Structure on a slant submanifold of a Kenmotsu manifold

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Abstract. We give an intrinsic characterization of slant submanifolds of a Kenmotsu manifold in terms of the induced metric and show that a slant submanifold of a Kenmotsu manifold is a Kenmotsu manifold. We also prove a theorem to obtain examples of slant submanifolds of Kenmotsu manifold.

Key words: Kenmotsu manifold, slant submanifold, mean curvature and sectional curvature.

§1. Introduction

The notion of a slant submanifold of an almost Hermitian manifold was introduced by Chen [6], [5]. Examples of slant submanifolds of $\mathbb{C}^2$ and $\mathbb{C}^4$ were given by Chen and Tazawa ([5], [8], [7]), while that of slant submanifolds of a Kaehler manifold were given by Maeda, Ohnita and Udagawa [?]. On the other hand, A. Lotta [13] has defined and studied slant submanifolds of an almost contact metric manifold. He has also studied the intrinsic geometry of 3-dimensional non-anti-invariant slant submanifolds of K-Contact manifolds [14]. Latter, L. Cabrerizo and others investigated slant submanifolds of a Sasakian manifold and obtained many interesting results ([3], [4]). Recently, we have studied slant submanifolds of Kenmotsu manifolds and trans-Sasakian manifolds ([10], [11]).

§2. Preliminaries

Let $\bar{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors $\{\varphi, \xi, \eta, g\}$, where $\varphi$ is a $(1,1)$ tensor field, $\xi$ a vector field, $\eta$ a 1-form and $g$ the Riemannian metric on $\bar{M}$. These tensors satisfy [1]

\[ \varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\varphi X) = 0 \quad \text{and} \]
\[ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \]
for any $X, Y \in T\mathcal{M}$, where $T\mathcal{M}$ denotes the Lie algebra of vector fields on $\mathcal{M}$. An almost contact metric manifold is called a Kenmotsu manifold if

$$\nabla_X \phi(Y) = g(\phi X, Y) \xi - \eta(Y) \varphi X \quad \text{and} \quad \nabla_X \xi = X - \eta(X) \xi$$

where $\nabla$ denotes the Levi-Civita connection on $\mathcal{M}$.

Let $M$ be an $m$-dimensional Riemannian manifold with induced metric $g$ isometrically immersed in $\mathcal{M}$. We denote by $TM$ the Lie algebra of vector fields on $M$ and by $T^\perp M$ the set of all vector fields normal to $M$. For any $X \in TM$ and $N \in T^\perp M$, we write

$$\varphi X = PX + FX, \quad \varphi N = tN + fN$$

where $PX$ (resp. $FX$) denotes the tangential (resp. normal) component of $\varphi X$, and $tN$ (resp. $fN$) denotes the tangential (resp. normal) component of $\varphi N$.

In what follows, we suppose that the structure vector field $\xi$ is tangent to $M$. Hence if we denote by $D$ the orthogonal distribution to $\xi$ in $TM$, we can consider the orthogonal direct decomposition $TM = D \oplus \xi$.

For each non zero $X$ tangent to $M$ at $x$ such that $X$ is not proportional to $\xi_x$, we denote by $\theta(X)$ the Wirtinger angle of $X$, that is, the angle between $\varphi X$ and $T_x\xi M$. The submanifold $M$ is called slant if the Wirtinger angle $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and $X \in T_xM - \{\xi_x\}$ ([13]). The Wirtinger angle $\theta$ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta$ equal to 0 and $\frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion.

Let $\nabla$ be the Riemannian connection on $M$. Then the Gauss and Weingarten formulae are

$$\nabla_X Y = \nabla_X Y + h(X, Y)$$

and

$$\nabla_X N = -A_N X + \nabla^\perp_X N$$

for $X, Y \in TM$ and $N \in T^\perp M$ of $\mathcal{M}$; $h$ and $A_N$ are the second fundamental forms related by

$$g(A_N X, Y) = g(h(X, Y), N)$$

and $\nabla^\perp$ is the connection in the normal bundle $T^\perp M$ of $M$.

The mean curvature vector $H$ is defined by $H = (\frac{1}{m}) \text{trace } h$. We say that $M$ is minimal if $H$ vanishes identically.
If $P$ is the endomorphism defined by (2.3), then
\begin{equation}
(2.7) \quad g(PX, Y) + g(X, PY) = 0
\end{equation}
Thus $P^2$, which is denoted by $Q$, is self-adjoint.

On the other hand, Gauss and Weingarten formulae together with (2.2) and (2.3) imply
\begin{align}
(2.8) \quad (\nabla_X P)Y &= A_{FY}X + th(X,Y) + g(Y, PX)\xi - \eta(Y)PX \\
(2.9) \quad (\nabla_X F)Y &= fh(X,Y) - h(X, PY) - \eta(Y)FX
\end{align}
for any $X, Y \in TM$.

We mention the following results for latter use:

**Theorem A.** [3] Let $M$ be a submanifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in TM$. Then, $M$ is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that
\begin{equation}
(2.10) \quad P^2 = -\lambda(I - \eta \otimes \xi)
\end{equation}
Furthermore, if $\theta$ is the slant angle of $M$, then $\lambda = \cos^2 \theta$.

**Corollary B.** [3] Let $M$ be a submanifold of an almost contact metric manifold $\overline{M}$ with slant angle $\theta$. Then for any $X, Y \in TM$, we have
\begin{align}
(2.11) \quad g(PX, PY) &= \cos^2 \theta(g(X, Y) - \eta(X)\eta(Y)) \\
(2.12) \quad g(FX, FY) &= \sin^2 \theta(g(X, Y) - \eta(X)\eta(Y))
\end{align}

**Lemma C.** [13] Let $M$ be a submanifold of an almost contact metric manifold $\overline{M}$ with slant angle $\theta$. Then, at each point $x$ of $M$, $Q|_D$ has only one eigenvalue $\lambda_1 = -\cos^2 \theta$.

§3. Intrinsic characterization of slant immersions of Kenmotsu manifolds

We now study intrinsic characterization of slant immersion of Kenmotsu manifold $\overline{M}$ in terms of slant angle of a slant submanifold $M$ and also the sectional curvature of arbitrary plane section of $M$ containing structure vector field $\xi$. We have:

**Lemma 3.1.** Let $M$ be a slant submanifold of a Kenmotsu manifold $\overline{M}$ such that structure vector field $\xi$ is tangent to $M$. Then curvature vector field associated to the metric induced by $\overline{M}$ on $M$ is given by
(3.1) \[ R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y) \]

Moreover,

(3.2) \[ R(\xi, X)\xi = X - \eta(X)\xi \]

and

(3.3) \[ R(X, \xi, \xi, X) = \eta^2(X) - g(X, X) \]

**Proof.** From equation (2.2), we have

(3.4) \[ \nabla_X \xi = X - \eta(X)\xi \]

for any \( X \in TM \). Further,

(3.5) \[ (\nabla_X P)Y = -\nabla_X \nabla_Y \xi + \nabla_{\nabla_X Y} \xi + 2\eta(X)\eta(Y)\xi - g(X, Y)\xi - \eta(Y)X \]

Substituting this formula in the definition of \( R(X, Y)\xi \) it is easy to get (3.1). Rewriting (3.1) for \( X = \xi \) and \( Y = X \) and using (3.5), we obtain

\[ R(\xi, X)\xi = X - \eta(X)\xi \]

which gives (3.3).

**Theorem 3.1.** Let \( M \) be an immersed submanifold of a Kenmotsu manifold \( \overline{M} \) such that \( \xi \) is tangent to \( M \). Then the following statements are equivalent:

(a) \( M \) is slant in \( \overline{M} \) with slant angle \( \theta \)

(b) For any \( x \) of \( M \) the sectional curvature of any 2-dimensional plane of \( T_x M \) containing \( \xi \) equals -1.

**Proof.** Assume that (a) is true. Then by Theorem (A), for any unit vector field \( X \in TM \) orthogonal to \( \xi \), we have

\[ QX = -\cos^2 \theta X \]

which by virtue of (3.3) yields

\[ R(X, \xi, \xi, X) = -1. \]

Let (b) hold and \( \cos \theta \neq 0 \). For any \( X \in TM \), we use the decomposition

\[ X = X_\perp + X_\parallel \]

where \( X_\parallel = g(X, \xi)\xi \). Then by the hypothesis

(3.6) \[ \frac{R(X_\parallel, \xi, X_\parallel, \xi)}{|X_\parallel|^2} = -1 \]

Now, if \( X \) is a unit vector field such that \( QX = 0 \), then from (3.3), we get

\[ |X_\parallel|^2 = -|X_\perp|^2 \]
that is, $|X^\perp|^2 = 0$ and hence $X = X_\xi$. This proves that
\begin{equation}
    \text{Ker}(Q) = \text{Span} \{\xi_x\}, \quad \forall x \in M. \tag{3.7}
\end{equation}
Moreover, let $X$ be a unit vector field such that $QX = \lambda_1X$, where $\lambda_1 : M \rightarrow \mathbb{R}$ is a smooth function and for any $x \in M, \lambda_1(x) = 0$. Obviously, $X$ is orthogonal to $\xi$, that is $X = X^\perp_\xi$ and using (3.3) and (3.6) it follows that $\lambda_1 = -\cos^2 \theta$.

We conclude that for any $x \in M$ the number $-\cos^2 \theta$ is the only eigenvalue of $Q$ different from 0. This fact together with (3.7) implies that $M$ is slant in $M$ with slant angle $\theta$.

Now, suppose that $\cos \theta = 0$ and let $X$ be an arbitrary unit vector field of eigenvectors of $Q$. Then $QX = \lambda_1X$, where $\lambda_1$ is a function on $M$. Now, equations (3.3) and (3.6) imply that $g(QX, X) = 0$, that is $\lambda_1 = 0$. Thus, we conclude that $Q = 0$, which means that $M$ is anti-invariant whereby proving (a).

\section*{§4. Structure on a slant submanifold}

In [5], Chen gives the notion of a Kaehlerian slant submanifold of an almost Hermitian manifold as a proper slant submanifold such that the tangential component $T$ of the almost complex structure $J$ is parallel, that is $\nabla T = 0$.

In fact, Kaehlerian slant submanifold is a Kaehlerian manifold with respect to the induced metric and with the almost complex structure given by $J = (\sec \theta)T$, where $\theta$ denotes the slant angle.

Let $M$ be a submanifold of a Kenmotsu manifold $\overline{M}$ such that $\xi$ is in $TM$. It is well known that if $M$ is an invariant submanifold, then the structure of $\overline{M}$ induces, in a natural way, a Kenmotsu structure over $M$. In this case the submanifold is usually called a Kenmotsu submanifold. The purpose of this paper is to study if we can obtain an induced Kenmotsu structure on a non-invariant slant submanifold.

In an almost contact case, we have

\begin{lemma}
Let $M$ be a non-anti-invariant slant submanifold of an almost contact metric manifold $\overline{M}$. Then, $M$ is an almost contact metric manifold with respect to the induced metric, with structure vector field $\xi$, and with almost contact structure given by $\mathcal{D} = (\sec \theta)P$, where $\theta$ denotes the slant angle of $M$.
\end{lemma}

\begin{proof}
By virtue of (2.10) and (2.11) we can show that $\mathcal{D}^2 X = -X + \eta(X)\xi$ and $g(\mathcal{D}X, \mathcal{D}Y) = g(X, Y) - \eta(X)\eta(Y)$, for any vector fields $X, Y \in TM$.

Now, we want to find an appropriate condition for $\nabla P$ so that it becomes possible to induce a Kenmotsu structure on $M$.

In [10] we have shown that for a proper slant submanifold of a Kenmotsu manifold
(4.1) \[(\nabla_X P)Y = -\eta(Y)PX + g(Y, PX)\xi\]
for any vector fields $X, Y \in TM$. In fact, the almost contact metric structure given by $\phi$ is a Kenmotsu structure, as from (4.1), we can see that
\[(\nabla_X \phi)Y = -\eta(Y)\phi X + g(Y, \phi X)\xi\]
for any vector fields $X, Y \in TM$.

From (2.8) and (2.9), for invariant and anti-invariant submanifolds of a Kenmotsu manifold, we have
(4.2) \[(\nabla_X P)Y = -\eta(Y)PX + g(Y, PX)\xi\]
(4.3) \[(\nabla_X F)Y = -\eta(Y)FX\]

In case of invariant and anti-invariant submanifolds $M$ of a Kenmotsu manifold, it is easy to show that the structure of $\mathcal{M}$ induces, in a natural way, a Kenmotsu structure over $M$. In this case the submanifold is usually called a Kenmotsu submanifold.

Therefore, we have

**Proposition 4.1.** A slant submanifold of a Kenmotsu manifold is a Kenmotsu manifold.

Also, from Theorem 3.1, it is clear that slant submanifold of a Kenmotsu manifold is a Kenmotsu manifold.

**§5. Examples of Slant submanifolds of Kenmotsu Manifolds**

We now give some examples of slant submanifolds of $\mathbb{R}^{2n+1}$ with almost contact structure $\{\phi_0, \xi, \eta, g\}$, which satisfy
\[(\nabla_X \phi_0)(Y) = g(\phi_0 X, Y)\xi - \eta(Y)\phi_0 X, \quad \nabla_X \xi = X - \eta(X)\xi\]
for $X, Y \in T\mathbb{R}^{2n+1}$.

The Kenmotsu structure on $\mathbb{R}^{2n+1}$ is
(5.1) \[\eta = dt, \quad \xi = \frac{\partial}{\partial t}\]
(5.2) \[g = \eta \otimes \eta + e^{2t}(\sum_{i=1}^n dx^i \otimes dx^i + dy^i \otimes dy^i)\]
(5.3) \[\phi_0(\sum_{i=1}^n (X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i}) + Z \frac{\partial}{\partial t}) = \sum_{i=1}^n (-Y_i \frac{\partial}{\partial x^i} + X_i \frac{\partial}{\partial y^i})\]
where \((x^i, y^i, t)\) are the Cartesian coordinates of \(\mathbb{R}^{2n+1} = C^n \times \mathbb{R}\).

Now, we prove the following result to obtain examples of slant submanifolds in \(\mathbb{R}^5(\varphi_0, \xi, \eta, g)\):

\[\textbf{Theorem 5.1. Let }\]
\[x(u, v) = (f_1(u, v), f_2(u, v), f_3(u, v), f_4(u, v))\]
\[\text{defines a slant surface } S \text{ in } C^2 \text{ with its usual Kaehlerian structure, such that } \frac{\partial}{\partial u} \text{ and } \frac{\partial}{\partial v} \text{ are non-zero and perpendicular to each other. Then}\]
\[y(u, v, w) = (f_1(u, v), f_2(u, v), f_3(u, v), f_4(u, v), w)\]
\[\text{defines a three dimensional slant submanifold } M \text{ in } \mathbb{R}^5(\varphi_0, \xi, \eta, g) \text{ with the same slant angle such that, if we put } e_1 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial u}, e_2 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial v}, \text{ then } \{e_1, e_2, \xi\} \text{ form an orthogonal basis of the tangent bundle of the submanifold.}\]

\[\textbf{Proof. Using } \{e_1, e_2, \xi\}, \text{ it is easy to show that } M \text{ is a three-dimensional submanifold of } \mathbb{R}^5. \text{ To prove that } M \text{ is slant, we write}\]
\[X = \lambda_1 e_1 + \lambda_2 e_2 + \eta(X) \xi, \quad \text{for } X \in \chi(M).\]
Then
\[\sqrt{|X|^2 - \eta^2(X)} = \sqrt{\lambda_1^2 + \lambda_2^2}\]
Now, since \(\{e_1, e_2, \xi\}\) is an orthogonal basis of \(\chi(M)\), using (2.5) we obtain
\[|PX|^2 = \frac{g^2(\varphi_0 X, e_1)}{g(e_1, e_1)} + \frac{g^2(\varphi_0 X, e_2)}{g(e_2, e_2)}\]
We may consider a vector field \(X_0 \in TS\) such that \(X_0 = \lambda_1 \frac{\partial}{\partial u} + \lambda_2 \frac{\partial}{\partial v}\) and denoting by \(J\) the usual almost complex structure of \(C^2\), we find that
\[g(\varphi_0 X, e_1) = g(JX_0, \frac{\partial}{\partial u}), \quad g(\varphi_0 X, e_2) = g(JX_0, \frac{\partial}{\partial v})\]
If \(TX_0\) is the tangent projection of \(JX_0\) and \(\theta\) is the slant angle of \(S\), then from (5.4) and (5.5), we get
\[\frac{|PX|}{\sqrt{|X|^2 - \eta^2(X)}} = \frac{|TX_0|}{|X_0|} = \cos \theta\]
Hence, \(M\) is a slant submanifold with the same slant angle \(\theta\).

By using the examples given in [6] and the above theorem, we now give some examples of slant submanifolds of Kenmotsu manifolds in \(\mathbb{R}^5(\varphi_0, \xi, \eta, g)\):
Example 5.1. For any \( \theta \in [0, \frac{\pi}{2}] \),
\[
x(u, v, w) = (u \cos \theta, u \sin \theta, v, 0, w)
\]
defines a three-dimensional minimal slant submanifold \( M \) with slant angle \( \theta \).

We may choose an orthonormal basis \( \{e_1, e_2, \xi\} \) such that
\[
e_1 = \frac{1}{c'}(\cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial z})
\]
\[
e_2 = \frac{1}{c'} \frac{\partial}{\partial y}, \quad \xi = \frac{\partial}{\partial \xi}
\]
Moreover, the vector fields
\[
e_1^* = \frac{1}{c'}(-\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial z})
\]
and
\[
e_2^* = \frac{1}{c'} \frac{\partial}{\partial y^2}
\]
form an orthonormal basis for \( T^\perp M \). Since \( \nabla_{e_1} e_1 = -\xi, \nabla_{e_2} e_2 = -\xi \) and \( \nabla \xi = 0 \), we get \( h(e_1, e_1) = h(e_2, e_2) = h(\xi, \xi) = 0 \). Therefore, the submanifold is minimal.

Example 5.2. For any constant \( k \),
\[
x(u, v, w) = (e^{ku} \cos u \cos v, e^{ku} \sin u \cos v, e^{ku} \cos u \sin v, e^{ku} \sin u \sin v, w)
\]
defines a three-dimensional slant submanifold \( M \) with slant angle \( \theta = \cos^{-1}\left(\frac{|k|}{\sqrt{1+k^2}}\right) \).

We may choose an orthonormal basis \( \{e_1, e_2, \xi\} \) such that
\[
e_1 = \frac{1}{c'}(e^{-ku} \frac{\partial}{\partial u})
\]
\[
e_2 = \frac{1}{c'}(e^{-ku} \frac{\partial}{\partial v}), \quad \xi = \frac{\partial}{\partial \xi}
\]
Then, by a straightforward computation we can show that it is a three-dimensional slant submanifold.

Example 5.3. For any positive constant \( k \),
\[
x(u, v, w) = (u, k \cos v, v, k \sin v, w)
\]
defines a three-dimensional non-minimal slant submanifold \( M \) with slant angle \( \theta = \cos^{-1}\left(\frac{1}{\sqrt{1+k^2}}\right) \).

Moreover, the following statements are equivalent:

(i) \( k = 0 \),  (ii) \( M \) is invariant  (iii) \( M \) is minimal.

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References


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