On the $m$-projective curvature tensor of a Kenmotsu manifold

S.K. Chaubey and R.H. Ojha

**Abstract.** In this paper, we study the properties of the $M$–projective curvature tensor in Riemannian and Kenmotsu manifolds.


**Key words**: Riemannian manifold; Kenmotsu manifold; $M$–projective curvature tensor; $\eta$–Einstein manifold and irrotational $M$–projective curvature tensor.

1 Introduction

Let $M_n$ be an $n$–dimensional differentiable manifold of differentiability class $C^{r+1}$ with a (1,1) tensor field $\phi$, the associated vector field $\xi$, a contact form $\eta$ and the associated Riemannian metric $g$. In 1958, Boothby and Wong [1] studied odd dimensional manifolds with contact and almost contact structures from topological point of view. Sasaki and Hatakeyama [14] re-investigated them using tensor calculus in 1961. In 1972, K. Kenmotsu studied a class of contact Riemannian manifold and call them Kenmotsu manifold [7]. He proved that if a Kenmotsu manifold satisfies the condition $R(X,Y).R = 0$, then the manifold is of negative curvature $-1$, where $R$ is the Riemannian curvature tensor of type $(1,3)$ and $R(X,Y)$ denotes the derivation of the tensor algebra at each point of the tangent space. In 1963, Kobayashi and Nomizu [8] shown that any two simply connected complete Riemannian manifolds of constant curvature $k$ are isometric to each other. A space form (i.e., a complete simply connected Riemannian manifold of constant curvature) is said to be elliptic, hyperbolic or euclidean according as the sectional curvature tensor is positive, negative or zero [2]. The properties of Kenmotsu manifold have been studied by several authors such as De [3], Sinha and Srivastava [15], Jun, De and Pathak [6], De and Pathak [4], De, Yildiz and Yaliniz [5], Çiğhan and De [12] and many others.

In this paper, we studied the properties of the Kenmotsu manifold equipped with $M$–projective curvature tensor. Section 2 consist the basic definitions of the Kenmotsu and $\eta$–Einstein manifolds. Section 3 is the study of $M$–projective curvature tensor in the Riemannian manifold and obtain the relation between different curvature tensors. In section 4, we prove that an $n$–dimensional Kenmotsu manifold $M_n$ is $M$–projectively flat if and only if it is either locally isometric to the hyperbolic space $H^n(-1)$ or $M_n$ has constant scalar curvature $-n(n-1)$. Section 5 deals
with \( W^* (\xi, X) \cdot R = 0 \) and obtain some interesting results. In section 6, we proved that the \( M \)-projective curvature tensor in an \( \eta \)-Einstein Kenmotsu manifold \( M_n \) is irrotational if and only if it is locally isometric to the hyperbolic space \( H^n (-1) \).

## 2 Preliminaries

If on an odd dimensional differentiable manifold \( M_n \), \( n = 2m + 1 \), of differentiability class \( C^{r+1} \), there exist a vector valued real linear function \( \phi \), a 1-form \( \eta \), the associated vector field \( \xi \) and the Riemannian metric \( g \) satisfying

\[
\phi^2 X = -X + \eta(X) \xi, \tag{2.1}
\]

\[
\eta(\phi X) = 0, \tag{2.2}
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X) \eta(Y), \tag{2.3}
\]

for arbitrary vector fields \( X \) and \( Y \), then \( (M_n, g) \) is said to be an almost contact metric manifold and the structure \( \{\phi, \eta, \xi, g\} \) is called an almost contact metric structure to \( M_n \) [9].

In view of (2.1), (2.2) and (2.3), we find

\[
\eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \quad \phi(\xi) = 0. \tag{2.4}
\]

If moreover,

\[
(D_X \phi)(Y) = -g(X, \phi Y)\xi - \eta(Y) \phi X, \tag{2.5}
\]

and

\[
D_X \xi = X - \eta(X) \xi, \tag{2.6}
\]

where \( D \) denotes the operator of covariant differentiation with respect to the Riemannian metric \( g \), then \( (M_n, \phi, \xi, \eta, g) \) is called a Kenmotsu manifold [7]. Also, the following relations hold in Kenmotsu manifold [5], [4], [6]

\[
R(X, Y) \xi = \eta(X) Y - \eta(Y) X, \tag{2.7}
\]

\[
R(\xi, X) Y = \eta(Y) X - g(X, Y) \xi, \tag{2.8}
\]

\[
S(X, \xi) = -(n - 1) \eta(X), \tag{2.9}
\]

\[
\eta(R(X, Y) Z) = \eta(Y) g(X, Z) - \eta(X) g(Y, Z), \tag{2.10}
\]

for arbitrary vector fields \( X, Y, Z \).

A Kenmotsu manifold \( (M_n, g) \) is said to be \( \eta \)-Einstein if its Ricci-tensor \( S \) takes the form

\[
S(X, Y) = ag(X, Y) + b\eta(X) \eta(Y) \tag{2.11}
\]
for arbitrary vector fields $X$, $Y$; where $a$ and $b$ are functions on $(M, g)$. If $b = 0$, then $\eta$–Einstein manifold becomes Einstein manifold. Kenmotsu [7] proved that if $(M, g)$ is an $\eta$–Einstein manifold, then $a + b = -(n - 1)$.

In view of (2.4) and (2.11), we have

$$QX = aX + b\eta(X)\xi,$$

where $Q$ is the Ricci operator defined by

$$S(X, Y) \overset{\text{def}}{=} g(QX, Y).$$

Again, contracting (2.12) with respect to $X$ and using (2.4), we have

$$r = na + b.$$  

Now, substituting $X = \xi$ and $Y = \xi$ in (2.11) and then using (2.4) and (2.9), we obtain

$$a + b = -(n - 1).$$

Equations (2.14) and (2.15) gives

$$a = \left(\frac{r}{n-1} + 1\right) \quad \text{and} \quad b = -\left(\frac{r}{n-1} + n\right).$$

3 The $M$-projective curvature tensor

In 1971, G. P. Pokhariyal and R. S. Mishra [13] defined a tensor field $W^*$ on a Riemannian manifold as

$$W^*(X, Y)Z = R(X, Y)Z - \frac{1}{2(\alpha - 1)}[S(Y, Z)X - S(X, Z)Y]$$

so that

$$'W^*(X, Y, Z, U) \overset{\text{def}}{=} g(W^*(X, Y)Z, U) = 'W^*(Z, U, X, Y)$$

and

$$'W_{ijkl}^* w_{ij}^k w^k = 'W_{ijkl} w_{ij}^k w^k,$$

where $'W_{ijkl}^*$ and $'W_{ijkl}$ are components of $'W^*$ and $'W$ respectively and $w_{kl}$ is a skew-symmetric tensor [10], [16]. Such a tensor field $W^*$ is known as $M$–projective curvature tensor. Second author [11], [10] defined and studied the properties of $M$–projective curvature tensor in Sasakian and Kähler manifolds. He has also shown that it bridges the gap between conformal curvature tensor, con-harmonic curvature tensor and con-circular curvature tensor on one side and $H$–projective curvature tensor on the other.

The Weyl projective curvature tensor $W$, con-circular curvature tensor $C$ and conformal curvature tensor $V$ are given by [9]

$$W(X, Y)Z = R(X, Y)Z - \frac{1}{n-1} \{S(Y, Z)X - S(X, Z)Y\}.$$
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\begin{equation}
C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)} \{ g(Y,Z)X - g(X,Z)Y \}
\end{equation}

\begin{equation}
V(X,Y)Z = R(X,Y)Z - \frac{1}{n-2} [ S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY ] + \frac{r}{(n-1)(n-2)} \{ g(Y,Z)X - g(X,Z)Y \}
\end{equation}

**Theorem 3.1.** The $M$-projective and Weyl projective curvature tensors of the Riemannian manifold $M_n$ are linearly dependent if and only if $M_n$ is an Einstein manifold.

**Proof.** We consider,

$$W^*(X,Y)Z = \alpha W(X,Y)Z,$$

where $\alpha$ being any non-zero constant. In view of (3.1) and (3.2), above equation becomes

\begin{align*}
(1 - \alpha)R(X,Y)Z &+ \left( \frac{\alpha}{n-1} - \frac{1}{2(n-1)} \right) \{ S(Y,Z)X - S(X,Z)Y \} \\
&- \frac{1}{2(n-1)} \{ g(Y,Z)QX - g(X,Z)QY \} = 0
\end{align*}

Contracting last result with respect to $X$, we get

\begin{equation}
S(Y,Z) = \frac{r}{n} g(Y,Z) \iff QY = \frac{r}{n} Y,
\end{equation}

which gives the first part of the theorem. In consequence of (3.1), (3.2) and (3.5), we obtain the converse part of the theorem. \qed

**Theorem 3.2.** The necessary and sufficient condition for a Riemannian manifold to be an Einstein manifold is that the $M$-projective curvature tensor $W^*$ and concircular curvature tensor $C$ are linearly dependent.

**Proof.** Let

$$W^*(X,Y,Z) = \alpha C(X,Y,Z),$$

where $\alpha$ is a non-zero scalar. In consequence of (3.1) and (3.3), above equation becomes

\begin{align*}
(1 - \alpha)R(X,Y,Z) &- \frac{1}{2(n-1)} [ S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY ] \\
&\quad - g(X,Z)QY + \frac{\alpha r}{n(n-1)} [ g(Y,Z)X - g(X,Z)Y ] = 0.
\end{align*}

Contracting above with respect to $X$, we obtain (3.5). Converse part is obvious from (3.1), (3.3) and (3.5). \qed

**Theorem 3.3.** A Riemannian manifold becomes an Einstein manifold if and only if conformal and $M$-projective curvature tensors of the manifold are linearly dependent.

The proof is straightforward as theorem (3.4).

**Corollary 3.4.** In an $n$-dimensional Riemannian manifold $M_n$, the following are equivalent

(i) $M_n$ is an Einstein manifold,
(ii) $M$-projective and Weyl projective curvature tensors are linearly dependent.
(iii) $M$-projective and Con-circular curvature tensors are linearly dependent.
(iv) $M$-projective curvature and conformal curvature tensors are linearly dependent.
4 \(M\)-projectively flat Kenmotsu manifolds

In view of \(W^* = 0\), (3.1) becomes

\[
(4.1) \quad R(X,Y)Z = \frac{1}{2(n-1)} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].
\]

Replacing \(Z\) by \(\xi\) in (4.1) and then using (2.4), (2.7) and (2.9), we obtain

\[
(n-1)(\eta(X)Y - \eta(Y)X) = \eta(Y)QX - \eta(X)QY.
\]

Again putting \(Y = \xi\) in the above relation and using (2.4) and (2.9), we have

\[
(4.2) \quad QX = -(n-1)X \iff S(X,Y) = -(n-1)g(X,Y)
\]

and

\[
(4.3) \quad r = -n(n-1).
\]

In consequence of (4.2), (4.1) becomes

\[
(4.4) \quad R(X,Y)Z = -\{g(Y,Z)X - g(X,Z)Y\}.
\]

A space form is said to be hyperbolic if and only if the sectional curvature tensor is negative [2]. Thus, we can state

**Theorem 4.1.** An \(n\)-dimensional Kenmotsu manifold \(M_n\) is \(M\)-projectively flat if and only if it is either locally isometric to the hyperbolic space \(H^n(-1)\) or \(M_n\) has constant scalar curvature \(-n(n-1)\).

5 An \(\eta\)-Einstein Kenmotsu manifold satisfying a certain condition

In view of (2.4), (2.8), (2.11), (2.12) and (2.16), (3.1) becomes

\[
(5.1) \quad W^*(\xi,X)Y = \frac{1}{2} \left\{ 1 + \frac{1}{n-1} \left( \frac{r}{n-1} + 1 \right) \right\} \{\eta(Y)X - g(X,Y)\xi\}.
\]

Now, we have

\[
\]

\[
- R(Y,W^*(\xi,X)Z)U - R(Y,Z)W^*(\xi,X)U.
\]

In consequence of \(W^*(\xi,X).R = 0\), (5.2) becomes

\[
W^*(\xi,X)R(Y,Z)U - R(W^*(\xi,X)Y,Z)U
\]

\[
- R(Y,W^*(\xi,X)Z)U - R(Y,Z)W^*(\xi,X)U = 0.
\]
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In view of (2.4), (2.7), (2.8), (2.10) and (5.1), last result becomes

\[
\frac{1}{2} \left\{ 1 + \frac{1}{n-1} \left( \frac{r}{n-1} + 1 \right) \right\} \left[ \eta(R(Y, Z)U)X - 'R(Y, Z, U, X)\xi \right.
\]
\[- \eta(Y)R(X, Z)U + g(X, Y)R(\xi, Z)U - \eta(Z)R(Y, X)U
\]
\[+ \ g(X, Z)R(Y, \xi)U - \eta(U)R(Y, Z)X + g(X, U)R(Y, Z)\xi \right]\ = 0,
\]

where

(5.2) \quad 'R(X, Y, Z, U) \overset{\text{def}}{=} g(R(X, Y)Z, U).

Taking inner-product of above with respect to the Riemannian metric \(g\) and then using (2.4) and (2.10), we have

\[
\frac{1}{2} \left\{ 1 + \frac{1}{n-1} \left( \frac{r}{n-1} + 1 \right) \right\} \left[ -'R(Y, Z, U, X) - g(X, Y)g(Z, U) + g(X, Z)g(Y, U) \right] = 0
\]

\[\implies \ 'R(Y, Z, U, X) = g(X, Z)g(Y, U) - g(X, Y)g(Z, U).
\]

Using (2.4) and (5.2) in the above equation, we obtain

(5.3) \quad R(Y, Z)U = g(Y, U)Z - g(Z, U)Y.

Contracting (5.3) with respect to the vector field \(Y\), we find

(5.4) \quad S(Z, U) = -(n-1)g(Z, U),

which gives

(5.5) \quad QZ = -(n-1)Z

and

(5.6) \quad r = -n(n-1).

In view of (2.4), (5.1), (5.3), (5.4) and (5.5), (5.2) gives \(W^* (\xi, X)R = 0\). Thus, consequently we state

**Theorem 5.1.** An \(n\)–dimensional \(\eta\)–Einstein Kenmotsu manifold \(M_n\) satisfies the condition \(W^* (\xi, X)R = 0\) if and only if either \(M_n\) is locally isometric to the hyperbolic space \(H^n(-1)\) or \(M_n\) has constant scalar curvature tensor \(-n(n-1)\).

In the light of Corollary (3.4) and theorem (4.1), theorem (5.1) state

**Corollary 5.2.** An \(n\)–dimensional \(\eta\)–Einstein Kenmotsu manifold \(M_n\) satisfies \(W^* (\xi, X)R = 0\) if and only if it is Conformally flat.
6 The irrotational $M$–projective curvature tensor

**Definition 6.1.** - Let $D$ be a Riemannian connection, then the rotation (Curl) of $M$–projective curvature tensor $W^*$ on a Riemannian manifold $M_n$ is defined as

$$\text{Rot}W^* = (D_UW^*)(X,Y)Z + (D_XW^*)(U,Y)Z + (D_YW^*)(X,U)Z - (D_ZW^*)(X,Y)U.$$  

(6.1)

In consequence of Bianchi’s second identity for Riemannian connection $D$, (6.1) becomes

$$\text{Rot}W^* = -(D_ZW^*)(X,Y)U.$$  

(6.2)

If the $M$–projective curvature tensor is irrotational, then curl $W^*$=0 and therefore

$$(D_ZW^*)(X,Y)U = 0,$$  

which gives


Replacing $U = \xi$ in (6.3), we have

$$(6.4) \quad D_Z(W^*(X,Y)\xi) = W^*(D_ZX,Y)\xi + W^*(X,D_ZY)\xi + W^*(X,Y)D_Z\xi.$$  

Now, substituting $Z = \xi$ in (3.1) and using (2.4), (2.7) and (2.9), we obtain

$$(6.5) \quad W^*(X,Y)\xi = k[\eta(X)Y - \eta(Y)X],$$  

where

$$(6.6) \quad k = \frac{1}{2} \left\{ 1 + \frac{1}{n-1} \left( \frac{r}{n-1} + 1 \right) \right\}.$$  

Using (6.5) in (6.4), we obtain

$$(6.7) \quad W^*(X,Y)Z = k[g(X,Z)Y - g(Y,Z)X].$$  

Also equations (3.1) and (6.7) gives

$$k[g(X,Z)Y - g(Y,Z)X] = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].$$  

Contracting above equation with respect to the vector $X$ and then using (6.6), we find

$$(6.8) \quad S(Y,Z) = -(n-1)g(Y,Z) \iff QY = -(n-1)Y,$$  

which gives

$$(6.9) \quad r = -n(n-1).$$  

In consequence of (3.1), (6.6), (6.7), (6.8) and (6.9), we can find

$$(6.10) \quad R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y].$$

Thus, we can state
Theorem 6.2. The $M$–projective curvature tensor in an \(\eta\)–Einstein Kenmotsu manifold \(M_n\) is irrotational if and only if it is locally isometric to the hyperbolic space \(H^n(−1)\).

Theorem 5.1 together with Theorem 6.2 lead to

Corollary 6.3. An \(n\)–dimensional \(\eta\)–Einstein Kenmotsu manifold \(M_n\) satisfies \(W^*(\xi, X).R = 0\) if and only if the \(M\)–projective curvature tensor is irrotational.

Corollary 6.4. The \(M\)–projective curvature tensor in an \(\eta\)–Einstein Kenmotsu manifold \(M_n\) is irrotational if and only if the manifold is conformally flat.

References

Authors’ address:

S.K. Chaubey and R.H. Ojha  
Department of Mathematics, Faculty of Science,  
Banaras Hindu University-221005, India.  
E-mail: sk22.math@yahoo.co.in ; rh_ojha@rediffmail.com