A short history of Convexity

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Abstract. The main goal of this short note is to show the importance of the notion of convexity and how it evolved over time: from relatively simple geometry to advanced applications in many areas of mathematics.

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1 Introduction

Convexity is a basic notion in geometry but also is widely used in other areas of mathematics. It is often hidden in other areas of mathematics: functional analysis, complex analysis, calculus of variations, graph theory, partial differential equations, discrete mathematics, algebraic geometry, probability theory, coding theory, crystallography and many other fields. Convexity plays an important role also in areas outside mathematics, such as physics, chemistry, biology and other sciences, but it is beyond the scope of this note to consider these applications.

It seems that the notion of convexity is underestimated in general. The purpose of this note is to show its great importance in mathematics and to encourage the reader to study this concept or apply it in her or his work. Maybe this paper will be helpful to serve as a starting point of elementary talks for undergraduate or graduate students. Actually each section of this note can be easily expanded into a longer paper or even a book. Already we have examples: Sections 7 and 8 into the book by Hörmander [13]; Section 9 into the books by Oda [18], Ewald [5] or paper [4].

More modern approach to convexity problems can be found in papers of the strong group of Balkan geometers, e.g., recent papers by Udrişte and Balan [23], Udrişte and Oprea [24], Pinheiro [19], Dogaru, Udrişte and Stamin [2], [3], and in the well-known book by C. Udrişte [22].

We give a short description of the history of convexity and how it influenced various fields of modern mathematics. Convexity was already considered by Greek philosophers, and probably can be traced to ancient Egypt and Babylon. Presumably this notion is not as old as that of numbers, but drawing basic geometric figures, like circles or triangles, goes back to the beginning of human civilization.

Many sources have been used in preparation of this note. Especially we should stress the paper by W. Fenchel [6], *Convexity through the ages*, which appeared in the proceedings *"Convexity and its Applications"* [6] (actually the original article

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appeared in Dutch in 1973), the book by Lars Hörmander [13], and the *Handbook of Convex Geometry*, vol. A and B, edited by P.M. Gruber and J.M. Willis [7].

2 Early geometry and Archimedes' definition of convexity

One of the most important contributions of the Greeks was their development of Geometry, culminating in Euclid's *"Elements"* (Euclid of Alexandria, ca 325 - ca 270 B.C.), a giant work containing thirteen books. Eight books are devoted to geometry, the other five to number theory but also with essential mixture of geometry and algebra. All the known geometric theorems at that time (about 300 B.C.) are presented in a logical fashion. Notice that the word *"geometry"* is made up of "geo", meaning the *earth*, and "metry" meaning *measurement of*, in Greek. The beginning of geometry can be traced back to Egyptian mathematics dating approximately 2000 B.C. and Babylonians at approximately the same time.

It is practically impossible to say who considered the notion of convexity first. For instance, triangles appeared already in the ancient Egypt and Babylon, in particular the right triangle and the Pythagorean theorem were known about a thousand years before Pythagoras (Pythagoras of Samos, ca 569 - ca 475 B.C.), when some clay tablets discovered contain lists of triplets of numbers, starting with (3, 4, 5), (5, 12, 13).

Five Platonic Solids. It was known to the ancient Greeks that there are only five regular convex polyhedra. Each regular polyhedron is made of congruent regular polygons. These five regular convex polyhedra are called *The five Platonic solids* (Plato, 427 - 347 B.C.) because Plato mentioned them in the *Timaeos*, but they were already known before, even in prehistoric times (see [9]). These solids are:

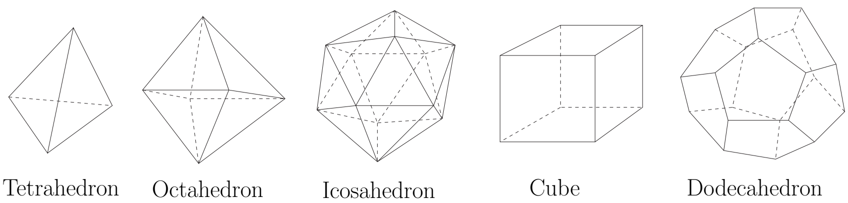

Figure 1: Five Platonic solids

made of triangles	tetrahedron	(4 faces)
	octahedron	(8 faces)
	icosahedron	(20 faces)
made of squares	cube	(6 faces)
made of pentagons	dodecahedron	(12 faces)

Archimedes' definition of convexity. It seems that the first more rigorous definition of convexity was given by Archimedes (Archimedes of Syracuse (Sicily), ca 287 - ca 212 B.C.) in his treatise *On the sphere and cylinder* [8] (p. 2). Here are his definitions.

Definition 2.1. (Archimedes) There are in a plane certain terminated bent lines, which either lie wholly on the same side of the straight lines joining their extremities, or have not part of them on the other side (see Fig. 2).

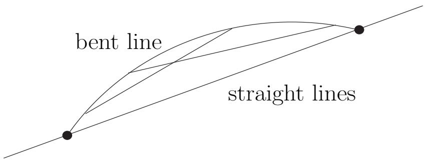

Figure 2: Archimedes' convexity

Definition 2.2. (Archimedes) I apply the term *concave in the same direction* to a line such that, if any two points on it are taken, either all the straight lines connecting the points fall on the same side of the line, or some fall on one and the same side while others fall on the line itself, but none on the other side (see Fig. 3).

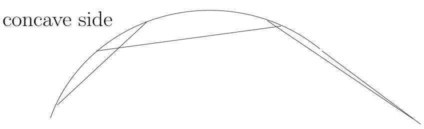

Figure 3: Archimedes' concavity

Analogously he defines a *convex surface* bounded by a plane curve. In the same treatise [8] (p. 4), Archimedes postulated the following property about the length of a convex arc, which we formulate in a more modern language:

Postulate of Archimedes. *If one of two convex arcs with common endpoints lies between the other one and the line joining the endpoints, the length of the first arc is smaller than that of the second one, see Fig. 4.*

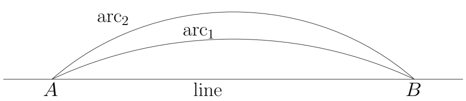

Figure 4: Length of convex arcs

These definitions and postulates of Archimedes were dormant for about two thousand years! Of course they were known to mathematicians, especially 17th century mathematicians. However, at that time calculus started to emerge and convexity problems were not of the first priority. Postulates of Archimedes required rigorous proofs. Some of them follow from the work of Cauchy (Augustin Louis Cauchy, 1789 - 1857). In 1841, Cauchy discovered the following interesting properties.

Theorem 2.1. (Cauchy, curves) *The perimeter of a closed convex curve equals π times the mean value of the lengths of the orthogonal projections of the curve onto the lines through a point (see Fig. 5).*

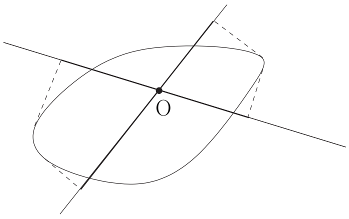

Figure 5: Perimeter of a curve and orthogonal projections

Theorem 2.2. (Cauchy, surfaces) *The surface area of a closed convex surface equals* 4 *times the mean value of the areas of the orthogonal projections of the surface onto the planes through a point.*

As an easy consequence of the first property we get the following corollary, which Cauchy explicitly mentions only in 1850.

Corollary 2.1. (Cauchy) *If a closed convex curve is contained in a circle, then its perimeter is smaller than that of the circle.*

For the circle of radius r, Theorem 2.1 is very easy to check. Namely the orthogonal projection of the circle on any line is $2r$. Take the lines passing through the center, and then the mean is obviously $2r$ as seen in Fig. 6. Cauchy's theorem says that the perimeter of the circle is $\pi \cdot 2r$, which agrees with the circumference formula. For the

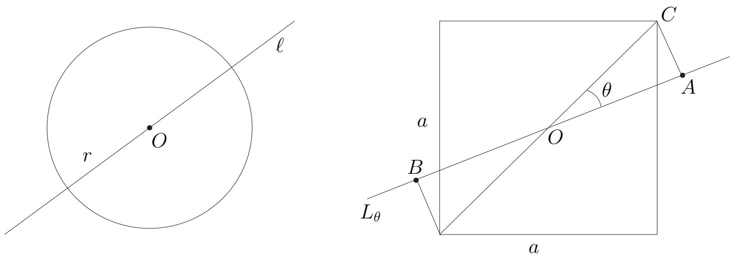

Figure 6: Circumference of the circle and square

square, we calculate

$$AB = a\sqrt{2}\cos\theta, \quad -\tfrac{\pi}{4} \leq \theta \leq \tfrac{\pi}{4}\,,$$

$$\text{mean} = \frac{1}{2\pi} \cdot 4 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a\sqrt{2}\cos\theta\, d\theta = \frac{4a}{\pi},$$

and Cauchy's theorem implies that the perimeter is $\frac{4a}{\pi} \cdot \pi = 4a$.

3 Euler (or Descartes'?) formula: $V - E + F = 2$

It is credited to Euclid that

Theorem 3.1. (Euclid) *The sum of the face angles at any vertex of a convex polyhedron P is less than 2π.*

Descartes' angular defect. The difference between this sum and 2π is called the *angular defect* at that vertex. If we sum the angular defects over all the vertices of P we obtain the *total angular defect* Δ of the polyhedron. Descartes (René Descartes, 1596 - 1650) proved that the *total angular defect* Δ *for any convex polyhedron is equal to* 4π. For example, there are 8 identical vertices on the cube and the angular defect at every vertex is $\pi/2$, so that the total angular defect $\Delta = 8 \cdot \frac{\pi}{2} = 4\pi$.

The manuscript of Descartes was lost and forgotten, and only a partial copy was found in 1860 among the papers of Leibniz (Gottfried Wilhelm von Leibniz, 1646 - 1716). But what actually happens, proving Descartes' formula we get

$$\Delta = 2\pi(V - E + F),$$

where V is the number of vertices, E of edges, and F of faces of a polyhedron. This formula was proved by Pólya (George Pólya, 1887 - 1985) in [20]. It is not clear whether Decartes knew his formula in this version or not. For sure, René Descartes could not know the Euler formula, but there was a small chance that Euler knew Descartes' result.

Euler's formula. Euler's formula was proved by Legendre (Adrien-Marie Legendre, 1752 - 1833) in 1794. The formula has been pronounced as "the first important event in topology" by Alexandrov (Pavel Sergeevich Aleksandrov, 1896 - 1982) and Hopf (Heinz Hopf, 1894 - 1971). The formula does not hold for arbitrary polyhedra and because of that it sparked many papers during the first half of the nineteenth century about its range of validity. At that time convexity gained an importance.

If we look at one-dimensional "polyhedra", i.e., just segments in the real line, we see that they always have two end-points, this means that the number of vertices V is 2. The number of 0-dimensional faces determines the "polyhedron".

If we count the number of vertices V and edges E for (convex) polygons in the plane, we see that always $V = E$, i.e., $V - E = 0$. Again, the number of 1-dimensional edges determines the polygon.

One can suspect that maybe this is also true in the three dimensional space, where a polyhedron has vertices, edges and faces. But a simple example shows that the pyramid and the prism are not equivalent, but have the same number of faces. A

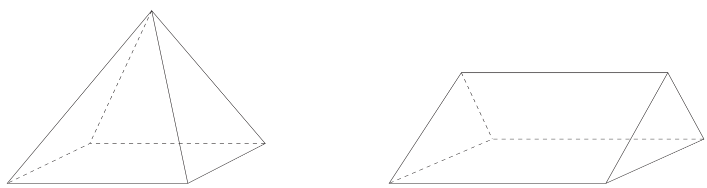

Figure 7: Pyramid and prism

well-known example of the occurrence of convexity is Euler's (Leonhard Euler, 1707 - 1783) formula on polyhedra discovered around 1752:

The numbers V of vertices, E of edges, and F of faces of a closed convex polyhedron satisfy $V - E + F = 2$.

Indeed it holds for basic examples, like Platonic solids:

	V	E	F	V-E+F
tetrahedron	4	6	4	2
cube	8	12	6	2
octahedron	6	12	8	2
dodecahedron	20	30	12	2
icosahedron	12	30	20	2

and can be easily checked for other convex polyhedra.

Sketch of proof of Euler's formula. There are at least a dozen different proofs of Euler's formula. Euler's formula actually is a particular case of the very important invariant of Euler characteristic in algebraic topology, giving relations between dimensions of some cohomology groups.

Sketch of the proof by induction on vertices. If a "polyhedron" G has only one vertex, each edge is a Jordan curve (roughly speaking, like a deformed circle with the vertex on it), so there are $E + 1$ faces and $V - E + F = 1 - E + (E + 1) = 2$. Otherwise, choose an edge e connecting two different vertices of G, and contract it. This decreases both the number of vertices and edges by one and the result then holds by induction.

Generalization of Euler's Formula. Euler's formula has been generalized for n-dimensional polyhedra by Schläfli (Ludwig Schläfli, 1814 - 1895) and proved by Poincaré (Jules Henri Poincaré, 1854 - 1912) in 1893:

> *If f_k denotes the number of k-dimensional faces of an n-dimensional polytop, then* $\sum_{k=0}^{n-1}(-1)^k f_k = 1 - (-1)^n$.

A difficult problem, which is open in full generality, is to find conditions on an n-tuple $f_0, \ldots, f_{n-1}$ of positive integers such that there exists a convex polytop in $\mathbb{R}^n$ with f_j faces of dimension j. The problem was solved by Steinitz (Ernst Steinitz, 1871 - 1928) in 1906 for $n = 3$.

4 More modern definition of convexity

The most common mathematical definition of a convex set (for simplicity restrict ourselves to the Euclidean space $\mathbb{R}^n$) is

Definition 4.1. (Convex sets) A set S in $\mathbb{R}^n$ is convex if with any two points p and q belonging to S the entire segment joining p and q lies in S.

We know that the points in the segment are of the form $tp + (1 - t)q$, where $0 \leq t \leq 1$. The above definition geometrically is very clear, but in analytical applications not very useful. Later on we give other definitions, more convenient in analysis.

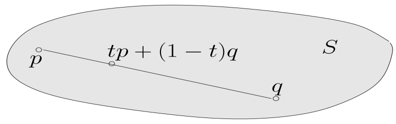

Figure 8: Convex set

Intimately related to convexity of sets is convexity of functions, however this notion appeared much later than the first one.

Definition 4.2. (Convex functions) Let V be a vector space and $S \subset V$ be a convex set. A function $f : S \longrightarrow \mathbb{R}$ is called convex if

$$f(\lambda p + \mu q) \leq \lambda f(p) + \mu f(q) \quad \text{for} \quad \lambda, \mu \geq 0, \;\; \lambda + \mu = 1, \;\; p, q \in S.$$

We see immediately, that if $V = \mathbb{R}$ and S is any interval in $\mathbb{R}$, then the above definition is equivalent to the property that the graph of f lies below the chord between any two points.

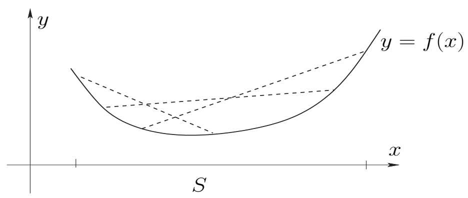

Figure 9: Convex function

To understand better the concept of convexity, we give equivalent definitions of convex functions and sets.

Definition 4.3. (Convex sets - revisited) Let V be a vector space over $\mathbb{R}$. A subset S of V is called convex if every line intersects S in an interval.

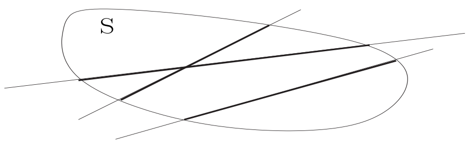

Figure 10: Convex sets - revisited

Definition 4.4. (Convex functions of one variable - revisited) A function $f : I \longrightarrow \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ is convex if for every compact interval $J \subset I$ with boundary ∂J, and every linear function $L = L(x) = ax$ we have

$$\sup_J (f - L) = \sup_{\partial J} (f - L).$$

Definition 4.5. (Convex functions of several variables - revisited) A function f defined on a convex set $S \subset V$ is convex if for any line ℓ the function f restricted to $\ell \cap S$ is convex.

In all these definitions, to define convexity we used linear functions or lines or segments of lines. The linear functions are the simplest non-trivial functions. Taking this point of view, it is clear that it is natural and reasonable to use various classes of functions to define other notions of convexity (with respect to these classes) as we shall see later on.

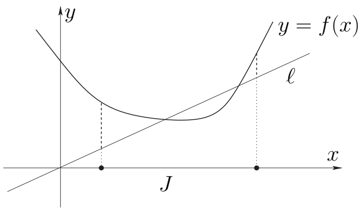

Figure 11: Convex functions - revisited

5 Minkowski's theorems

Interests in convexity from Number Theory. Minkowski's interest (Hermann Minkowski, 1864 - 1909) in convexity came from number theory. In his theory of the reduction of positive definite quadratic forms he needed a theorem of Ch. Hermite of 1850 (Charles Hermite, 1822 -1901). In Minkowski's geometrical formulation it states that an ellipsoid in $\mathbb{R}^n$ with center at the origin and volume greater than a number depending only on n contains at least one latice point different from the origin. In 1891 Minkowski realized that actually convexity and symmetry are the properties that are essential in the proof.

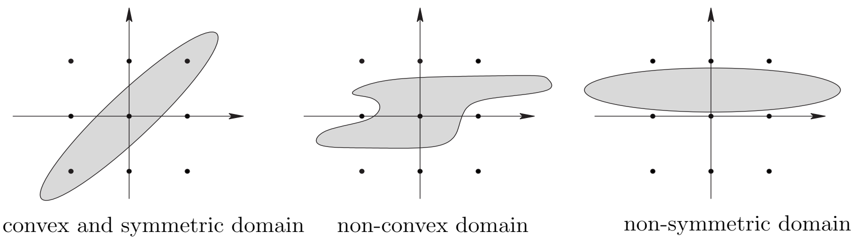

Figure 12: Illustration of the Minkowski theorem

Theorem 5.1. (Minkowski, 1891) *Every closed bounded convex set in $\mathbb{R}^n$ with the origin as center and volume greater than 2^n contains at least one lattice point different from the origin.*

By specializing and improving the theorem, Minkowski found many important results in number theory. He founded the branch of it to which he gave the name *"Geometry of Numbers"*.

Isoperimetric problems. Isoperimetric problems usually mean to find minima or maxima subject to some conditions, for instance, maximize the area if the perimeter is fixed. For a nice discussion of classical isoperimetric problems see the paper by Talenti [21] and parts in two books by Frank Morgan [16], [17].

Besides applications of convexity to number theory, Minkowski made many other important contributions to the theory of convex sets. Here we mention only few of them. From Minkowski's work follows the classical "isoperimetric inequality" for convex bodies, namely

> *Among all convex bodies with given volume the ball and only it has minimal surface area.*

Another, more surprising result is the following (1897):

> *Given unit vectors $\vec{v}_1, \ldots, \vec{v}_k$ in $\mathbb{R}^3$ and positive numbers $a_1, \ldots, a_k$. Then there exists a convex polyhedron the faces of which are normal to the v_j and have areas a_j if and only if $a_1\vec{v}_1 + \ldots + a_k\vec{v}_k = 0$.*

Extreme points and linear programming. Convex polyhedra, convex sets given by linear inequalities in $\mathbb{R}^n$ (classically in $\mathbb{R}^3$), naturally appear in physical problems. For instance Fourier (Jean Baptiste Joseph Fourier, 1768 - 1830) needed convex polyhedra in his work on statics. Also he specified conditions for the solvability of linear inequalities and developed an algorithm for the determination of all solutions in the paper *"Solution d'une question particulière du calcul de inégalités"* in 1826. More systematical studies of convex polyhedra are given in the book by H. Minkowski *"Geometry of Numbers"*, the main part appeared in 1896 [15]. In his book he considers systems of the form $Ax \geq 0$, where A is a real $m \times n$ matrix and x a vector, and where the inequality means that each entry is non-negative.

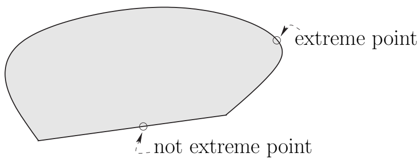

Figure 13: Extreme and non-extreme points

Later on, Minkowski introduced the notion of *"extreme point"* of a convex set. This is a point of the set which is not interior to any segment belonging to the set. Equivalently, the set remains convex if the point is removed. He shows that a closed and bounded convex set in $\mathbb{R}^3$ is the convex hull of the set of its extreme points. More precisely, every point of the set belongs to a (possibly degenerate) tetrahedron the vertices of which are extreme points.

In 1940, M. Krein (Mark Grigorievich Krein, 1907 - 1989) and D. Milman (David Milman, 1912 - 1982) succeeded in generalizing this to a large class of infinite-dimensional topological vector spaces in a slightly weaker form:

Theorem 5.2. (Krein and Milman, 1940) *A compact convex set is the closure of the convex hull of its extreme points.*

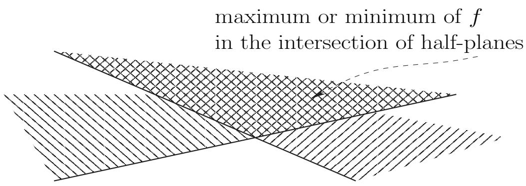

Figure 14: Rough explanation of linear programming

The work by Minkowski and other mathematicians was a prelude to *linear programming theory*, which emerged in late forties of the 20th century (works of Leonid Vitalievich Kantorovich, 1912 - 1986, and George Dantzig, born 1914). Linear programming deals with the optimization of a linear objective function f subject to linear equality or inequality constraints. More precisely, given a set in $\mathbb{R}^n$ which is an intersection of half-spaces or hyperpanes (like planes in $\mathbb{R}^3$), find the maximum and minimum values of $f(x_1, \ldots, x_n) = a_1x_1 + \ldots + a_nx_n$ on this set (see Fig. 14).

Minkowski's distance function, norms, and Hahn-Banach theorem. As a precursor of the norm in vector spaces, Minkowski introduced the *distance function of a convex set*, namely if S is a convex set in $\mathbb{R}^n$ containing the origin O in its interior, then the function is defined as follows: Let $x \in \mathbb{R}^n \setminus \{O\}$ and ξ be the point

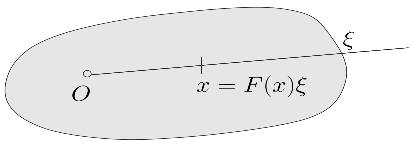

Figure 15: The distance function $F(x)$

at which the half-line from O through x intersects the boundary of S. Then define $F(x) > 0$ as the number for which $x = F(x)\xi$; for the origin, define $F(O) = 0$, see Fig. 15. The function is convex and positively homogeneous, that is, $F(tx) = tF(x)$ for $t \geq 0$. Clearly, $F(x) \leq 1$ if and only if $x \in S$. Every function with these properties is the distance function of a convex body. In his book *"Geometry of Numbers,"* Minkowski uses the distance function F of a convex set in $\mathbb{R}^n$ to define a metric, namely $\mathrm{dist}(x, y) = F(y - x)$. If $F(-x) = F(x)$, then F satisfies all conditions of a norm. In more details the distance function is described in [1].

In 1922, S. Banach (Stefan Banach, 1892 - 1945) introduced the notion of what is now called a Banach space, a normed vector space which, considered as a metric space, is complete. A few years later, in 1927 H. Hahn (Hans Hahn, 1879 - 1934) and independently in 1929 S. Banach proved a basic theorem in functional analysis (but not only), now called the Hahn-Banach theorem. It has a lot in common with convex sets. Here we formulate a version of the theorem.

Theorem 5.3. (Hahn-Banach) *Let F be a positively homogeneous* convex *function on a real vector space V, and let g be a linear functional defined on a linear subspace U of V such that $g(x) \leq F(x)$ for $x \in U$. Then g can be extended to a linear functional G on V such that $G(x) \leq F(x)$ for $x \in V$.*

Minkowski's inequality. As an example of a norm, we mention the p-norm $\|(a_1, \ldots, a_n)\|_p := \left(\sum_{j=1}^n |a_j|^p\right)^{1/p}$ and famous Minkowski's inequality. Minkowski's inequality plays a major role in l^p and L^p spaces in functional analysis. It can be derived from the Hölder inequality and all this is related to convex functions; see [13] pp. 8 - 12.

Let $a_j \geq 0$, $b_j \geq 0$ for $j = 1, 2, \ldots, n$. Then

$$\left(\sum_{j=1}^n (a_j + b_j)^p\right)^{1/p} \leq \left(\sum_{j=1}^n a_j^p\right)^{1/p} + \left(\sum_{j=1}^n b_j^p\right)^{1/p}, \qquad p \geq 1,$$

with strict inequality unless $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are linearly dependent.

6 Convexity with respect to a family of functions

Let $U \subset \mathbb{R}^n$ be a domain (i.e., open and connected) and let $\mathscr{F}$ be a family of real-valued functions on U. Let K be a compact subset of U. Then the *convex hull of K with respect to $\mathscr{F}$* is defined by

$$\hat{K}_{\mathscr{F}} \equiv \left\{ x \in U : \; f(x) \leq \sup_{t \in K} f(t) \quad \text{for all} \quad f \in \mathscr{F} \right\}$$

We say that U is *convex* with respect to $\mathscr{F}$ provided $\hat{K}_{\mathscr{F}}$ is compact in U whenever K is. When the functions in $\mathscr{F}$ are complex-valued, then $|f|$ replaces f in the definition of $\hat{K}_{\mathscr{F}}$.

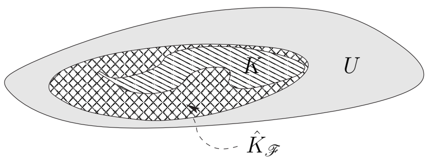

Figure 16: Convex hull with respect to a family of functions

A simple exercise follows immediately from the classical definition of convexity:

Let $U \subset \mathbb{R}^n$ and let $\mathscr{F}$ be the family of real linear functions. Then U is convex with respect to $\mathscr{F}$ if and only if U is geometrically convex.

Another simple exercise, any domain in $\mathbb{R}^n$ is convex with respect to the family $\mathscr{F} = \mathscr{C}$ of all continuous functions on U. Of course taking more complicated families $\mathscr{F}$ it is more difficult to check convexity of a domain U. However this approach is quite uniform and allows us to see that geometric convexity is a particular case of a more general picture.

Subharmonic and Plurisubharmonic Functions

In the next section we shall see what role plurisubharmonic functions will play in the definition of the so-called pseudo-convex domains - the most important class of domains in complex analysis. First we need some definitions of classes of functions.
Harmonic functions are functions that satisfy the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0. \tag{6.1}$$

There are many monographs devoted to this class of functions and the theory is well developed. If $n = 1$, the Laplace equation becomes $d^2u/dx^2 = 0$ and the solution is $u(x) = ax + b$ on each interval where it is considered. So it is reasonable to say that geometric convexity is convexity with respect to one-variable harmonic functions.
Subharmonic functions. We shall see a complete analogue between convex functions and subharmonic functions, where the role of linear functions is replaced by harmonic functions (compare Def. 4.4).

Definition 6.1. A function u defined in an open subset U of $\mathbb{R}^n$ with values in $[-\infty, \infty)$ is called *subharmonic* if

1. u is upper semi-continuous, i.e., $\limsup_{U \ni x \to p} f(x) \leq f(p)$ for $p \in U$;

2. for every compact subset K of U and every continuous function h on K which is harmonic in the interior of K, the inequality $u \leq h$ is valid in K if it holds on the boundary of K. In other words (actually more generally),

$$\sup_K (u-h) = \sup_{\partial K} (u-h).$$

Especially if the dimension $n = 2$, the theory of harmonic and subharmonic functions is interesting and is intimately related to analytic functions. A complex-valued function $f = f(z)$ defined in a domain U in $\mathbb{C}$ is called *complex analytic* if locally it can be expanded into a power series in z, i.e.,

$$f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j$$

in a small disk around z_0. If we identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$, then the real and imaginary parts of a complex analytic function are harmonic; conversely, in a simply connected domain, each harmonic function is the real part of some holomorphic function and also the imaginary part of some holomorphic function.

Plurisubharmonic functions generalize subharmonic functions to several variables:

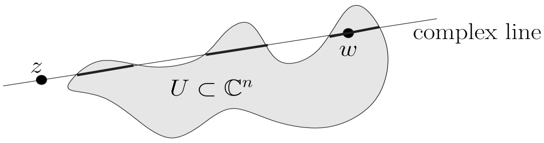

Figure 17: Domain of a plurisubharmonic function

Definition 6.2. (Plurisubharmonic Functions) A function u with values in $[-\infty, \infty)$ defined in an open set $U \subset \mathbb{C}^n$ is called *plurisubharmonic* if its restriction to any complex line (where it is defined) is subharmonic. More precisely:

1. u is upper semi-continuous;

2. For arbitrary z and w in $\mathbb{C}^n$ the function $\tau \longrightarrow u(z + \tau w)$ is subharmonic in the open subset of $\mathbb{C}$ where it is defined.

7 Convexity in real and complex analysis

Convexity in real analysis. The standard definition of convexity which is given at the beginning of this note, is of little use from the point of view of analysis. The definition is non-quantitative, nonlocal, and not formulated in the language of functions. Here we give a differential characterization of convexity, more useful in analysis.

Assume that $U \subset \mathbb{R}^N$ is a domain with C^2 boundary and ρ is a defining function for U, i.e., $\rho = 0$ and grad$\rho \neq 0$ on ∂U. Fix $p \in \partial U$. We say that ∂U is (*weakly*) *convex* at p if

(7.1)

$$\sum_{j,k=1}^{N} \frac{\partial^2 \rho(p)}{\partial x_j \partial x_k} u_j u_k \geq 0 \text{ for all } u = (u_1, \ldots, u_N),\ u_1 \frac{\partial \rho(p)}{\partial x_1} + \cdots + u_N \frac{\partial \rho(p)}{\partial x_N} = 0.$$

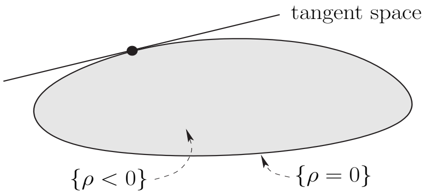

Figure 18: Convex domain in terms of the defining function ρ

We say that ∂U is *strongly convex* at p if the inequality is strict whenever $u \neq 0$. If ∂U is convex (resp. strongly convex) at each boundary point, then we say that U is convex (resp. strongly convex). We note that vectors $(u_1, \ldots, u_N)$ satisfying the condition as in (7.1) can be interpreted as vectors from the tangent space to the boundary at p.

Convexity in Complex Analysis. In complex analysis, where we deal with complex analytic functions, a natural question appears: what is the natural notion of convexity there? As we know, convexity is not preserved under complex analytic homeomorphisms. Because of that, we cannot use geometrically convex domains, i.e., domains convex with respect to the class of linear functions. However, we can save something from this, as explained later on in this section about eliminating the failure of real analysis.

One way to get right "convex" domains in complex analysis is to repeat the more general definition from Section 6 for some classes of functions. Since in complex analysis of several variables, the class of complex analytic functions and the class of plurisubharmonic functions play main roles, therefore we define:

An open set U in $\mathbb{C}^n$ is *pseudoconvex* if for any compact set K, $K \subset U$, we have

$$\hat{K} = \{z \in U;\ u(z) \leq \sup_K u \quad \text{for all plurisubharmonic } u \text{ in } U\} \Subset U.$$

The domain U is *holomorphically convex* if $\hat{K}_{\mathrm{Hol}} \Subset U$ for any $K \Subset U$ (see Fig. 16), where the *hull of holomorphy of* K is

$$\hat{K}_{\mathrm{Hol}} \equiv \left\{ z \in U : \ |f(z)| \leq \sup_{\zeta \in K} |f(\zeta)| \quad \text{for all} \quad f \in \mathrm{Hol}(U) \right\}.$$

It appears that there is equivalence of these two types of convexity: U is pseudoconvex if and only if U is holomorphically convex. Pseudoconvex domains are the most important and most natural to consider in complex analysis of several variables. The reader can consult several excellent books in this subject, e.g., Gunning and Rossi [10], Hörmander [12], or Krantz [14].

Convex sets by eliminating the failure of real analysis. As we know, convexity is not preserved under complex analytic homeomorphisms. We want to understand analytically where the failure lies. We look at the quadratic form (7.1) and write it in complex coordinates. Of course, now $N = 2n$, $\mathbb{C}^n \sim \mathbb{R}^{2n}$, $z = (z_1, \ldots, z_n) \sim$

$(x_1, y_1, \ldots, x_n, y_n)$, and $w_j = u_j + iv_j$. Then after simple computations we get

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial x_j \partial x_k} u_j u_k + 2 \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial x_j \partial y_k} u_j v_k + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial y_j \partial y_k} v_j v_k$$

$$(7.2) \qquad = 2\mathbf{R} \left(\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k} w_j w_k \right) + 2 \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \,.$$

Actually what is calculated above, the real Hessian, when written in complex coordinates, decomposes rather naturally into two Hessian-like expressions. The first term in (7.2) is not invariant under complex analytic homeomorphisms, however the second term in (7.2) is.

It turns out that the domains U with C^2 boundary, which are given by the defining function ρ that satisfy the condition

$$(7.3) \quad \sum_{j,k=1}^{n} \frac{\partial^2 \rho(p)}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \geq 0 \ \text{ for all } \ w = (w_1, \ldots, w_n), \ w_1 \frac{\partial \rho(p)}{\partial z_1} + \cdots + w_n \frac{\partial \rho(p)}{\partial z_n} = 0,$$

are the right "convex" domains in complex analysis. They are called pseudoconvex domains. Actually we see that these domains are defined in a very similar way as convex domains in real spaces; compare (7.1) and (7.3).

Tubular domains. It is interesting to consider pseudoconvex domains in $\mathbb{C}^n$ in the shape of a tube: $\Omega = \omega \times \mathbb{R}^n$, where $\omega \subset \mathbb{R}^n$. Here $\mathbb{C}^n$ is naturally split into the real part and the imaginary part, namely $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$. It turns out that *such domains are pseudocovex or holomorphically convex domains if and only if ω is convex in the classical geometric sense.* This is the so-called Bochner tube theorem.

8 Convexity and partial differential equations

Congruence of isometric surfaces. We have already seen convexity with respect to harmonic or plurisubharmonic functions. Historically, first relations between convexity and partial differential equations seem to be at the end of the eighteenth and beginning of nineteenth century when J. Fourier considered equilibrium problems for mechanical systems and statics problems. Another interesting example, which was developed over almost 150 years, started with Cauchy in 1812:

> *If corresponding faces of two isomorphic convex polyhedra are congruent, the polyhedra are (properly or improperly) congruent.*

Cauchy's result shows in particular that a convex polyhedron cannot be deformed continuously in such a way that each face remains congruent with itself. In 1845 Cauchy claimed that an immediate consequence of his theorem on polyhedra is that a closed convex surface is rigid in the sense that it does not admit isometric deformations.

Minkowski's result, formulated in Section 5, is related to the determination of a smooth convex body by its Gauss curvature. These problems were taken up by

H. Weyl in 1916 (Hermann Klaus Hugo Weyl, 1885 - 1955), some of his proofs are not complete. Later on, H. Lewy in 1938 (Hans Lewy, 1904 - 1988) based on his results on elliptic partial differential equations of Monge-Ampère type solved Weyl's problem with analytic data. In 1953 L. Nirenberg proved Weyl's statement completely. At the same time (1952) A.V. Pogorelov proved that every closed convex surface is uniquely determined, up to orthogonal transformations, by its intrinsic metric, completing what Cauchy started.

Convexity with respect to differential operators. This is a difficult subject related to existence of solutions of partial differential equations in some domains in $\mathbb{R}^n$ or $\mathbb{C}^n$. It is described precisely in the book by L. Hörmander [13], Ch. 6, and requires an extensive knowledge of partial differential equations.

Let P be a polynomial in n-variables with complex coefficients. We say that an open set U, $U \subset \mathbb{R}^n$, is P-convex if the equation $P(\partial/\partial x_1, \ldots, \partial/\partial x_n)u = f$ has a solution $u \in C^\infty(U)$ for every $f \in C^\infty(U)$.

It turns out that some classes of partial differential equations have solutions only in domains with some type of convexity properties. This already can be seen in complex analysis (Section 7), where the so-called $\overline{\partial}$-problem has solutions only in the pseudo-convex domains.

9 Convexity in algebraic geometry

Since Algebraic Geometry deals with geometry from an algebraic point of view, it is natural that convexity plays a very important role. Here, of course, there is no space to describe precisely this beautiful area, only we give just one nice application of convexity in the theory of toric varieties.

Toric varieties (if we consider them over the field $\mathbb{C}$) are complex analytic varieties of special kind. They were formally introduced in the beginning of 1970's. By a complex torus we mean $T_n = (\mathbb{C}^*)^n$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Very roughly speaking, a *toric variety* of dimension n is a compactification of the complex torus T_n. It appears that it is possible to compactify the torus in many different ways. Some compactifications can be described beautifully in geometric and combinatorial way.

Here we give a very schematic description of toric varieties. For detailed treatment, see the textbooks [5], [18] and expository paper [4].

Strongly convex rational polyhedral cone. A subset $\sigma_{\mathbb{R}}$ of $\mathbb{R}^n$ is called a *convex rational polyhedral cone* if there exists a finite number of elements $n_1, n_2, \ldots, n_s$ in $\mathbb{Z}^n$ such that

$$\sigma_{\mathbb{R}} = \mathbb{R}_{\geq 0} n_1 + \ldots + \mathbb{R}_{\geq 0} n_s = \{a_1 n_1 + \ldots + a_s n_s \mid a_i \in \mathbb{R}_{\geq 0} \text{ for all } i\} \subset \mathbb{R}^n$$

$$\sigma_{\mathbb{R}} \cap (-\sigma_{\mathbb{R}}) = \{0\}.$$

Definition of the dual cone. The *dual cone* to $\sigma_{\mathbb{R}} \subset \mathbb{R}^n$ is defined as

$$\check{\sigma}_{\mathbb{R}} := \{y \in \mathbb{R}^n \mid \langle y, x\rangle \geq 0 \text{ for all } x \in \sigma_{\mathbb{R}}\}.$$

Lattice cones. By lattice cones we mean

$$\sigma_{\mathbb{Z}} = \sigma_{\mathbb{R}} \cap \mathbb{Z}^n, \qquad \check{\sigma}_{\mathbb{Z}} = \check{\sigma}_{\mathbb{R}} \cap \mathbb{Z}^n.$$

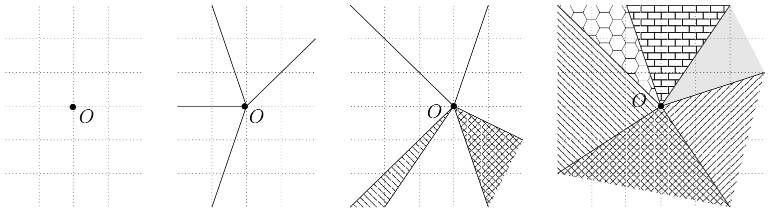

Figure 19: Examples of fans in $\mathbb{R}^2$

Fan. A *fan* in $\mathbb{R}^n$ is a nonempty collection Δ of strongly convex rational polyhedral cones in $\mathbb{R}^n$ satisfying the following conditions:

(a) Every face of any $\sigma \in \Delta$ is contained in Δ.

(b) For any $\sigma,\ \sigma' \in \Delta$ the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

How to get a toric variety from a fan. Schematically the procedure is the following

$$\Delta = \{\sigma_{\mathbb{R}}\} \rightsquigarrow \{\check{\sigma}_{\mathbb{R}}\} \rightsquigarrow \{\check{\sigma}_{\mathbb{Z}}\} \rightsquigarrow \{\text{affine toric varieties}\} \rightsquigarrow \text{glue} \rightsquigarrow \text{toric variety}$$

Here we describe the process, up to the affine toric varieties only, on an example. Let $\sigma_{\mathbb{R}}$ be the strongly convex rational polyhedral cone in $\mathbb{R}^2$ generated over $\mathbb{R}_{\geq 0}$ by

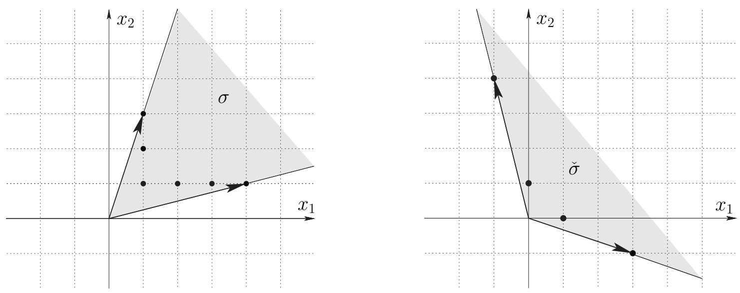

Figure 20: Cone and its dual spanned and generated

$n_1 = [1,\ 3]$ and $n_2 = [4, 1]$ as in Fig. 20. The dual cone $\check{\sigma}_{\mathbb{R}}$ is spanned over $\mathbb{R}_{\geq 0}$ by vectors $m_1 = [-1, 4]$ and $m_2 = [3, -1]$. Additional two vectors are needed to generate the cone $\check{\sigma}_{\mathbb{Z}}$ over $\mathbb{Z}_{\geq 0}$, namely $[1, 0]$ and $[0, 1]$. So the lattice vectors that generate the dual cone are $[-1, 4]$, $[3, -1]$, $[1, 0]$, $[0, 1]$.

Now the procedure is

$$\begin{array}{ll} m_1 = -e_1 + 4e_2 & u(m_1) = z_1 \\ m_2 = e_2 & u(m_2) = z_2 \\ m_3 = e_1 & u(m_3) = z_3 \\ m_4 = 3e_1 - e_2 & u(m_4) = z_4 \end{array} \quad \begin{cases} m_1 + m_3 = 4m_2 \\ m_2 + m_4 = 3m_3 \end{cases} \Longrightarrow \begin{cases} z_1 z_3 = z_2^4 \\ z_2 z_4 = z_3^3 \end{cases} \text{ in } \mathbb{C}^4$$

The affine toric variety is given by two equations in $\mathbb{C}^4$.

At the end we mention that starting with a convex polytope in $\mathbb{R}^n$ with vertices from $\mathbb{Z}^n$ it is possible to determine a fan and to construct a compact toric variety that can be embedded into a projective space. Convexity of the polytope is essential.

10 Conclusions

As it can be seen from this short note, convexity appears in many areas of mathematics. Many other areas are left, not to mention more applied sciences, simply because of lack of space. With more details, this paper can grow to hundreds of pages or even to a series of books. The author hopes that the reader has been stimulated into researching the aspect of convexity on her or his own. If it is the case, the goal of this note is achieved.

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