Conformal gradient vector fields

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Abstract. In this paper we prove that a compact Riemannian manifold with positive constant scalar curvature is isometric to a sphere provided that it admits a nonzero conformal gradient vector field.

Key words: Scalar curvature; conformal vector field; conformal gradient vector field; isometry to a sphere.

1 Introduction

Spheres have many interesting geometrical properties among the class of compact connected Riemannian manifolds. That is why, it is an important issue to classify spheres (cf. [1], [2], [4], [5], [6]). An interesting property is the existence of nonconstant functions \( f \) on \( S^n(c) \) which satisfies \( \nabla_X \text{grad } f = -cfX \), where \( \text{grad } f \) is the gradient of \( f \) and \( \nabla_X \) is the covariant derivative operator with respect to the smooth vector \( X \).

Obata showed that a complete connected Riemannian manifold that admits a non constant solution of this differential equation is necessarily isometric to \( S^n(c) \) (cf. [5]). Deshmukh and Alsalamy [3] gave an answer for the question: "under what conditions does an \( n \)-dimensional compact and connected Riemannian manifold that admits a nonzero conformal gradient vector field has to be isometric to a sphere \( S^n(c) \)?", by giving certain bounds for the Ricci curvature which involves the first nonzero eigenvalue of the Laplacian operator on \( M \). In this paper we will provide an answer to this question by restricting the scalar curvature to be positive constant as follows:

**Theorem.** Let \((M, g)\) be an \( n \)-dimensional compact connected Riemannian manifold of positive constant scalar curvature \( n(n-1)c \). If \( M \) admits a non zero conformal gradient vector field, then \( M \) is isometric to the \( n \)-sphere \( S^n(c) \).

2 Preliminaries

Let \((M, g)\) be a Riemannian manifold with Lie algebra \( \mathfrak{X}(M) \) of smooth vector fields on \( M \). A vector field \( X \in \mathfrak{X}(M) \) is said to be conformal if it satisfies

\[
\mathcal{L}_X g = 2\varphi g
\]
for a smooth function $\varphi : M \to R$, where $\mathcal{L}_X$ is the Lie derivative with respect to $X$. If $u = \text{grad } f$ is the gradient of a smooth function $f$ on $M$ and $u$ is a conformal vector field, then it follows from (2.1) that a conformal vector field $u$ satisfies

$$\nabla_X u = \varphi X, \quad X \in \mathfrak{X}(M)$$

For a sphere $S^n(c)$, there exists a non constant function $\varphi \in C^\infty(S^n(c))$ which satisfies

$$\nabla_X \nabla \varphi = -c \varphi X$$

where $\nabla \varphi$ is the gradient of $\varphi$ and $\nabla_X$ is the covariant derivative operator with respect to the smooth vector $X$.

The following result is an immediate consequence of the equation (2.2):

**Lemma 2.1** Let $u$ be a conformal gradient vector field on a compact Riemannian manifold $(M, g)$. Then, for $\varphi = n^{-1} \text{div } u$,

$$\int_M \varphi dv = 0.$$ 

For a smooth function $f$ on $M$, define an operator $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $AX = \nabla_X \nabla f$, where $\nabla f$ is the gradient of $f$. The Ricci operator $Q$ is a symmetric operator defined by

$$\text{Ric}(X, Y) = g(Q(X), Y), \quad X, Y \in \mathfrak{X}(M)$$

where $\text{Ric}$ is the Ricci tensor of the Riemannian manifold, and hence from the definition of the operator $A$ we have the following relation

$$R(X, Y)\nabla f = (\nabla A)(X, Y) - (\nabla A)(Y, X)$$

where $(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$. Note that using (2.4) we have the following lemma which gives an important property of the operator $A$.

**Lemma 2.2** Let $(M, g)$ be Riemannian manifold and $f$ be a smooth function on $M$. Then the operator $A$ corresponding to the function $f$ satisfies

$$\sum_{i=1}^{n} (\nabla A) (e_i, e_i) = \nabla (\Delta f) + Q(\nabla f)$$

where $\{e_1, e_2, \ldots, e_n\}$ be a local orthonormal frame on $M$.

**Lemma 2.3** Let $u$ be a conformal gradient vector field on an $n$-dimensional Riemannian manifold $(M, g)$. Then the operator $Q$ satisfies

$$Q(u) = -(n - 1) \nabla \varphi,$$

where $\nabla \varphi$ is the gradient of the smooth function $\varphi = n^{-1} \text{div } u$. 
3 Proof of the Theorem

For an $n$-dimensional compact connected Riemannian manifold $(M, g)$ of positive constant scalar curvature $S = n(n - 1)c$, we have that $\Delta \varphi = -nc\varphi$ and from Lemma 2.3 that $\Delta f = nf$. These two relations imply $\Delta \varphi = -c\Delta f$ that is $\Delta(\varphi + cf) = 0$. Thus $\varphi = -cf + \alpha$, where $\alpha$ is a constant. Consequently $\nabla \varphi = -c\nabla f$, which gives $\nabla_X \nabla \varphi = -c\nabla_X \nabla f = -c\varphi_X$ that is $\varphi$ satisfies the Obata’s differential equation. We claim that $\varphi$ is not constant. If $\varphi$ is a constant, it will imply that $f$ is a constant which in turn will imply that $u = 0$ and that leads to a contradiction as the statement of the Theorem requires that $u$ is nonzero vector field. Hence by Obata’s theorem we get that $M$ is isometric to the $n$-sphere $S^n(c)$.

Remark. The compactness condition in the theorem essential, as for the the Riemannian manifold $(\mathbb{R}^n, g)$ where $g$ is the Riemannian metric defined by $g = \frac{1}{1 + \|x\|^2} \langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the Euclidean metric on $\mathbb{R}^n$, choosing $u$ to be the position vector field, the scalar curvature $S$ of $(\mathbb{R}^n, g)$ is a positive constant but the manifold is not isometric to a sphere $S^n(c)$.

References


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