

On mixed super quasi-Einstein manifolds

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Abstract. In this paper, mixed super quasi-Einstein manifolds ($MS(QE)_n$) have been defined. The existence theorem and an example have been provided and the relations between the associated scalars have been established. As well, manifolds of mixed super quasi-constant curvature are defined, and it is shown that quasi conformally flat, conformally flat, conharmonically flat and projectively flat $MS(QE)_n$ are manifolds of mixed super quasi-constant curvature. Further, super quasi-umbilical hypersurfaces of Riemannian manifolds have been defined and it is proved that a super quasi-umbilical hypersurface of a Euclidean space is $MS(QE)_n$.

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1 Introduction

The notion of quasi-Einstein manifold was introduced in [4] by M. C. Chaki and R. K. Maity. A non-flat Riemannian manifold (M^n, g) , $(n \geq 3)$ is a *quasi-Einstein manifold* if its Ricci tensor S satisfies the condition

$$(1.1) \quad S(X, Y) = ag(X, Y) + bA(X)A(Y)$$

and is not identically zero, where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

$$(1.2) \quad g(X, U) = A(X), \quad \forall X \in TM,$$

U being a unit vector field.

Here a and b are called the associated scalars, A is called the associated 1-form and U is called the generator of the manifold. Such an n -dimensional manifold will further be denoted by $(QE)_n$. As well, in [8], U.C. De and G.C. Ghosh defined the generalized quasi-Einstein manifolds. A non-flat Riemannian manifold is called *generalized quasi-Einstein manifold* if its Ricci-tensor is non-zero and satisfies the condition

$$(1.3) \quad S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y),$$

where a, b and c are non-zero scalars and A, B are two 1-forms such that

$$(1.4) \quad g(X, U) = A(X) \text{ and } g(X, V) = B(X),$$

U and V being unit vectors which are orthogonal, i.e.,

$$(1.5) \quad g(U, V) = 0.$$

The vector fields U and V are called the generators of the manifold. This type of manifold will be denoted by $G(QE)_n$.

In [5], Chaki introduced the super quasi-Einstein manifold, denoted by $S(QE)_n$, where the Ricci tensor is not identically zero and satisfies the condition

$$(1.6) \quad \begin{aligned} S(X, Y) = & ag(X, Y) + bA(X)A(Y) + \\ & + c[A(X)B(Y) + A(Y)B(X)] + dD(X, Y), \end{aligned}$$

where a, b, c and d are scalars such that b, c, d are nonzero, A, B are two nonzero 1-forms defined as (1.4) and U, V are mutually orthogonal unit vector fields, D is a symmetric $(0, 2)$ tensor with zero trace which satisfies the condition

$$(1.7) \quad D(X, U) = 0, \quad \forall X.$$

Here a, b, c, d are called the associated scalars, A, B are called the associated main and auxiliary 1-forms respectively, U, V are called the main and the auxiliary generators and D is called the associated tensor of the manifold.

Recently in [1], A. Bhattacharyya and T. De introduced the notion of mixed generalized quasi-Einstein manifold. A non-flat Riemannian manifold is called *mixed generalized quasi-Einstein manifold* if its Ricci tensor is non-zero and satisfies the condition

$$(1.8) \quad \begin{aligned} S(X, Y) = & ag(X, Y) + bA(X)A(Y) + cB(X)B(Y) + \\ & + d[A(X)B(Y) + B(X)A(Y)], \end{aligned}$$

where a, b, c, d are non-zero scalars,

$$(1.9) \quad g(X, U) = A(X) \text{ and } g(X, V) = B(X),$$

$$(1.10) \quad g(U, V) = 0,$$

A, B are two non-zero 1-forms, U and V are unit vector fields corresponding to the 1-forms A and B respectively. If $d = 0$, then the manifold reduces to a $G(QE)_n$. This type of manifold is denoted by $MG(QE)_n$.

A Riemannian manifold is said to be a *manifold of generalized quasi-constant curvature* ([8]) if the curvature tensor \mathcal{R} of type $(0, 4)$ satisfies the condition

$$(1.11) \quad \begin{aligned} \mathcal{R}(X, Y, Z, W) = & p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ & + q[g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W) \\ & + g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z)] \\ & + s[g(X, W)D(Y)D(Z) - g(X, Z)D(Y)D(W) \\ & + g(Y, Z)D(X)D(W) - g(Y, W)D(X)D(Z)], \end{aligned}$$

where p, q, s are scalars, and T, D are non-zero 1-forms; ρ and $\bar{\rho}$ are unit orthogonal vector fields, such that

$$(1.12) \quad g(X, \rho) = T(X) \text{ and } g(X, \bar{\rho}) = D(X)$$

and

$$(1.13) \quad g(\rho, \bar{\rho}) = 0.$$

A Riemannian manifold is said to be a manifold of *mixed generalized quasi-constant curvature* ([1]) if the curvature tensor \mathcal{R} of type $(0, 4)$ satisfies the condition

$$(1.14) \quad \begin{aligned} \mathcal{R}(X, Y, Z, W) = & p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + q[g(X, W)A(Y)A(Z) \\ & - g(Y, W)A(X)A(Z) + g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W)] \\ & + s[g(X, W)B(Y)B(Z) - g(Y, W)B(X)B(Z) + g(Y, Z)B(X)B(W) \\ & - g(X, Z)B(Y)B(W) + t\{A(Y)B(Z) + B(Y)A(Z)\}g(X, W) \\ & - \{A(X)B(Z) + B(X)A(Z)\}g(Y, W) + \{A(X)B(W) \\ & + B(X)A(W)\}g(Y, Z) - \{A(Y)B(W) + B(Y)A(W)\}g(X, Z)], \end{aligned}$$

where p, q, s, t are scalars, A, B are non-zero 1-forms, ρ and $\bar{\rho}$ are orthonormal unit vector fields corresponding to A and B , which are defined as (1.12) and (1.13) and

$$(1.15) \quad g(\mathcal{R}(X, Y)Z, W) = \mathcal{R}(X, Y, Z, W).$$

This paper deals with mixed super quasi-Einstein manifold and mixed super quasi-constant curvature which are defined as follows

Definition 1. A non-flat Riemannian manifold (M^n, g) , ($n \geq 3$) is called *mixed super quasi-Einstein manifold* if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$(1.16) \quad \begin{aligned} S(X, Y) = & ag(X, Y) + bA(X)A(Y) + cB(X)B(Y) \\ & + d[A(X)B(Y) + B(X)A(Y)] + eD(X, Y), \end{aligned}$$

where a, b, c, d, e are scalars of which $b \neq 0, c \neq 0, d \neq 0, e \neq 0$ and A, B are two non zero 1-forms such that

$$(1.17) \quad g(X, U) = A(X) \text{ and } g(X, V) = B(X), \quad \forall X \in TM$$

U, V being mutually orthogonal unit vector fields, D is a symmetric $(0, 2)$ tensor with zero trace which satisfies the condition

$$(1.18) \quad D(X, U) = 0, \quad \forall X \in TM.$$

Here a, b, c, d, e are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, U, V are called the main and the auxiliary generators and

D is called the associated tensor of the manifold. We denote this type of manifold $MS(QE)_n$.

Definition 2. A Riemannian manifold is said to be a manifold of *mixed super quasi-constant curvature* if the curvature tensor R of type $(0, 4)$ satisfies the condition (1.19)

$$\begin{aligned} R(X, Y, Z, W) = & m[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + p[g(X, W)A(Y)A(Z) \\ & - g(Y, W)A(X)A(Z) + g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W)] \\ & + q[g(X, W)B(Y)B(Z) - g(Y, W)B(X)B(Z) + g(Y, Z)B(X)B(W) \\ & - g(X, Z)B(Y)B(W) + s[\{A(Y)B(Z) + B(Y)A(Z)\}g(X, W) \\ & - \{A(X)B(Z) + B(X)A(Z)\}g(Y, W) + \{A(X)B(W) \\ & + B(X)A(W)\}g(Y, Z) - \{A(Y)B(W) + B(Y)A(W)\}g(X, Z)] \\ & + t[g(Y, Z)D(X, W) - g(X, Z)D(Y, W) \\ & + g(X, W)D(Y, Z) - g(Y, W)D(X, Z)] \end{aligned}$$

where m, p, q, s, t are scalars. A, B are non-zero 1-forms defined in (1.17) and U, V are mutually orthogonal unit vector fields, D is a symmetric $(0, 2)$ tensor defined in (1.18).

In this paper, the existence theorem of $MS(QE)_n$ has been proved with an example and it is shown that the scalars $a + b$ and $a + c + eD(V, V)$ are the Ricci curvatures in the directions of the vector fields U and V respectively.

We have also studied quasi conformally flat, conformally flat, conharmonically flat and projectively flat $MS(QE)_n$. Lastly we have shown that a super quasi umbilical hypersurface of a Euclidean space is $MS(QE)_n$.

2 Preliminaries

In an n -dimensional ($n > 2$) Riemannian manifold the covariant quasi-conformal curvature tensor is defined as [3]

$$\begin{aligned} \tilde{C}(X, Y, Z, W) = & aR(X, Y, Z, W) + b[S(Y, Z)g(X, W) - S(X, Z)g(Y, W) \\ (2.1) \quad & + g(Y, Z)g(QX, W) - g(X, W)g(QY, W)] \\ & - \frac{r}{n} \left[\frac{\alpha}{n-1} + 2b \right] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \end{aligned}$$

where

$$(2.2) \quad g(\tilde{C}(X, Y)Z, W) = \tilde{C}(X, Y, Z, W).$$

If $\alpha = 1$ and $b = -\frac{1}{n-2}$ then (2.1) reduces to conformal curvature tensor, where

$$\begin{aligned} C(X, Y, Z, W) = & R(X, Y, Z, W) - \frac{1}{n-2} [S(Y, Z)g(X, W) - S(X, Z)g(Y, W) \\ (2.3) \quad & + g(Y, Z)S(X, W) - g(X, W)S(Y, W)] \\ & + \frac{r}{(n-1)(n-2)} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \end{aligned}$$

The conharmonic curvature tensor is denoted by $H(X, Y, Z, W)$ and in a $V_n (n > 2)$ it is defined as

$$(2.4) \quad \begin{aligned} H(X, Y, Z, W) = & \mathcal{R}(X, Y, Z, W) - \frac{1}{n-2}[g(X, W)S(Y, Z) - g(X, Z)S(Y, W) \\ & + g(Y, Z)S(X, W) - g(Y, W)S(X, Z)] \end{aligned}$$

and L be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S , where

$$(2.5) \quad g(LX, Y) = S(X, Y) \quad \forall X, Y \in TM.$$

The projective curvature tensor is denoted by $\tilde{P}(X, Y, Z, W)$ and in a $V_n (n > 2)$ it is defined as

$$(2.6) \quad \begin{aligned} \tilde{P}(X, Y, Z, W) = & \mathcal{R}(X, Y, Z, W) \\ & - \frac{1}{n-1}[S(Y, Z)g(X, W) - S(Y, W)g(X, Z)]. \end{aligned}$$

3 Existence theorem of a mixed super quasi-Einstein manifold

In this section we state and prove the existence theorem of a mixed super quasi-Einstein manifold.

Theorem 1. *If the Ricci-tensor S of a Riemannian manifold satisfies the relation*

$$(3.1) \quad \begin{aligned} S(X, W)S(Y, Z) - S(Y, W)S(X, Z) = & \mu[S(Y, W)g(Z, X) + S(Z, X)g(Y, W)] \\ & + \beta[g(X, W)g(Y, Z) - g(Y, W)g(X, Z)] \\ & + \gamma[g(Y, Z)D(X, W) - g(X, Z)D(Y, W) \\ & + g(X, W)D(Y, Z) - g(Y, W)D(X, Z)] \end{aligned}$$

where μ, β, γ are non-zero scalars and D is a symmetric $(0, 2)$ tensor with zero trace which satisfies the condition $D(X, U) = 0, \forall X$ then the manifold is mixed super quasi-Einstein manifold.

Proof. Let U be a vector field defined by

$$(3.2) \quad g(X, U) = T(X) \quad \forall X \in TM.$$

Putting $X = W = U$ in (3.1), we obtain

$$(3.3) \quad \begin{aligned} S(U, U)S(Y, Z) - S(Y, U)S(U, Z) = & \mu[S(Y, U)g(Z, U) + S(Z, U)g(Y, U)] \\ & + \beta[g(U, U)g(Y, Z) - g(Y, U)g(Z, U)] \\ & + \gamma[g(Y, Z)D(U, U) - g(U, Z)D(Y, U) \\ & + g(U, U)D(Y, Z) - g(Y, U)D(U, Z)]. \end{aligned}$$

Now using (2.5) and (3.2) in the above equation , we get

$$(3.4) \quad \begin{aligned} \bar{\alpha}S(Y, Z) - T(LY)T(LZ) = & \mu[T(LY)T(Z) + T(LZ)T(Y)] \\ & + \rho[|U|^2g(Y, Z) - T(Y)T(Z)] \\ & + \lambda[|U|^2D(Y, Z)] \end{aligned}$$

where $S(U, U) = \bar{\alpha}$ and $T(LY) = g(LY, U) = S(Y, U)$.

Therefore

$$(3.5) \quad \begin{aligned} S(Y, Z) = & \alpha T(LY)T(LZ) + \mu\alpha[T(LY)T(Z) + T(LZ)T(Y)] \\ & + \rho\alpha[|U|^2g(Y, Z) - T(Y)T(Z)] + \lambda\alpha[|U|^2D(Y, Z)]. \end{aligned}$$

Taking $\alpha = \frac{1}{\bar{\alpha}}$ and $T(LY) = P(Y)$ in (3.30), we get

$$(3.6) \quad \begin{aligned} S(Y, Z) = & \rho\alpha[|U|^2g(Y, Z)] + (-\rho\alpha)T(Y)T(Z) + \alpha P(Y)P(Z) \\ & + \mu\alpha[P(Y)T(Z) + P(Z)T(Y)] + \lambda\alpha[|U|^2D(Y, Z)]. \end{aligned}$$

Therefore in view of (3.6) the manifold is mixed super quasi-Einstein manifold. \square

4 Example of a mixed super quasi-Einstein manifold

A manifold of mixed super quasi-constant curvature defined by (1.19) is a mixed super quasi-Einstein manifold. Putting $X = W = e_i$ in (1.19), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i, 1 \leq i \leq n$, we get

$$(4.1) \quad \begin{aligned} S(Y, Z) = & \{(n-1)m + p + q + 2s + tr_1\}g(Y, Z) \\ & + p(n-2)A(Y)A(Z) + q(n-2)B(Y)B(Z) \\ & + s(n-2)\{A(Y)B(Z) + A(Z)B(Y)\} + t(n-2)D(Y, Z) \end{aligned}$$

where $r_1 = D(e_i, e_i)$.

From (4.1) we see that the manifold is $MS(QE)_n$.

5 The associated scalars of a $MS(QE)_n$ ($n \geq 3$)

In this section we consider $MS(QE)_n$ ($n \geq 3$) with associated scalars a, b, c, d, e , associated main and auxiliary 1-forms A, B , main and auxiliary generators U, V and associated symmetric $(0, 2)$ tensor D .

So (1.16), (1.17) and (1.18) will hold. Since U and V are mutually orthogonal unit vector fields, we have

$$g(U, U) = 1, \quad g(V, V) = 1 \quad \text{and} \quad g(U, V) = 0.$$

Further,

$$(5.1) \quad \text{trace}D = 0$$

$$(5.2) \quad D(X, U) = 0 \quad \forall \quad X \in TM.$$

In virtue of (1.17), $g(U, V) = 0$ can be expressed as

$$(5.3) \quad A(V) = B(U) = 0.$$

Now contracting X and Y in (1.16), we get

$$(5.4) \quad r = na + b + c$$

where r is the scalar curvature.

Again from (1.17) we have

$$(5.5) \quad S(U, U) = a + b$$

$$(5.6) \quad S(V, V) = a + c + eD(V, V)$$

and

$$(5.7) \quad S(U, V) = d.$$

If X is a unit vector field, then $S(X, X)$ is the Ricci-curvature in the direction of X . Hence from (5.5) and (5.6) we can state that $a + b$ and $a + c + eD(V, V)$ are the Ricci curvature in the directions of U and V respectively. From (2.5) we have

$$g(LX, Y) = S(X, Y).$$

We also consider

$$(5.8) \quad g(\ell X, Y) = D(X, Y).$$

Further, let d_1^2 and d_2^2 denote the squares of the lengths of the Ricci-tensor S and the associated tensor D . Then

$$(5.9) \quad d_1^2 = S(Le_i, e_i)$$

and

$$(5.10) \quad d_2^2 = D(\ell e_i, e_i)$$

where $\{e_i\}$, $i = 1, 2, \dots, n$ is an orthonormal basis of the tangent space at a point of $MS(QE)_n$. Now from (1.16) we get

$$(5.11) \quad \begin{aligned} S(Le_i, e_i) = & (n-1)a^2 + a^2 + b^2 + c^2 \\ & + 2ab + 2bc + ceD(V, V) + eS(\ell e_i, e_i). \end{aligned}$$

Now from (1.16), we have

$$(5.12) \quad S(\ell e_i, e_i) = eD(\ell e_i, e_i).$$

From (5.11) and (5.12) it follows that

$$(5.13) \quad \begin{aligned} S(Le_i, e_i) = & (n-1)a^2 + a^2 + b^2 + c^2 \\ & + 2ab + 2bc + ceD(V, V) + e^2D(\ell e_i, e_i). \end{aligned}$$

Hence

$$(5.14) \quad d_1^2 = (n-1)a^2 + (a+b+c)^2 - 2ca + ceD(V, V) + e^2(d_2)^2.$$

We consider $eD(V, V) = 2a$, i.e., $D(V, V) = \frac{2a}{e} = k$. From (5.14) we can write $d_1^2 - e^2(d_2)^2 = (n-1)a^2 + (a+b+c)^2 > 0$ and hence

$$(5.15) \quad e < d_1/d_2,$$

which concludes the proof. \square .

From (5.15) we can state the following theorem

Theorem 2. *In $MS(QE)_n$ ($n \geq 3$) the scalars $a+b$ and $a+c+eD(V, V)$ are the Ricci-curvature in the direction of the generators U and V respectively and the associated scalar e is less than the ratio which the length of Ricci-tensor S bears the length of the associated tensor D .*

6 Quasi conformally flat $MS(QE)_n$ ($n > 3$)

Let R be the curvature tensor of type (1, 3) of a quasi conformally flat $MS(QE)_n$ ($n \geq 3$). Then

$$(6.1) \quad \begin{aligned} \mathcal{R}(X, Y, Z, W) = & \frac{r}{na} \left[\frac{a'}{n-1} + 2b \right] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ & - \frac{b'}{a} [g(Y, Z)S(X, W) - g(X, Z)S(Y, W) \\ & + g(X, W)S(Y, Z) - g(Y, W)S(X, Z)] \end{aligned}$$

where \mathcal{R} is defined in (1.15).

Using (1.16), in (6.1), we obtain

$$\begin{aligned}
\mathcal{R}(X, Y, Z, W) = & \left\{ \frac{r}{n\alpha} \left[\frac{\alpha'}{n-1} + 2b \right] - \frac{2ab'}{\alpha'} \right\} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
& - \frac{bb'}{\alpha} [g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W)] \\
& + g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)] \\
& - \frac{cb'}{\alpha} [g(Y, Z)B(X)B(W) - g(X, Z)B(Y)B(W)] \\
& + g(X, W)B(Y)B(Z) - g(Y, W)B(X)B(Z)] \\
(6.2) \quad & - \frac{db'}{\alpha} [g(Y, Z)\{A(X)B(W) + A(W)B(X)\} \\
& - g(X, Z)\{A(Y)B(W) + A(W)B(Y)\} \\
& + g(X, W)\{A(Y)B(Z) + A(Z)B(Y)\} \\
& - g(Y, W)\{A(X)B(Z) + A(Z)B(X)\}] \\
& - \frac{eb'}{\alpha} [g(Y, Z)D(X, W) - g(X, Z)D(Y, W)] \\
& + g(X, W)D(Y, Z) - g(Y, W)D(X, Z)].
\end{aligned}$$

Now the equation (6.2) can be written as

$$\begin{aligned}
\mathcal{R}(X, Y, Z, W) = & a_1 [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
& + b_1 [g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W)] \\
& + g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)] \\
& + c_1 [g(Y, Z)B(X)B(W) - g(X, Z)B(Y)B(W)] \\
& + g(X, W)B(Y)B(Z) - g(Y, W)B(X)B(Z)] \\
(6.3) \quad & + d_1 [g(Y, Z)\{A(X)B(W) + A(W)B(X)\} \\
& - g(X, Z)\{A(Y)B(W) + A(W)B(Y)\} \\
& + g(X, W)\{A(Y)B(Z) + A(Z)B(Y)\} \\
& - g(Y, W)\{A(X)B(Z) + A(Z)B(X)\}] \\
& + e_1 [g(Y, Z)D(X, W) - g(X, Z)D(Y, W)] \\
& + g(X, W)D(Y, Z) - g(Y, W)D(X, Z)]
\end{aligned}$$

where

$$(6.4) \quad a_1 = \left\{ \frac{r}{n\alpha} \left[\frac{\alpha'}{n-1} + 2b \right] - \frac{2ab'}{\alpha'} \right\}, \quad b_1 = -\frac{bb'}{\alpha}, \quad c_1 = -\frac{cb'}{\alpha}, \quad d_1 = -\frac{db'}{\alpha} \quad \text{and} \quad e_1 = -\frac{eb'}{\alpha}.$$

Let U^1 be the $(n-1)$ dimensional distribution, orthogonal to U , in a quasi conformally flat $MS(QE)_n$. Then $g(X, U) = 0$ if $X \in U^1$. Hence from (6.3) we get the

following properties of R

$$\begin{aligned}
 R(X, Y, Z) = & \lambda[g(Y, Z)X - g(X, Z)Y] \\
 & + c_1[g(Y, Z)B(X) - g(X, Z)B(Y)]V \\
 (6.5) \quad & + c_1[g(Z, V)g(Y, V)X - g(X, V)g(Z, V)Y] \\
 & + d_1[g(Y, Z)B(X) - g(X, Z)B(Y)]U \\
 & + e_1[D(Y, Z)X - D(X, Z)Y + g(Y, Z)\ell X - g(X, Z)\ell Y]
 \end{aligned}$$

when $X, Y, Z \in U^1$ and

$$(6.6) \quad R(X, U, U) = \lambda X + c_1 g(X, V)V + e_1 \ell X \quad \text{when } X \in U^1$$

where $\lambda = \left\{ \frac{na+b+c}{na^2} \left[\frac{a'}{n-1} + 2b \right] - \frac{2ab'}{a^2} \right\}$ and c_1, d_1 and e_1 have values given by (6.4). \square

We can therefore state

Theorem 3. *A quasi conformally flat mixed super quasi-Einstein manifold ($n > 3$) is a manifold of mixed super quasi-constant curvature and the curvature tensor R of type (1, 3) satisfies the properties given by (6.5) and (6.6).*

Corollary 1. *A conformally flat mixed super quasi-Einstein manifold ($n > 3$) is a manifold of mixed super quasi-constant curvature and the curvature tensor R of type (1, 3) satisfies the property $R(X, U, U) = \lambda X + c_1 B(X)V + e_1 \ell X$ when $X \in U^1$, where $\lambda = \frac{(n-2)a-b-c}{(n-1)(n-2)}, c_1 = \frac{c}{n-2}$ and $e_1 = \frac{e}{n-2}$.*

Corollary 2. *A conharmonically flat mixed super quasi-Einstein manifold ($n > 3$) is a manifold of mixed super quasi-constant curvature and the curvature tensor R of type (1, 3) satisfies the property $R(X, U, U) = \lambda X + c^1 B(X)V + e^1 \ell X$ when $X \in U^1$ where $\lambda = \frac{2a}{n-1}, c^1 = \frac{c}{n-2}$ and $e^1 = \frac{e}{n-2}$.*

7 Projectively flat $MS(QE)_n$ ($n > 3$)

Let R be the curvature tensor of type (1, 3) of a projectively flat $MS(QE)_n$ ($n \geq 3$). Then

$$(7.1) \quad \mathcal{R}(X, Y, Z, W) = \frac{1}{n-1} [g(X, W)S(Y, Z) - g(Y, W)S(X, Z)]$$

where \mathcal{R} is defined earlier.

From (1.16) and (7.1), we get

$$\begin{aligned}
 \mathcal{R}(X, Y, Z, W) = & \frac{a}{n-1} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
 & + \frac{b}{n-1} [g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)] \\
 (7.2) \quad & + \frac{c}{n-1} [g(X, W)B(Y)B(Z) - g(Y, W)B(X)B(Z)] \\
 & + \frac{d}{n-1} [g(X, W)\{A(Y)B(Z) + A(Z)B(Y)\} \\
 & - g(Y, W)\{A(X)B(Z) + A(Z)B(X)\}] \\
 & + \frac{e}{n-1} [g(X, W)D(Y, Z) - g(Y, W)D(X, Z)].
 \end{aligned}$$

Let U^1 be the $(n-1)$ dimensional distribution, orthogonal to U , in a projectively flat $MS(QE)_n$. Then $g(X, U)=0$ if $X \in U^1$. Hence from (7.2) we get the following properties of R

$$\begin{aligned}
 R(X, Y, Z) = & a_2 [g(Y, Z)X - g(X, Z)Y] \\
 & + b_2 [A(Y)A(Z)X - A(X)A(Z)Y] \\
 & + c_2 [B(Y)B(Z)X - B(X)B(Z)Y] \\
 & + d_2 [\{A(Y)B(Z) + A(Z)B(Y)\}X \\
 & - \{A(X)B(Z) + A(Z)B(X)\}Y] \\
 & + e_2 [D(Y, Z)X - D(X, Z)Y]
 \end{aligned}
 \tag{7.3}$$

when $X, Y, Z \in U^1$, $a_2 = \frac{a}{n-1}$, $b_2 = \frac{b}{n-1}$, $c_2 = \frac{c}{n-1}$, $d_2 = \frac{d}{n-1}$ and $e_2 = \frac{e}{n-1}$. Also,

$$R(X, U, U) = (a_2 + b_2)X - b_2g(X, U)U - d_2g(X, V)U
 \tag{7.4}$$

for all $X \in U^1$. □

Thus we can state

Theorem 4. *In projectively flat $MS(QE)_n$ ($n > 3$) the curvature tensor R of type (1, 3) satisfies the properties given by (7.3) and (7.4).*

8 Hypersurfaces of the Euclidean space

Let M^n be a hypersurface of the Euclidean space E^{n+1} , with the metric tensor \tilde{g} of M^n induced by E^{n+1} .

The Gauss equation of M^n in E^{n+1} can be written as

$$\tilde{g}(\tilde{R}(X, Y)Z, W) = \tilde{g}(H(X, W), H(Y, Z)) - \tilde{g}(H(Y, W), H(X, Z))
 \tag{8.1}$$

where \tilde{R} is the Riemannian curvature tensor corresponding to the induced metric \tilde{g} . H is the second fundamental tensor of M^n (orthonormal to M^n) and X, Y, Z, W are vector fields tangent to M^n .

If A_ξ is the (1, 1) tensor corresponding to the normal valued second fundamental tensor \tilde{H} , then we have [6]

$$\tilde{g}(A_\xi(X), Y) = g(H(X, Y), \xi)
 \tag{8.2}$$

where ξ is the unit normal vector field and X, Y are tangent vector fields.

Let H_ξ be the symmetric (0, 2) tensor associated with A_ξ in the hypersurface defined by

$$\tilde{g}(A_\xi(X), Y) = H_\xi(X, Y).
 \tag{8.3}$$

A hypersurface of a Riemannian manifold (M^n, g) is called *quasi-umbilical* [6] if its second fundamental tensor has the form

$$H_\xi(X, Y) = \alpha\tilde{g}(X, Y) + \beta\omega(X)\omega(Y)
 \tag{8.4}$$

where ω is 1-form. The vector field corresponding to the 1-form ω is a unit vector field, and α, β are scalars. If $\alpha = 0$ (resp. $\beta = 0$ or $\alpha = \beta = 0$) holds then M^n is called cylindrical (resp. *umbilical or geodesic*).

A hypersurface of a Riemannian manifold (M^n, g) is called *generalized quasi-umbilical* [6] if its second fundamental tensor has the form

$$(8.5) \quad H_\xi(X, Y) = \alpha \tilde{g}(X, Y) + \beta \omega(X)\omega(Y) + \gamma \delta(X)\delta(Y)$$

where α, β, γ are scalars. The vector fields corresponding to 1-forms ω and δ are unit vector fields. If $\alpha = \beta = \gamma = 0$, M^n is called geodesic. If $\alpha = \gamma = 0$ or $\alpha = \beta = 0$, M^n is called cylindrical. Also M^n is called umbilical when $\beta = \gamma = 0$.

In this section, we define super quasi-umbilical hypersurface of a Riemannian manifold.

Definition 3. A hypersurface of a Riemannian manifold (M^n, g) is called *super quasi-umbilical* if its second fundamental tensor has the form

$$(8.6) \quad H_\xi(X, Y) = \alpha \tilde{g}(X, Y) + \beta \omega(X)\omega(Y) + \gamma \delta(X)\delta(Y) + \rho D(X, Y)$$

where α, β, γ are scalars. The vector fields corresponding to 1-forms ω and δ are unit vector fields. If $\alpha = \beta = \gamma = \rho = 0$, M^n is called geodesic. If $\alpha = \gamma = \rho = 0$ or $\alpha = \beta = \rho = 0$, M^n is called cylindrical. Also M^n is called umbilical when $\beta = \gamma = \rho = 0$.

Now from (8.2), (8.3) and (8.5), we get

$$(8.7) \quad \begin{aligned} g(H(X, Y), \xi) = & \alpha g(X, Y)g(\xi, \xi) + \beta \omega(X)\omega(Y)g(\xi, \xi) \\ & + \gamma \delta(X)\delta(Y)g(\xi, \xi). \end{aligned}$$

Since ξ is the only unit normal vector, (8.7) reduces to

$$(8.8) \quad H(X, Y) = \alpha g(X, Y)\xi + \beta \omega(X)\omega(Y)\xi + \gamma \delta(X)\delta(Y)\xi.$$

Let us assume that the hypersurface is super quasi-umbilical. Then in view of (8.8) we get

$$(8.9) \quad H(X, Y) = \alpha g(X, Y)\xi + \beta \omega(X)\omega(Y)\xi + \gamma \delta(X)\delta(Y)\xi + \rho D(X, Y)\xi.$$

From (8.1), (8.2) and (8.5) it follows that

$$\begin{aligned}
\tilde{g}(\tilde{R}(X, Y)Z, W) = & \alpha^2\{g(X, W)g(Y, Z) - g(Y, W)g(X, Z)\} \\
& + \alpha\beta\{g(X, W)\omega(Y)\omega(Z) + g(Y, Z)\omega(X)\omega(W) \\
& - g(Y, W)\omega(X)\omega(Z) - g(X, Z)\omega(Y)\omega(W)\} \\
& + \alpha\gamma\{g(X, W)\delta(Y)\delta(Z) + g(Y, Z)\delta(X)\delta(W) \\
& - g(Y, W)\delta(X)\delta(Z) - g(X, Z)\delta(Y)\delta(W)\} \\
& + \beta\gamma\{\omega(X)\omega(W)\delta(Y)\delta(Z) + \omega(Y)\omega(Z)\delta(X)\delta(W) \\
& - \omega(Y)\omega(W)\delta(X)\delta(Z) - \omega(X)\omega(Z)\delta(Y)\delta(W)\} \\
(8.10) \quad & + \alpha\rho[g(X, W)D(Y, Z) + g(Y, Z)D(X, W) \\
& - g(Y, W)D(X, Z) - g(X, Z)D(Y, W)] \\
& + \beta\rho\{D(X, W)\omega(Y)\omega(Z) + D(Y, Z)\omega(X)\omega(W) \\
& - D(Y, W)\omega(X)\omega(Z) - D(X, Z)\omega(Y)\omega(W)\} \\
& + \gamma\rho\{D(X, W)\delta(Y)\delta(Z) + g(Y, Z)\delta(X)\delta(W) \\
& - D(Y, W)\delta(X)\delta(Z) - D(X, Z)\delta(Y)\delta(W)\} \\
& + \{D(X, W)D(Y, Z) - D(Y, W)D(X, Z)\}.
\end{aligned}$$

On contraction to (8.10) we get

$$\begin{aligned}
(8.11) \quad \bar{S}(Y, Z) = & [\alpha^2(n-2) + \alpha\beta + \alpha\gamma + \alpha\rho r_1]g(Y, Z) + [(n-2)\alpha\beta + \beta\gamma + \beta\rho r_1]\omega(Y)\omega(Z) \\
& + [(n-2)\gamma\alpha + \beta\gamma + \gamma\rho r_1]\delta(Y)\delta(Z) - \beta\gamma[\omega(Y)\delta(Z) + \delta(Y)\omega(Z)] \\
& + [\rho^2(r_1-2) + \alpha\rho(n-2) + \beta\rho + \gamma\rho]D(Y, Z)
\end{aligned}$$

which shows that the manifold is a mixed super quasi-Einstein manifold. \square

Thus we can state the following theorem.

Theorem 5. *A super quasi-umbilical hypersurface of a Euclidean space is a mixed super quasi-Einstein manifold.*

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