Klein geometries, Lie differentiation and spin

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Abstract. The issue is considered of whether one needs to have a concept of non-effective elementary geometry in order to accommodate spin. We show that, if spin seems to be a subtle concept in need of special treatment when dealing with the concept of Klein geometry, it may be so because (a) one ignores the possibility of doing quantum mechanics with differential forms and (b) one fails to consider that the expression for the Lie operator in the ring of differential forms is not simply the covariant form of the corresponding expression on the ring of functions. There are extra terms, which happen to be zero for translational but not rotational symmetries. They constitute the spin terms. We also point out at a related erratum in transcribing the concept of elementary geometry in one of our papers in a Proceedings volume of the Balkan Geometry Society.

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1 Introduction

In a recent paper [15], we attempted to put together in concise form the concept of what Clifton [5] and Sharpe [11] respectively call elementary and Klein geometries. Coincidentally, the first one did so in a paper [5] presented by Chern, who also authored the foreword to that book by Sharpe [11]. Both concepts target the same mathematical structure, namely the holonomic spaces whose non-holonomic generalizations constitute the differentiable manifolds endowed with conformal, projective, affine, Weyl, Euclidean ... connections. For both authors, the defined concept is a pair, (G, G_0), of a Lie group G and a closed subgroup G_0. Clifton requires that no subgroups of G_0 other than the identity be normal in G. He went on to state that, by virtue of this condition, "G operates effectively to the right on the set $F = G/G_0$ of right cosets mod G_0". Recall that the action of a group on a set is called effective if only the unit element of the group leaves each and every element of the set unchanged.

In our aforementioned paper, we inadvertently referred to an elementary or Klein geometry as a pair "of a group G and a *normal* subgroup". We overlooked to mention that G must be a Lie group, possibly because we only deal with such groups. But readers would certainly realize this oversight.

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Sharpe did not require the absence of subgroups of G_0 that are normal in G. He required G/G_0 to be connected and went on to define the kernel of a Klein geometry G/G_0 as the largest subgroup K of G_0 that is normal in G. He further defined "effective Klein geometry" if $K = 1$, and locally effective if K is discrete. It our inadvertently erroneous transcription of Clifton's definition, we had been more restrictive than Sharpe, given our additional requirement that G_0 be normal in G. This requirement, however, goes in the wrong direction, as will become clear with the example below. What makes this issue important for physics is the fact that Sharpe invokes spin when dealing with the concept of effective/non-effective geometry. We do not feel authoritative enough to pass judgement on this issue. We wish, however, to illustrate Kähler's important finding in 1962 that spin is a consequence of the thesis that an electron is represented by an inhomogeneous differential form that belongs to a Clifford algebra, rather than to one of its ideals as is the case in the Dirac theory.

For a practical perspective, consider the familiar affine geometries, and the special cases of Euclidean and Lorentz-Einstein-Minkowski geometries. The general element of the affine group can be given by the block matrix [15]

$$\left[\begin{array}{c|c} 1 & \mathrm{T} \\ \hline 0 & \mathrm{L} \end{array}\right], \tag{1.1}$$

where T is a row matrix $\left[A^0, ..., A^3\right]$ representing a translation, L is a square block of entries $L_m^{\ l}$ representing a linear transformation with non-null determinant (or a rotation or pseudo-rotation in the aforementioned particular cases). The action of (1.1) on the column

$$\left\{ \begin{array}{c} \mathbf{Q} \\ \mathbf{a}_m \end{array} \right\} \tag{1.2}$$

constituted by a fixed point $\mathbf{Q}$ and a fixed basis $\mathbf{a}_m$ yields the general basis $\mathbf{e}_m$ at the general point $\mathbf{P}$:

$$\left\{ \begin{array}{c} \mathbf{P} \\ \mathbf{e}_m \end{array} \right\} = \left[\begin{array}{c|c} 1 & \mathrm{T} \\ \hline 0 & \mathrm{L} \end{array}\right] \left\{ \begin{array}{c} \mathbf{Q} \\ \mathbf{a}_m \end{array} \right\}. \tag{1.3}$$

A translation, of course, belongs to G but not to G_0; L belongs to G_0. Consider the product $ghg^{-1}(h \in G_0)$. We get

$$\left[\begin{array}{c|c} 1 & \mathrm{T} \\ \hline 0 & \mathrm{I} \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \mathrm{L} \end{array}\right] \left[\begin{array}{c|c} 1 & \text{-T} \\ \hline 0 & \mathrm{I} \end{array}\right] = \left[\begin{array}{c|c} 1 & -A^l + A^m L_m^{\ l} \\ \hline 0 & L \end{array}\right]. \tag{1.4}$$

We thus see in this most relevant geometry for physicists that G_0 is not a normal subgroup in G. As we said, our inadvertent requirement of normal subgroup goes in an undesirable direction.

Whereas Clifton requires that the elementary geometry be effective, Sharpe, who does not cite any source with regards to this issue, does not make that requirement. Sharpe goes on to ask "Why not define Klein geometries to be *effective*?" (emphasis in original). He goes on to give a first reason as to why not, namely that doing so eliminates the subtle phenomenon of spin. In other words, he appears to be saying that spin requires non-effective Klein geometries. The brevity of his comment does not allow one to know what exactly he had in mind; he does not deal with that notion in the book in question or, apparently, anywhere else. In a letter of August 29, 2007 to JGV, Professor Sharpe accepts both possibilities, namely that spin can sometimes be included entirely within the context of Cartan geometries by allowing ineffective models, and also as a structure additional to the Cartan geometry.

2 The relation of spin to the Lie derivatives for rotations

We intend to show in this paper that "the subtlety of spin" may be just an accident in the history of physics. The accident is that physicists created the notion of spinor, the mathematical concept to which the notion of spin is attached, at a time when they did not know how best to represent the state of the electron. Our justification for such a claim is to be found in seminal work by the great mathematician E. Kähler [9],[8],[7], better known for his namesake manifolds and for his generalization of Cartan's theory of exterior differential systems. He extended Cartan's exterior calculus, making what is the language of geometry suitable also for quantum mechanics [7]. In Kähler's own words [[9], section 26]: "The spin of the electron will be interpreted as the need to represent the state of an electron by a state differential form rather than by a state function." We hope that geometers in the long overdue approach to differential geometry proposed by Sharpe can then decide how to best use this claim in further polishing the concept of Klein geometries.

The spinorial representations had been discovered by Cartan in 1913 [3] before physicists used them. But spin there is buried in non recognizable form, a not surprising fact in view of the title of the paper: "The projective groups which do not leave invariant any flat manifold". Spinors became a explicit subject in Cartan's work only in 1937 [1], by which time the physicists' practical concept of spinor had become all pervasive in quantum mechanics, whether relativistic or not. Spin would not be such a subtle concept if mathematics had been much more developed than it was in the golden decades of theoretical physics, 1900-1930.

Apart from that result on the interpretation of spin, the Kähler calculus produces other results of great potential impact, like the fact that positrons emerge with the same sign of the energy as electrons. For all that, it suffices to deal with scalar-valued differential forms. Kähler actually did not develop any application for non-scalar valuedness. The present authors have suggested the replacement of tensor-valuedness with the more convenient Clifford-valuedness [18], as well as new lines for theoretical developments [15],[17],[14],[13].

Recently, we have made the point that concepts in differential geometry such as vector fields and differential forms have to be viewed from a perspective which is more like Cartan and Kähler's [14],[13]. In section 3, we make a similar case for Lie derivatives. In a 1922 book, starting from the concept of infinitesimal operator acting on a function, Cartan reaches in less than four pages a theorem which is nowadays known as Cartan's magic theorem for Lie derivatives of differential forms, which incidentally permits to easily compute them [2]. This is in sharp contrast with the modern approach, where it appears that the concept of Lie derivative of a vector field is the pre-eminent concept from which everything else derives. And yet, the concept of Lie derivative of a vector field is not even mentioned by Cartan. If one realizes that 1922 is the year when he announced in a series of letters to the French academy his work on the general theory of connections (hence the theory of tensor-valued differential forms), the suggestion that Cartan did not view the concept of Lie derivative of a vector field as a significant geometric concept is not speculation; one need only check in his work on geometry the total absence of that concept. It would not have escaped his attention that the 1-parameter groups of symmetries with which one associates Lie derivatives lack geometric significance for vector fields because the equality of vectors at different points is governed by the affine connection. It does not impede, however, that Lie differentiation of vector fields be used in intermediate steps of some approach to the equations of structure of affine and Euclidean connections, though the final equations have nothing to do with Lie differentiation [4], [16].

In section 3, we present different derivations of explicit forms to compute the Lie derivative of scalar-valued differential forms, namely the derivation of the formula by Cartan [2] (i.e. the magic theorem), and the derivations by Kähler [9] and Slebodzinski [12] of a more common formula for their computation. Each derivation is instructive in its own way, and all of them have a flavor different from that of modern treatments of the subject. The terms which will evolve into the spin term(s) for the appropriate Lie derivative associated with rotations (sections 4 and 5) are already present in those formulas. In section 4, we further develop for manifolds endowed with a metric the expression for the Lie derivative of differential forms. In section 5, we continue the specialization when the Lie operator leaves the metric invariant. In section 6, we specialize those formulas to the rotation operator. Hence, spin does not appear to be a structure additional to the Cartan geometries that generalize the Klein geometries, or additional to the Klein geometries themselves. Section 7 shows how the Kähler approach outshines the theory of representations in allowing for the possibility in principle of obtaining additional eigenvalues related to spacetime structure.

3 The Lie derivative of scalar-valued differential forms

Cartan extended the concept of Lie derivative from the ring of functions to the ring of differential forms [2]. Let A be the operator $\alpha^i \dfrac{\partial}{\partial x^i}$ on functions. Let u be an arbitrary scalar-valued differential form. Cartan shows that $Adx = d(Ax)$, and considers as obvious that the action of A on u is well defined. But he does not even try a direct computation of Au using $Adx = d(Ax)$. He, however, defines the action of u on A as

$$u(A) = \alpha^i \frac{\partial u}{\partial (dx^i)}, \tag{3.1}$$

where the terms arising in the partial differentiation of products alternate in sign, i.e.

$$\frac{\partial (u^1 \wedge u^2 \wedge \ldots)}{\partial u^1} = u^2 \wedge u^3 \wedge ..., \quad \frac{\partial (u^1 \wedge u^2 \wedge \ldots)}{\partial u^2} = -u^1 \wedge u^3 \wedge \ldots \tag{3.2}$$

He then proceeds to prove the magic theorem

$$Au = (du)(A) + d[u(A)], \tag{3.3}$$

which we have written in a form appropriate to easily connect, later in the section, with a more common formula for obtaining the Lie derivative of differential forms [9]- [7]. We use parenthesis around du to emphasize that exterior differentiation precedes the action on A. It is worth pointing out that, although $\alpha^i \frac{\partial}{\partial x^i}$ evokes one of the modern definitions of tangent vector, neither Cartan nor Kähler consider tangent vectors as differential operators, but rather as passive objects (say, equivalence classes of curves at a point).

It is of interest to repeat Cartan's proof of the magic theorem, if only because this is a rare occasion where Cartan uses differential forms as antisymmetric multilinear functions of vectors rather than integrands (functions of hypersurfaces). He refers to δ in δf as the symbol for an *indeterminate* operation, and to A in Af as the symbol for a *determinate* operation. Thus, for example, if u is a differential 2-form, the formula

$$du(\delta, \delta', \delta") = \delta u(\delta', \delta") - \delta' u(\delta, \delta") + \delta" u(\delta, \delta'). \tag{3.4}$$

illustrates his use of the symbol δ for those familiar with the concept of antisymmetric multilinear functions of vectors (Cartan does not, however, mention the term vector in his treatment of Lie differentiation). Replacing δ with A allows him to write

$$du(A, \delta', \delta") = Au(\delta', \delta") - \delta' u(A, \delta") + \delta" u(A, \delta'). \tag{3.5}$$

He does not consider this to be an equation with differential forms until he rewrites it as

$$(du)(A, \delta) = A[u(\delta)] - d[u(A, \delta)], \tag{3.6}$$

the last two terms on the right hand side of (3.5) becoming the last term of (3.6). Equation (3.3), whose right hand side is very easy to compute using equations (3.1) and (3.2), is just a small illustration of one of several deep results on Lie differentiation of differential forms by Cartan in 1922, long before the work on this subject by de Donder, Sledbozinski, van Dantzig, Yano, etc.

We shall now produce two derivations of the standard expression for Au. Their interest lies in that they provide two different perspectives of what is to be done with dx^i $(i \neq n)$ when we partial differentiate scalar functions with respect to, say, x^n. The fact that the coordinates x^i $(i \neq n)$ remain constant in differentiating the coefficients of the differential forms does not mean that the dx^i $(i \neq n)$ are to be set to zero. This may appear to be somewhat confusing. We shall now see why it is not so.

We start with Kähler's derivation [9]. Given the chart $\{x\}$, he considers an overlapping chart $\{y\}$ constructed as follows. In the system of ordinary differential equations

$$\frac{dx^i}{dy^n} = \alpha^i(x^1, ..., x^n), \quad (i = 1, ..., n), \tag{3.7}$$

let $y^1, ..., y^{n-1}$ be $n-1$ independent first integrals not additive to y^n. The use of the symbol y^n for the independent variable then become obvious since, together with those first integrals, $\{y\}$ is a new coordinate system where the Lie operator A becomes simply $\partial/\partial y^n$. Kähler pulls u to the y coordinate system and identifies Au with $\partial u/\partial y^n$. He thus obtains

$$
\begin{aligned}
u &= \frac{1}{p!} a_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} = \\
&\frac{1}{p!} a_{i_1 \ldots i_p} \frac{\partial x^{i_1}}{\partial y^{k_1}} \wedge \ldots \wedge \frac{\partial x^{i_p}}{\partial y^{k_p}} dy^{k_1} \wedge \ldots \wedge dy^{k_p}. \qquad (3.8)
\end{aligned}
$$

He uses that, for each of the factors $\frac{\partial x^{i_r}}{\partial y^{k_r}} dy^{k_r}$,

$$
\sum_{k_r} \frac{\partial}{\partial y^n} \left(\frac{\partial x^{i_r}}{\partial y^{k_r}} dy^{k_r} \right) = \frac{\partial x^{i_r}}{\partial y^n}. \qquad (3.9)
$$

After performing the $\partial/\partial y^n$ differentiation and some manipulations to return to the x coordinate system, he obtains

$$
Au = \frac{1}{p!} (A a_{i_1 \ldots i_p}) dx^{i_1} \wedge \ldots \wedge dx^{i_p} + d\alpha^i \wedge \frac{\partial u}{\partial (dx^i)}, \qquad (3.10)
$$

which is the formula universally considered to give the Lie derivative of a differential form. Although the formula (3.3) is then considered a theorem, the magic theorem, one should not overlook the fact that so is Eq. (3.10). It arises from "first principles" rather than constituting a definition.

The simple example of the 1-form $a_i dx^i$ will illustrate that the two terms on the right hand side of Eq. (3.3) do not correspond to the two terms on the right hand side of (3.10). Only their sums coincide. Using Eq. (3.3), we get

$$
A(a_i dx^i) = \alpha^k (a_{j,k} - a_{k,j}) dx^j + \alpha^k a_{k,j} dx^j + a_i \alpha^i_{,j} dx^j. \qquad (3.11)
$$

The second and third terms on the right hand side of (3.11) cancel each other out. The remaining two terms are, for $a_i dx^i$, the same as those on the right hand side of (3.10).

In Kähler's derivation there is no motivation for interpretations of the nature of the objects, called differential forms, that are the subject of his calculus. In other words, it is immaterial whether u represents a cochain or an antisymmetric multilinear function of vectors.

Assume finally that the Lie operator is simply $\partial/\partial x^n$. The $d\alpha^i$ are all zero. Only the first term on the right of (3.10) survives. This means that viewing the x^i $(i \neq n)$ as constants when partial differentiating with respect to x^n does not render zero the dx^i in the differential forms. We are "dragging the integrands" along the coordinate line x^n. We are not cancelling those integrands since the $dx^i = 0$ $(i \neq n)$ does not apply to them, and do not, therefore, vanish. The following derivation by Sledbozinski [12] of equation (3.10) will illustrate how the two terms on its right hand side come together in modifying those integrands as they are dragged along.

In order to avoid clutter in his derivation without summations over repeated indices, Sledbozinski makes r be two. To preserve the flavor of his derivation, we shall make only minor use of notation using such summations. Given the integrand $u = (1/2) a_{ij}(x) dx^i \wedge dx^j$, he parametrizes its domain of integration Δ by

$x^i = x^i(v^1, v^2)$. Let t be the parameter of the Lie group. Let $\xi^i(v^1, v^2, t)$ be the pull-back under the transformations in the group. The pull-back of the integral of $(1/2)a_{ij}dx^i \wedge dx^j$ then goes to

$$I(t) \equiv \frac{1}{2}\int_\Delta a_{ij}(\xi)\frac{\partial \xi^i}{\partial v^1}\frac{\partial \xi^j}{\partial v^2}dv^1 \wedge dv^2. \tag{3.12}$$

It is only a matter now of differentiating with respect to t under the integral sign, taking into account that

$$\left|\frac{\partial}{\partial t}a_{ij}(\xi)\right|_{t=0} = \left|\frac{\partial \xi^k}{\partial t}\frac{\partial}{\partial \xi^k}a_{ij}(\xi)\right|_{t=0} = \alpha^k\frac{\partial a_{ij}(x)}{\partial x^k}, \tag{3.13}$$

and that

$$\left|\frac{\partial}{\partial t}\frac{\partial \xi^i}{\partial v^A}\right|_{t=0} = \left|\frac{\partial}{\partial v^A}\frac{\partial \xi^i}{\partial t}\right|_{t=0} = \frac{\partial \alpha^i}{\partial v^A} = \frac{\partial \alpha^i}{\partial x^k}\frac{\partial x^k}{\partial v^A}. \tag{3.14}$$

In this way, one obtains for $u = (1/2)a_{ij}(x)dx^i \wedge dx^j$ that

$$\left|\frac{dI(t)}{dt}\right|_{t=0} \equiv \int Au, \tag{3.15}$$

with Au given by expression (3.10) specialized to $u = (1/2)a_{ij}(x)dx^i \wedge dx^j$. What this derivation shows is that the domain of integration in the x plane has changed, but not in the v plane. We can state this in the following alternative way. If we consider the x's as functions of the v's without change of domain, the coefficients of $dx^i \wedge dx^j$ in the integrand (i.e. in the differential form) have changed in the process of having the latter being dragged along.

4 Lie operators on metrically endowed manifolds

Equation (3.3) is in ostensibly covariant form. Equation (3.10) is not, in the sense that the terms on the right hand side are not covariant; only their sum is. When there is a metric, one can obtain still another expression for Au which is the sum of two covariant terms, as Kähler showed. He proceeded as follows. He introduced the concept of covariant derivative d_iu of u, for any tensor-valued differential form on any differentiable manifold endowed with a metric-compatible affine connection [9], [8] (Kähler uses only the Levi-Civita connection, but his concept is readily generalizable to any affine connection). The Clifford product $dx^i \vee d_iu$ yields the exterior and interior derivatives of u, denoted du and δu, through

$$dx^i \vee d_iu = du + \delta u, \tag{4.1}$$

where

$$du \equiv dx^i \wedge d_iu, \tag{4.2}$$

$$\delta u \equiv dx^i \cdot d_iu. \tag{4.3}$$

Kähler shows that, when the connection is Levi-Civita's, δu is the coderivative.

Equation (3.10) did not require that the differentiable manifold be endowed with a metric. If we have a metric, we have a Clifford algebra of differential forms. Said structure allows us to rewrite Eq. (3.10) as the following sum of two invariant forms:

$$Au = \alpha^i d_i u + (d\alpha)^i \wedge e_i u, \tag{4.4}$$

where $e_i u$ is short for $\dfrac{\partial u}{\partial(dx^i)}$, and where $(d\alpha)^i$ is defined as $d(\alpha^i) + \omega^i_k \alpha^k$. Although Kähler uses the Levi-Civita connection, the same computation can be performed with any other metric compatible affine connection.

Notice that the last term in Eq. (4.4) vanishes when u is a 0-form, f. Hence, although the first term on the right, $\alpha^i d_i u$, is always covariant and is simply the Lie derivative when u is the 0-form f, it does not generalize the concept of Lie derivative from the ring of functions to the ring of differential forms. The covariant operator$(d\alpha)^i \wedge e_i$ completes the expression for the Lie operator acting on functions, so that it becomes the Lie operator on differential forms. This is the key to why there is a concept of spin. We shall also see that the extra term is null for Lie derivatives associated with time and space translations.

5 Killing operators for differential forms

As can be expected, we use the term Killing operator to refer to Lie operators which, by virtue of a property of the metric, have a Killing equation associated with them. One first constructs the differential form

$$\alpha \equiv \alpha_k dx^k = g_{ik}\alpha^i dx^k, \tag{5.1}$$

attached to the Lie operator which, when acting on scalar-valued 0-forms, takes the form $\alpha^i \dfrac{\partial}{\partial x^i}$. We need the specific form of the covariant derivative of a differential 1-form α $(= \alpha_k dx^k)$. We obtain it from "first principles" by writing α in terms of non-coordinate bases of (soldering) forms, $\alpha = \alpha_k \omega^k$. One then exterior differentiates this equation and gets $d\omega^k$ from the first equation of structure with null torsion to obtain

$$d_l \alpha = (\alpha_{h,l} - \alpha_k \Gamma^{\cdot k}_{hl}) dx^h, \tag{5.2}$$

where the $\Gamma^{\cdot k}_{hl}$ are the Christoffel symbols. We have used that, in particular, the equation thus obtained applies when $\omega^k = dx^k$. Defining $d_l \alpha_h$ as the coefficient of dx^h in the expression for $d_l \alpha$, we get

$$d_l \alpha_h = \alpha_{h,l} - \alpha_k \Gamma^{\cdot k}_{hl}, \tag{5.3}$$

which are the components in the tensor calculus of the covariant derivative of a vector field of covariant components α_h. We readily have:

$$\begin{aligned} d_i\alpha_k + d_k\alpha_i &= \alpha_{k,i} + \alpha_{i,k} - \alpha_l(\Gamma^{\cdot l}_{ki} + \Gamma^{\cdot l}_{ik}) = \\ &= (\alpha^j g_{jk})_{,i} + (\alpha^j g_{ji})_{,k} - 2\alpha^l \Gamma_{kli}. \end{aligned} \tag{5.4}$$

In coordinate systems where $\alpha^i = 0$ for $i = 1, \cdots, n-1$, and $\alpha^n = 1$, the last equation becomes

$$d_i\alpha_k + d_k\alpha_i = g_{kn};_i + g_{in};_k - 2\Gamma_{kni} = g_{ik};_n . \tag{5.5}$$

If the metric does not depend on x^n, the Killing relation

$$d_i\alpha_k + d_k\alpha_i = 0, \tag{5.6}$$

results. One can then show [9] that equation (4.4) can be given the form

$$Au = \alpha^i d_i u + \frac{1}{4} d\alpha \vee u - \frac{1}{4} u \vee d\alpha. \tag{5.7}$$

The last two terms constitute the extra contribution absent in the Lie derivative of 0-forms.

6 The rotation operator for differential forms

Consider next the 1-parameter group of rotations in Euclidean 3-space. In terms of Cartesian coordinates, we define

$$w_i \equiv dx^k \wedge dx^\ell = dx^i \vee w = w \vee dx^i, \tag{6.1}$$

where (i, k, ℓ) is $(1, 2, 3)$ or a cyclic permutation thereof, and where

$$w \equiv dx^1 \wedge dx^2 \wedge dx^3 = dx^1 \vee dx^2 \vee dx^3. \tag{6.2}$$

The w_i satisfy the relations

$$w_i \vee w_i = -1 \tag{6.3}$$

$$w_i \vee w_k = -w_k \vee w_i = -w_\ell. \tag{6.4}$$

These equations are similar to the relations satisfied by the spin matrices σ^i. One readily obtains $d\alpha(i) = 2w_i$, where the $\alpha(i)$ are the three 1-forms $x^k dx^\ell - x^\ell dx^k$ that correspond to the operators $x^k\partial_\ell - x^\ell\partial_k$, as per equation (5.1). Thus the three operators A_i that one reads from equation (4.4) can now be written as

$$A_i = x^k \frac{\partial}{\partial x^\ell} - x^\ell \frac{\partial}{\partial x^k} + \frac{1}{2} w_i \vee \_ - \frac{1}{2} \_ \vee w_i, \tag{6.5}$$

where the underscore shows where to place the differential form upon which the Lie operator is made to act.

We shall compare (6.5) with a corresponding result which uses differential forms, but in the abstract context of group theory representations. In $\sum_{i=1}^{2} f_i(x) d\xi_i$, let $f_i(x)$ be defined on R_3 and let $\xi^i \in C_2$. The ξ^i form a basis on which $SU(2)$ acts. The generators of infinitesimal displacements (rotations) on that differential form are given by [6]

$$J_i = x^k \frac{\partial}{\partial x^\ell} - x^\ell \frac{\partial}{\partial x^k} + \frac{i}{2}\sigma_i. \tag{6.6}$$

The unnecessary variety of concepts from group representation theory entering the description of what J_i acts upon and the also unnecessary appearance of the unit imaginary (not present in (6.5)) speaks of the far greater naturalness and transparency of the Kähler approach to the formal treatment of spin.

7 Concluding remarks

The Kähler approach to spin shows that it has to do with the dragging of a differential form along curves defined by a 1-parameter group of rotations. Hence, it has to do with a local property. It is thus seemingly unrelated to the global property of existence of four connected components of the Lorentz group. The same conclusion appears to follow from the fact that the representations of the Lorentz group that are unitary, thus infinite dimensional, are determined by mass and energy though an argument that does not involve global properties of the Lorentz group [10]. For completeness purposes, we show with the Kähler calculus how spin, which is related to rotational symmetry, connects with mass (i.e. rest energy), which is related to time translations. We need to make use of an equation known as the Kähler equation; it supersedes Dirac's, and plays the central role in applications of the Kähler calculus.

Consider the constant idempotents

$$\epsilon^{\pm} \equiv \frac{1}{2} \mp \frac{i}{2}dt, \qquad \tau^{\pm} = \frac{1}{2}(1 \pm idx \wedge dy), \tag{7.1}$$

built with signature (-1,1,1,1). A differential c is said to be constant if $d_i c$ equals zero for all i. If u is a solution of a Kähler equation, $\partial u = a \vee u$, so is $u \vee c$ for the same equation, i.e. $\partial(u \vee c) = a \vee (u \vee c)$. This type of property can be expected of other related equations, like those for strict harmonic, $\partial u = 0$, and harmonic differentials, $\partial\partial u = 0$.

The differential forms $\tau^{\pm} \vee \epsilon^{\pm}$ and $\tau^{\pm} \vee \epsilon^{\mp}$ are four mutually annulling, constant, primitive idempotents (An idempotent is said to be primitive if it cannot be written as a sum $A + B$ of two mutually annulling idempotents). Because of those properties, any solution u of a Kähler equation can be written as a sum

$$u = u \vee \tau^{+} \vee \epsilon^{+} + u \vee \tau^{-} \vee \epsilon^{+} + u \vee \tau^{+} \vee \epsilon^{-} + u \vee \tau^{-} \vee \epsilon^{-}, \tag{7.2}$$

of four spinorial (meaning here members of minimal ideals) solutions of the same Kähler equation. Again, this property may apply to the solutions of other equations on the same grounds. In general, the four solutions are not independent; they are entangled by the presence of the common factor u, which has 32 independent real components. If there were rotational and time reflection symmetries, u could be written as a sum

$$u = u_1 \vee \tau^{+} \vee \epsilon^{+} + u_2 \vee \tau^{-} \vee \epsilon^{+} + u_3 \vee \tau^{+} \vee \epsilon^{-} + u_4 \vee \tau^{-} \vee \epsilon^{-}, \tag{7.3}$$

of four independent solutions, each dependent on only eight real components [8]. The positive and negative values of energy are represented here by ϵ^{+} and ϵ^{-}, except that they rather represent positive and negative charge, but with the same sign for energy. Opposite energies for particles and antiparticles are a spurious effect of Dirac's theory. τ^{+} and τ^{-} are associated with opposite signs of internal rotation in the electron, or in the positron. Hence equation (7.3) appears to be related to the fact that the unitary representations of the inhomogeneous Lorentz group are what they are known to be. The fact that in representation theory there are no more eigenvalues associated with external symmetries seems to be a reflection of the fact that the idempotents $\tau^{\pm} \vee \epsilon^{\pm}$ and $\tau^{\pm} \vee \epsilon^{\mp}$ are primitive. If they were not, there would be room for further decomposition into ideals and concomitant emergence of additional spacetime eigenvalues. Of course, this is a limitation of representation theory. Here again, the Kähler calculus allows one to see how one could in principle give rise to a richer structure of spacetime related idempotents, as we now intimate.

The argument above concerns solutions u of a Kähler equation, $\partial u = a \vee u$, where a is scalar-valued and, as a result, also is u. But energy-momentum is vector-valued. This is indicative of the fact that, if the basic equation of quantum physics is a Kähler equation, the input differential form should be of valuedness other than scalar, so that the vector-valuedness of energy-momentum may emerge spontaneously from the basic equation of quantum mechanics. Since ∂ is scalar-valued, the solution u must be an inhomogeneous differential form if a not scalar-valued. One thus may have the tensor product of two algebras, namely the one for differential forms that we have dealt with in this paper, and, ideally, another Clifford algebra, namely one of valuedness. It is then a matter of considering either the primitive idempotents or corresponding concepts in the tensor product of the two Clifford algebras, not in just one of them. To make matters even more complicated, it may be the case that the role of the idempotents is perhaps to be played by some other concept in the Clifford algebra of valuedness. The issue of effective elementary geometries may or may not be closed by the considerations of the present note; the issue of the number of spacetime eigenvalues defining a quantum state certainly is not closed if the Kähler calculus is the true calculus of the physics.

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