Immersed hypergroups

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Abstract. In this paper the notion of immersed hypergroup is considered. It is proved that the product of two immersed hypergroups is an immersed hypergroup. We show that any open subset of immersed hypergroup is an immersed hypergroup, and the image of any immersed hypergroup is an immersed hypergroup. Quotient hypergroups are considered and it is proved that they are immersed hypergroups.


Key words: Immersed hypergroup, quotient hypergroup, hypergroupoid.

1 Introduction

Theory of hypergroups have brought many bright results in mathematics [1, 4, 5, 6]. In this paper we would like to consider hypergroups which are also smooth manifolds and their join operations create immersed submanifolds. More precisely we have the next definition.

Definition 1.1. A hypergroup $H$ is called an immersed hypergroup, if $H$ is smooth real manifold and for all $a, b \in H$, $ab$ is an immersed submanifold of $H$.

We will prove the following results:
(1) If $H_1$ and $H_2$ are immersed hypergroups, then $H_1 \times H_2$ is an immersed hypergroup.
(2) If $H$ is an immersed hypergroup, then any open subset $U$ of $H$ is an immersed hypergroup.
(3) If $H$ is an immersed hypergroup and $\Psi$ is an immersion on $H$, then $\Psi(H)$ is an immersed hypergroup.

If $H$ is an immersed hypergroup and $\{A(x)\}$ is a family of open subsets of a differentiable manifold $N$, then we will present two methods to construct new immersed hypergroups.

2 Immersed hypergroups

We begin this section with an example of an immersed hypergroup.

Example 2.1. Let $H$ be the set of polynomials $P_n : [0, 1] \rightarrow [0, 1]$ such that $x \mapsto a_n x^n + ... + a_0$, where $a_i \in [0, 1]$ and additions and multiplications are in the...
mode 1.

$H$ with the norm $\|P_n(x)\| = \text{sup}\{|P_n(x)| : x \in [0, 1]\}$ is a Banach space, which is also
an infinite dimensional real manifold.

If $P_n(x) = a_0 + a_1x + \ldots + a_nx^n$ and $P_m(x) = b_0 + b_1x + \ldots + b_mx^m$, then we define
$P_n(x)P_m(x) = \{a_0b_0 + a_1c_1x + \ldots + c_{m+n-1}x^{m+n-1} + a_nb_m : c_i \in [0, 1]\} \equiv [0, 1]^{m+n-1}$.

$H$ with the above differentiable structure is a hypergroup.

**Example 2.2.** Let $H$ be the set of $k$–dimensional subspaces of $\mathbb{R}^n$, with $k \leq n$, if $a, b \in H$, then we define $ab = \{c | c$ is a $k$-dimensional subspace of $a + b\}$. $H$ with this join operation and the above differentiable structure is an immersed
hypergroup.

**Theorem 2.1.** Let $H_1$ and $H_2$ be immersed hypergroups. Then $H_1 \times H_2$ is an
immersed hypergroup.

**Proof.** We know that $H = H_1 \times H_2$ is a smooth manifold. We only show that for given
$(a, b), (b', b'') \in H$, $a_1b_1 \times a_2b_2$ is an immersed submanifold of $H$.

Since $i_1 : a_1b_1 \to H_1$ and $i_2 : a_2b_2 \to H_2$ are immersions, then the map $i : a_1b_1 \times a_2b_2 \to H_1 \times H_2$ defined by
$i(c_1, c_2) = (i_1(c_1), i_2(c_2))$ is an immersion as well. Because it is a one-to-one map and $di(c_1, c_2) = di_1(c_1) \times di_2(c_2)$, which is a one-to-one
map. \hfill $\Box$

**Theorem 2.2.** If $H$ is an immersed hypergroup. Then any open subset $U$ of $H$
with the join operation $\ast_U$ on $U \times U$ defined by $a \ast_U b = ab \cap U$ is an immersed
hypergroup.

**Proof.** $U$ with the atlas $A_U = (\text{smooth charts} (V, \Theta) \text{ of } H : V \subset U)$ is a smooth
manifold. Moreover, it is a hypergroup [1, 4]. If $a, b \in H$, then $i : ab \to H$ is an immersion. Hence $i_U : ab \cap U \to U$ is also an immersion. Thus $U$ is an immersed
hypergroup. \hfill $\Box$

**Theorem 2.3.** If $H$ is an immersed hypergroup, and $\Psi : H \to \mathbb{R}^k$ is a smooth
one-to-one immersion, then $\Psi(H)$ with the join operation $\Psi(a)\Psi(b) = \Psi(ab)$ is an
immersed hypergroup, where the smooth structure of $\Psi(H)$ is the smooth structure
induced by $\Psi$.

**Proof.** $\Psi(H)$ is a hypergroup, because $\Psi(a)\Psi(H) = \Psi(aH) = \Psi(Ha) = \Psi(H)\Psi(a) = \Psi(H)$, for all $a \in H$. Then $\Psi(a)\Psi(b)c = \Psi(ab)c = \Psi(ab)\Psi(c) = \Psi(a)\Psi(b)\Psi(c)$, for all $a, b, c \in H$. Moreover, $\Psi(H)$ with the smooth structure created by $\Psi$ as a diffeomorphism, is a smooth manifold.

Let $\Psi(a)$ and $\Psi(b) \in \Psi(H)$ be given. Then there exists an immersion $i : ab \to H$. Hence the mapping $\Psi_0 : ab \to \mathbb{R}^k$ is an immersion, and $\Psi(ab) = \Psi_0(ab)$ is an
immersed submanifold of $\mathbb{R}^k$. \hfill $\Box$

Suppose $(H, o)$ is a hypergroupoid ([1]) and assume that $\{A(x)\}$ is a family of
pairwise disjoint non-empty sets and $K_H = \cup_{x \in H}A(x)$. We define $g(a) = x$ if and
only if $a \in A(x)$, for all $a \in K_H$. Now in $K_H$ we have the hyperoperation:

$a \circ b = \cup_{z \in g(a)\circ g(b)}A(z)$, where $(a, b) \in K_H^2$.

It is well known that $(H, o)$ is a hypergroup if and only if $(K_H, \circ)$ is a hypergroup
([1]).

**Theorem 2.4.** Let $(H, o)$ be an immersed hypergroup and let $\{A(x)\}$ be a family
of pairwise disjoint non-empty open subsets of a differentiable manifold $N$. Then $K_H$
is an immersed hypergroup.
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Proof. Since $A(x)$ is open, $K_H$ is an open submanifold of $N$. Thus $a \circ b$ is also an open submanifold of $K_H$, for all $(a,b) \in K_H^2$. So it is an immersed submanifold of $K_H$. □

3 Quotient immersed hypergroups

Let $M$ be a differentiable manifold and $R \subseteq M \times M$ be an equivalence relation on $M$. Moreover let $M/R$ be the set of the equivalence classes of $M$ with respect to $R$. Let $P : M \to M/R$ be the natural projection $m \mapsto [m]$. We define the quotient topology on $M/R$, i.e., $U \subseteq M/R$ is open if and only if $P^{-1}(U)$ is open in $M$. Thus $P : M \to M/R$ is a continuous map. Moreover, for any continuous map $F : M \to N$, there exists an unique continuous map $\tilde{F} : M/R \to N$ such that $F = \tilde{F} \circ P$. Hence we have the following commutative diagram:

In general, $M/R$ may not be a manifold. We now assume that $M/R$ has a differentiable structure such that $P : M \to M/R$ is a submersion. Since $P$ is a continuous map, for all open set $U$ in $M/R$, $P^{-1}(U)$ is an open set in $M$. Moreover, $P$ is an open map, because it is a submersion [4]. Thus $U = P(P^{-1}(U))$ is open in $M/R$, for all open sets $U$ in $M/R$. Therefore a subset $U$ in $M/R$ is open if and only if $P^{-1}(U)$ is open in $M$, i.e., the topology on $M/R$ is the quotient topology.

Moreover, if the map $F$ from $M$ into a differentiable manifold $N$ is differentiable, then the map $\tilde{F} : M/R \to N$ is also differentiable, and the differentiable structure on $M/R$ is unique [4].

$M/R$ is called the quotient manifold of $M$ with respect to $R$, if it is allows a differentiable structure such that $P : M \to M/R$ is a submersion. In this case, the equivalence relation is called regular. In fact, $R$ is regular if and only if $R$ is a closed submanifold of $M \times M$ and the relations are defined by $P_1 (m_1, m_2) = m_1$ and $P_2 (m_1, m_2) = m_2$ are submersions.

Let $(H, o)$ be a hypergroupoid and let $\rho$ be an equivalence relation on $H$. $\rho$ is called regular on the right if the following implication holds:

$apb$ then for all $u \in H, x \in aou$ and there exists $y \in bau$ such that $xpy$ and for all $y \in bau$, there exists $x \in aou$ such that $xpy$.

Similarly, the regularity on the left can be defined.

Now we state the main theorem of this section.

**Theorem 3.1.** Let $(M, o)$ be an immersed hypergroup and $\rho$ be a left and right regular equivalence relation on $M$. Moreover, let $P : M \to M/\rho$ be a submersion. Then $M/\rho$ with respect to the join operation, $[x] \circ [y] = \{[z] : z \in xoy\}$ is an immersed hypergroup.
Proof. Since $M$ is an immersed hypergroup, there exists an immersion $i : xoy \to M$, for all $x, y \in M$. Let $[x]$ and $[y] \in M/\rho$ be given. Then the inclusion map $i : x_0y_0 \to M$ is an immersion, for each $x_0 \in [x]$ and $y_0 \in [y]$. The differentiable structure of $M/\rho$ implies for given $[m] \in [x] \otimes [y]$ there exist the open sets $V$ and $W$ of $M$ and a diffeomorphism $\beta : V/\rho V \to W$, where $\rho V = \rho \cap (V \times V)$.

Now the inclusion map $i : ([x] \otimes [y]) \cap V/\rho V \to M/\rho$ is equal to the map $i \circ \beta$ restricted to $([x] \otimes [y]) \cap V/\rho V$. Thus it is an immersion.

Moreover, since $\rho$ is left and right regular equivalence relation, then $M/\rho$ is a hypergroup [1]. Thus $M/\rho$ is an immersed hypergroup.

References


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