously then we have \( a = -b \) and \( a = (1 - n)b \) which gives \( n = 2 \), which is contradiction the fact that \( n > 2 \).

**Theorem 4.7.** A pseudo projectively flat \((\text{WRS})_n (n > 2)\) is a quasi-Einstein manifold.

**Proof.** Since the manifold is pseudo projectively flat \((\text{WRS})_n \) then from (1.3)
\[
^{t}K(X, Y, Z, W) = - \frac{b}{a} [R(Y, Z)g(X, W) - R(X, Z)g(Y, W)]
\]
\[
+ \frac{r}{an} \left[ \frac{a}{n-1} + b \right] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].
\]
Now using (4.7) in above equation we get
\[ 'K(X, Y, Z, W) = p [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + q [g(X, W)H(Y)H(Z) - g(Y, W)H(X)H(Z)], \]
where \( p = \frac{r}{n} \left[ \frac{1}{n - 1} + \frac{b}{a} \right] \), \( q = -\frac{br}{a} \). Now substituting \( X = W = e_i \) and then taking summation over \( i, 1 \leq i \leq n \), we get
\[ R(Y, Z) = p' g(Y, Z) + q' H(Y)H(Z), \]
where \( p' = (n - 1)p \), \( q' = (n - 1)q \). Hence the manifold is quasi-Einstein [2]. □

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References


Authors’ address:
J. P. Jaiswal and R. H. Ojha
Department of Mathematics, Faculty of Science,
Banaras Hindu University Varanasi, U.P. 221005, India.
E-mail: jaipjai_m@rediffmail.com, rh_ojha@rediffmail.com