The tangents and the chords of the Poincaré circle

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Abstract. After defining the tangent and chord concepts, we determine the chord length and tangent lines of Poincaré circle and the relationship between the tangent lines and the chords of a Poincaré circle.

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1 Introduction

The Poincaré upper half plane geometry has been introduced by Henri Poincaré. The Poincaré upper half plane \( \mathbb{H} \) is the upper half plane of the Euclidean analytical plane \( \mathbb{R}^2 \). Although the points in Poincaré upper half plane are same as the points in the upper half plane of the Euclidean analytical plane \( \mathbb{R}^2 \), the lines and the distance function between any two points are different. The lines in the Poincaré upper halfplane are defined by

- half lines: \( \quad \left\{ (x, y) \in \mathbb{R}^2 \mid x = a, \ y > 0, \ a \in \mathbb{R}, \ a \text{ is a constant} \right\} \)
- half circles: \( \quad \left\{ (x, y) \in \mathbb{R}^2 \mid (x - c)^2 + y^2 = r^2, \ y > 0, \ c, r \in \mathbb{R}, c, r \text{ are constants} \right\} \).

If \( A = (x_1, y_1) \) and \( B = (x_2, y_2) \) are any two points in \( \mathbb{H} \) then the Poincaré distance between these points is given by

\[
    d_H(A, B) = \begin{cases} 
    \ln \left( \frac{y_2}{y_1} \right), & x_1 = x_2 \\
    \ln \left( \frac{y_2(x_1 - c + r)}{y_1(x_2 - c + r)} \right), & x_1 \neq x_2,
    \end{cases}
\]

where

\[
    c = \frac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2(x_2 - x_1)}
\]

\[
    r = \sqrt{(x_1 - c)^2 + y_1^2} = \sqrt{(x_2 - c)^2 + y_2^2}
\]

The Poincaré upper half plane geometry is a non-Euclidean, since it fails to satisfy the parallel postulate within the thirteen axioms \([2, 3, 5]\) of the Euclidean plane geometry. In Poincaré upper half plane geometry, the lines and the function of distance...
are different, therefore, it seems interesting to study the Poincaré analogues of the topics that include the concept of distance in the Euclidean geometry. A few of such topics have been studied by some authors, see [1, 3, 4, 5, 6, 7, 8]. One of these is a study in [5, pp. 80–81] that is about every Poincaré circle in the upper half plane is an Euclidean circle. In this work, I defined tangent and chord concepts. Then, the chord length and tangent lines of Poincaré circle and the relationship between the tangent lines and the chords of a Poincaré circle have been determined.

In [5], Stahl showed the following results concerning Poincaré circle and Euclidean circle.

**Proposition 1.1.** If a circle Euclidean center \((h, k)\) and a Euclidean radius \(r\), then it has the hyperbolic center \((H, K)\), and the hyperbolic radius \(R\), where

\[
H = h, \quad K = \sqrt{k^2 - r^2}, \quad R = \frac{1}{2} \ln \left( \frac{k + r}{k - r} \right)
\]

and

\[
h = H, \quad k = K \cosh (R), \quad r = K \sinh (R)
\]

**Theorem 1.** Every Euclidean circle in the upper half-plane is also a hyperbolic circle. Since the equation of a circle with Euclidean center \((h, k)\) and an Euclidean radius \(r\) is \((x - h)^2 + (y - k)^2 = r^2\) the equation of a circle in the Poincaré upper half plane \(H\) is

\[
(x - H)^2 + (y - K \cosh (R))^2 = K^2 \sinh^2 (R)
\]

**Proposition 1.2.** Let \(q\) is a circle with Euclidean center \((h, k)\) and Euclidean radius \(r\). The hyperbolic length of \(q\) is

\[
\zeta = \frac{2\pi r}{\sqrt{k^2 - r^2}} = \frac{2\pi r}{K}
\]

2 The Chords of Poincaré Circle

**Definition 2.1.** A line segment that joins two points on the Poincaré circle is called as chord.

![Fig.1. Some chords](image1)
![Fig.2. Some diameters](image2)
If \( X_1 \) and \( X_2 \) are points on the Poincaré circle \( Q (X_1 \neq X_2) \) then the length of the chord \( X_1X_2 \) is equal to the length of Poincaré line which is joined the points \( X_1 \) and \( X_2 \) on the Poincaré circle \( Q \).

Let \( X_1 = (x_1, y_1) \) and \( X_2 = (x_2, y_2) \) \((y_1, y_2 > 0)\) are points on the Poincaré circle \( Q \) with hyperbolic center \((H, K)\) and hyperbolic radius \( R \). Since the equation of Poincaré circle is

\[
(x - H)^2 + (y - K \cosh (R))^2 = K^2 \sinh^2 (R)
\]

(see Theorem 1) the coordinates of the points \( X_1 \) and \( X_2 \) are defined

\[
X_1 = \left(x_1, K \cosh (R) \pm \sqrt{K^2 \sinh^2 (R) - (x_1 - H)^2}\right)
\]

\[
X_2 = \left(x_2, K \cosh (R) \pm \sqrt{K^2 \sinh^2 (R) - (x_2 - H)^2}\right)
\]

(A) If \( x_1 \neq x_2 \).

(i) Let \( y_1 \neq y_2 \). Then

\[
y_1 = K \cosh (R) + \sqrt{K^2 \sinh^2 (R) - (x_1 - H)^2}
\]

or

\[
y_1 = K \cosh (R) - \sqrt{K^2 \sinh^2 (R) - (x_1 - H)^2}
\]

and

\[
y_2 = K \cosh (R) + \sqrt{K^2 \sinh^2 (R) - (x_2 - H)^2}
\]

or

\[
y_2 = K \cosh (R) - \sqrt{K^2 \sinh^2 (R) - (x_2 - H)^2}
\]

and

\[
d_H (X_1, X_2) = \left| \ln \left( \frac{y_2 (x_1 - d + s)}{y_1 (x_2 - d + s)} \right) \right|
\]

\[
d = \frac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2 (x_2 - x_1)}
\]

\[
s = \sqrt{(x_1 - d)^2 + y_1^2} = \sqrt{(x_2 - d)^2 + y_2^2}.
\]

(ii) Let \( y_1 = y_2 \)

\[
d_H (X_1, X_2) = \left| \ln \left( \frac{x_1 - d + s}{x_2 - d + s} \right) \right|
\]

\[
d = \frac{x_1 + x_2}{2}
\]

\[
s = \sqrt{(x_1 - d)^2 + y_1^2}
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(B) If $x_1 = x_2$.

\[
\begin{align*}
X_1 &= \left( x_1, K \cosh (R) - \sqrt{K^2 \sinh^2 (R) - (x_1 - H)^2} \right), \\
X_2 &= \left( x_2, K \cosh (R) + \sqrt{K^2 \sinh^2 (R) - (x_2 - H)^2} \right)
\end{align*}
\]

\[
d_H (X_1, X_2) = \ln \left( \frac{y_2}{y_1} \right) = \ln \left( \frac{K \cosh (R) + \sqrt{K^2 \sinh^2 (R) - (x_1 - H)^2}}{K \cosh (R) - \sqrt{K^2 \sinh^2 (R) - (x_1 - H)^2}} \right)
\]

Also, since the hyperbolic length of a circle is $2\pi R/\sqrt{k^2 - r^2}$ the length of Poincaré arc $X_1X_2$ is

\[
d_H (X_1, X_2) = \frac{2\pi r}{360\sqrt{k^2 - r^2}}
\]

(see Proposition 1.2).

**Definition 2.2.** If the length of a chord is equal to the length of a diameter then it is called as long chord.

(C) If $X_1 = A_1$ and $X_2 = A_2$ then the length of a long chord is

\[
d_H (X_1, X_2) = \ln \left( \frac{y_2}{y_1} \right) = \ln \left( \frac{K \cosh (R) + \sqrt{K^2 \sinh^2 (R)}}{K \cosh (R) - \sqrt{K^2 \sinh^2 (R)}} \right) = \ln \frac{k + r}{k - r} = 2R
\]

(see the following figure).

(D) If $X_1 = B_1$ and $X_2 = B_2$ then the length of a long chord is as follows.

(i) Let $d = H$ ($x_1 \neq x_2$ and $y_1 = y_2$). Then

\[
\begin{align*}
d_H (X_1, X_2) &= \ln \left( \frac{x_1 - d + s}{x_2 - d + s} \right) = \ln \left( \frac{x_1 - H + s}{x_2 - H + s} \right) \\
s &= \sqrt{(x_1 - d)^2 + y_1^2} = \sqrt{(x_1 - H)^2 + y_1^2}
\end{align*}
\]
(ii) Let $d \neq H$ ($x_1 \neq x_2$ and $y_1 \neq y_2$). Then

$$d_H (X_1, X_2) = \ln \left( \frac{y_2 (x_1 - d + s)}{y_1 (x_2 - d + s)} \right)$$

$$d = \frac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2 (x_2 - x_1)}$$

$$s = \sqrt{(x_1 - d)^2 + y_1^2} = \sqrt{(x_2 - d)^2 + y_2^2}$$

(see the following figures).

In two cases, since we can find to the hyperbolic reflection that transforms respectively the points $B_1$ and $B_2$ on to the points $A_1$ and $A_2$ [5], the distance between the points is

$$d_H (X_1, X_2) = \ln \left( \frac{k + r}{k - r} \right) = 2R$$

3 The Tangents of Poincaré Circle

In the Poincaré upper half plane, the tangent line is Euclidean line that is a line through a point on a circle [3]. We know that the tangent line that is a line through every point on a circle is perpendicular to diameter. Now, we can examine this condition.

Let $Q$ is a circle with hyperbolic center $C = (H, K)$ and hyperbolic radius $R$.

1 $x = H + R$ is the tangent line that is a line through a point $A_1 = (x_1, y_1) = (H + R, K)$ and is perpendicular to diameter. The chord that is perpendicular to this tangent is the line $H + RL_P$ that is center $M = (H + R, 0)$ and radius $P$ [4]. If $C = (H, K)$ is replaced in equation of line then

$$\sqrt{P^2 - (H + (H + R))^2} = \sqrt{P^2 - R^2} \neq K$$

2 $x = H - R$ is the tangent line that is a line through a point $A_2 = (H - R, K)$ and is perpendicular to diameter. The chord that is perpendicular to this tangent
is the line $H - RL_S$ that is center $M = (H - R, 0)$ and radius $S$. If $C = (H, K)$ is replaced in equation of line then

$$(x - (H - R))^2 + y^2 = S^2$$

$$\sqrt{S^2 - (H - (H - R))^2} = \sqrt{S^2 - R^2} \neq K$$

In two cases, center isn’t on the chord and also the chord isn’t diameter. But the chord that is perpendicular to tangent is long chord and $\|H - X\| = 2R = d_H(X, A_1) = d_H(X, A_2)$, $X \in Q$ and $d_H(C, A_1) = R = d_H(C, A_2)$.

(3) $y = K + R$ is the tangent line that is a line through a point $A_3 = (H, K + R)$ and is perpendicular to diameter. As the same as $y = K - R$ is the tangent line that is a line through a point $A_4 = (H, K - R)$ and is perpendicular to diameter. In two cases the chord that is perpendicular to this tangents is the line $HL$ and for the point $C = (H, K)$ since $x = H$ the chord is diameter. In this condition $d_H(C, A_3) = R = d_H(C, A_4)$.

**Corollary 3.1.** The tangents $x = H + R$, $x = H - R$, $y = K + R$ and $y = K - R$ are perpendicular to the Poincaré diameter.

**References**


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