Projective curvature inheritance in an $NP - F_n$

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Abstract. The concept of projective curvature inheritance in Finsler space have been studied by S.P. Singh [6]. In the present investigation, our aim is to study the Projective curvature inheritance in an $NP - F_n$. Corresponding results for contra and concurrent vector fields are rendered intuitive.

Key words: Finsler spaces, $NP - F_n$, $RNP - F_n$, $SNP - F_n$, projective motion, contra and concurrent vector field.

§ 1. Preliminaries

K.Yano [4] defined a set of parameters

\[ \Pi^i_{kh} = \frac{\dot{x}^i}{n+1} G_{kh}^r, \]

which form a connection called the normal projective connection. The functions $\Pi^i_{kh}$, $G^i_{kh}$ and $G^i_{jkh}$ are symmetric in their lower indices and are positively homogeneous of degree 0, 0 and $-1$ respectively in their $\dot{x}^i$’s. The functions $G^i_{jk}$ are the Berwald’s connection parameters. The derivatives $\dot{\Pi}^i_{jkh}$, denoted by $\Pi^i_{jkh}$, is given by

\[ \dot{\Pi}^i_{jkh} = G^i_{jkh} - \frac{1}{n+1} (\delta^i_j G^r_{kh} + \dot{x}^i G^r_{jkh}), \]

are symmetric in $k$ and $h$ only and are positively homogeneous of degree $-1$ in directional arguments. Therefore, the following relation which will be used in our discussion follow from (1.1) and (1.2)

\[ \begin{aligned} 
\text{a)} \quad & \Pi^i_{jkh} \dot{x}^k = \Pi^i_{jkh} \dot{x}^k = G^i_{kh}, \\
\text{b)} \quad & \Pi^i_{kh} = G^i_{kh}, \\
\text{c)} \quad & \dot{x}^j \Pi^i_{jkh} = 0, \\
\text{d)} \quad & \Pi^i_{kh} = \frac{2}{n+1} G^i_{ikh}, \\
\text{e)} \quad & \Pi^i_{jki} = G^i_{jki} = \Pi^i_{jki}. 
\end{aligned} \]

The normal projective covariant derivative of a vector field $X^i(x, \dot{x})$ is defined by
\begin{equation}
\nabla_k X^i = \partial_k X^i - (\partial_j X^i)\Pi^j_{kh} \dot{x}^h + X^j \Pi^i_{jk},
\end{equation}
where $\partial_k = \partial/\partial x^k$, $\partial_k = \partial/\partial \dot{x}^k$, and preserve the vector character of $X^i$.

In particular, this derivative vanishes for $\dot{x}^i$. The corresponding curvature tensor $N^i_{jkh}(x, \dot{x})$ as called by K.Yano, the normal projective curvature tensor, is given by
\begin{equation}
N^i_{jkh} = 2\{\partial_j \Pi^i_{kh} + \Pi^j_{lh} \Pi^i_{km} \dot{x}^m + \Pi^i_{lj} \Pi^j_{kl}\}.
\end{equation}

**Definition 1.1.** The manifold $F_n$ with normal projective connection parameters $\Pi^i_{kh}$ and the normal projective curvature tensor $N^i_{jkh}$ is termed as normal projective Finsler manifold and is usually denoted by $NP - F_n$.

It is worth mentioning that the normal projective curvature tensor is skew-symmetric in $j$, $k$ indices and is a homogeneous function of degree 0 in $\dot{x}^i$’s, so in the light of (1.5) it is fairly easy to observe that
\begin{equation}
\begin{aligned}
a) & \quad N^i_{jkh} = -N^i_{kjh}, \\
b) & \quad \partial_l N^i_{jkh} \dot{x}^l = 0.
\end{aligned}
\end{equation}

The contraction of $N^i_{jkh}$ with respect to $i$, $j$ ; $i$, $k$ and $i$, $h$ give
\begin{equation}
\begin{aligned}
a) & \quad N^i_{ikh} = N^i_{kh}, \\
b) & \quad N^i_{jih} = -N^i_{ijh} = -N^i_{jh}, \\
b) & \quad N^i_{jkh} = 2N[ij],
\end{aligned}
\end{equation}
respectively, where $[ij]$ represent the skew-symmetric part.

P.N. Pandey [5] has shown the following relationship between the normal projective curvature tensor $N^i_{jkh}$ and the Berwald’s curvature tensor $H^i_{jkh}$.
\begin{equation}
N^i_{jkh} = H^i_{jkh} - \frac{\dot{x}^i}{n+1} \partial_l H^r_{kh}.
\end{equation}

The covariant derivative gives rise to the commutation formula
\begin{equation}
2\nabla_j \nabla_k X^i = N^i_{jkh} X^k - (\partial_l X^i)N^i_{jkh} \dot{x}^l
\end{equation}
together with the normal projective curvature tensor $N^i_{jkh}$. In term of contracted tensor $N_{kh} = N^i_{ikh}$ there is defined a tensor
\begin{equation}
M_{kh} = -\frac{1}{n^2 - 1}(nN_{kh} + N_{kk})
\end{equation}
and the Weyl’s projective curvature tensor $W^i_{jkh}$ is given by K.Yano [4]
\begin{equation}
W^i_{jkh} = N^i_{jkh} + 2\{\delta^i_j M^i_{kh} - M^i_{[jk]} \delta^i_{kh}\}.
\end{equation}
The projective curvature tensor also satisfies
\[
\begin{align*}
\text{(a)} & \quad \partial_j \partial_l W^{i}_{kh} = 0, \\
\text{(b)} & \quad W^{i}_{ikh} = -W^{i}_{khi} = 0, \\
\text{(c)} & \quad W^{i}_{khi} = 0.
\end{align*}
\]
(1.12)

The commutation formulae for any general tensor, involving the curvature tensor, are given as follows
\[
2 \nabla_k \nabla_h T^{i}_{j} = N^{i}_{kh} T^{j}_{l} - N^{i}_{lk} T^{j}_{h} - (\partial_l T^{i}_{j}) N^{i}_{km} \dot{x}^{m},
\]
(1.13)

\[
(\partial_j \nabla_h - \nabla_j \partial_h) T^{i}_{l} = \Pi^{i}_{jkh} T^{j}_{l} - \Pi^{i}_{lkj} T^{j}_{h} - \Pi^{i}_{kmj} \dot{x}^{m} (\partial_l T^{i}_{j}).
\]
(1.14)

The Lie-derivative of a tensor $T^i_j$ and the connection coefficients $\Pi^i_{jk}$ defined by an infinitesimal transformation
\[
\dot{x}^i = x^i + \varepsilon v^i (x)
\]
are characterized by K.Yano [4]
\[
\mathcal{L} T^i_j = v^h (\nabla_h T^i_j) - T^h_j (\nabla_h v^i) + T^i_h (\nabla_j v^h) + (\partial_l T^i_j) (\nabla_s v^h) \dot{x}^s
\]
and
\[
\mathcal{L} \Pi^i_{jk} = \nabla_j \nabla_k v^i - N^i_{hjk} v^h + \Pi^i_{hjk} (\nabla_l v^h) \dot{x}^l
\]
respectively.

The commutation formulae with respect to Lie-derivative and other for any tensor $T^i_{jk}$ are given by
\[
\mathcal{L} (\nabla(T^i_{jk})) - \nabla (\mathcal{L} T^i_{jk}) = \mathcal{L} \Pi^i_{jkh} T^j_{h} - \mathcal{L} \Pi^i_{lkj} T^j_{h} - \mathcal{L} \Pi^i_{kmj} \dot{x}^{m} (\partial_l T^{i}_{j})
\]
and
\[
\dot{\partial}_l (\mathcal{L} T^i_{jk}) = \mathcal{L} (\dot{\partial}_l T^i_{jk}) = 0.
\]
(1.18)

The Lie-derivative of the normal projective curvature tensor $N^i_{kjh}$ is expressed in the form
\[
\nabla_k (\mathcal{L} \Pi^i_{jkh}) - \nabla_j (\mathcal{L} \Pi^i_{kh}) = \mathcal{L} N^i_{kjh} + (\mathcal{L} \Pi^r_{km}) \dot{x}^{m} \Pi^i_{rjh}
\]
(1.20)

In view of the infinitesimal transformation (1.15) K.Yano [4] defined a projective motion, if there exists a homogeneous scalar function $p$ of degree one in $\dot{x}^i$’s satisfying
\[
\mathcal{L} \Pi^i_{jk} = 2 \partial_{(j} \Pi^i_{k)h}, \quad \Pi^i_{j} = \dot{\partial}_h p
\]
(1.21)
where \((jh)\) represents the symmetric part. For the homogeneity of \(p_k\) and \(p_{jk}\), they satisfy the conditions
\[
(1.22) \quad a)p_k \dot{x}^k = p, \quad b)p_{jk} \dot{x}^k = 0.
\]

§ 2. Projective \(N\)-curvature inheritance

S.P. Singh [6] defined the projective \(H\)-curvature inheritance as an infinitesimal transformation with respect to which the Lie-derivative of Berwald’s curvature tensor \(H_{jkh}\) satisfies a relation of the form
\[
(2.23) \quad \mathcal{L}H_{jkh}^i = \alpha H_{jkh}^i,
\]
where \(\alpha(x)\) is non-zero scalar function.

In the present paper, we consider the infinitesimal transformation (1.15) which admits the projective motion in an \(NP - F_n\). Now we define and study the cases under which the infinitesimal transformation (1.15) defines a projective \(N\)-curvature inheritance in \(NP - F_n\).

\textbf{Definition 2.2.} In an \(NP - F_n\), if the normal projective curvature tensor field \(N_{jkh}^i\) satisfies the relation
\[
(2.24) \quad \mathcal{L}N_{jkh}^i = \alpha N_{jkh}^i,
\]
where \(\alpha(x)\) is non zero scalar function and \(\mathcal{L}\) denotes Lie-derivative defined by the infinitesimal transformation (1.15), which admits the projective motion. The transformation (1.15) is called projective \(N\)-curvature inheritance in the light of (2.24).

Contracting with respect to the indices \(i\) and \(j\) (2.24) yields
\[
(2.25) \quad \mathcal{L}N_{kh} = \alpha N_{kh}.
\]
In view of 2.25, the projective inheritance is called the projective Ricci-like \(N\)-curvature inheritance. Employing (1.21) in the equation (1.20), we arrive at
\[
(2.26) \quad 2\{\delta^i_j (\nabla_k p_h) + \delta^i_h (\nabla_k p_{jh})\} = \mathcal{L}N_{jkh}^i + 2\delta^i_{[k} p_{m]} \dot{x}^m \Pi_{r]jh}^i - 2\delta^i_{[k} p_{m]} \dot{x}^m \Pi_{r]kh}^i.
\]
In view of (1.22), (1.3c) and (2.24) the above equation reduces to
\[
(2.27) \quad \alpha N_{jkh}^i = 2\{\delta^i_j (\nabla_k p_h) + \delta^i_h (\nabla_k p_{jh}) + p\Pi_{[jkh]}^i\}.
\]

Above discussion leads us to the following theorem.

\textbf{Theorem 2.1.} An \(NP - F_n\), admitting projective \(N\)-curvature inheritance, the normal projective curvature tensor field can not be expressed in terms of homogeneous scalar function \(p(x, \dot{x})\) given in the form (2.27).
If we contract (2.27) with respect to indices \(i\) and \(j\) and make use of (1.3d), (1.3e) and (1.7a), it is fairly easy to arrive at

\[
\alpha N_{kh} = n\nabla_k p_h - \nabla_h p_k - (\frac{n-1}{n+1}) p G_{ikh}.
\] (2.28)

Thus, we can state:

**Corollary 2.1.** An \(NP - F_n\), admitting projective Ricci-like \(N\)-curvature inheritance, \(\text{Ricci-like normal projective curvature tensor} N_{kh}\) cannot be expressed in terms of only homogeneous scalar function \(P(x, \dot{x})\) given in the form (2.28).

Using (1.19) for \(N^i_{jkh}\), we get

\[
\dot{\partial}_l (\mathcal{L} N^i_{jk}) = \mathcal{L} (\dot{\partial}_l N^i_{jk}),
\] (2.29)

which in view of (2.24) and (2.29) reduces to

\[
\alpha (\dot{\partial}_l N^i_{jk}) = \mathcal{L} (\dot{\partial}_l N^i_{jk}),
\] (2.30)

where \(\alpha\) is scalar function.

Hence we can state:

**Lemma 2.1.** An \(NP - F_n\), admitting projective \(N\)-curvature inheritance, the partial derivative of the normal projective curvature tensor satisfies the inheritance property (2.30).

Contracting (2.30) with respect to indices \(i, j\) and then using (1.7a), we get

\[
\alpha (\dot{\partial}_l N_{kh}) = \mathcal{L} (\dot{\partial}_l N_{kh}).
\] (2.31)

Accordingly, we can state:

**Lemma 2.2.** An \(NP - F_n\), admitting projective Ricci-like \(N\)-curvature inheritance, the partial derivative of the Ricci-like normal projective curvature tensor satisfies the inheritance property (2.31).

Now using the commutation formula (1.13) for the normal projective curvature tensor \(N^i_{jkh}\), we have

\[
\mathcal{L} \nabla_{\dot{x}} N^i_{jkh} = N^i_{lmr} N^r_{jkh} - N^r_{lmj} N^i_{rkh} - N^r_{lmk} N^i_{jrh} - N^r_{lmh} N^i_{jkr} - (\dot{\partial}_r N^i_{jkh}) N^r_{lmn} \dot{x}^n.
\] (2.32)

Applying Lie-derivative operator to both sides of the above equation, we get

\[
\mathcal{L} \mathcal{L} \nabla_{\dot{x}} N^i_{jkh} = 2\alpha (N^i_{lmr} N^r_{jkh} - N^r_{lmj} N^i_{rkh} - N^r_{lmk} N^i_{jrh} - N^r_{lmh} N^i_{jkr} - (\dot{\partial}_r N^i_{jkh}) N^r_{lmn} \dot{x}^n)
\] (2.33)

In view of (2.24) and Lemma 2.1 along with the fact \(\mathcal{L} \dot{x} = 0\). The above equation together with (2.32) simplifies to yield

\[
\mathcal{L} \nabla_{\dot{x}} N^i_{jkh} = 2\alpha \nabla_{\dot{x}} \nabla_{\dot{x}} N^i_{jkh}.
\] (2.34)

Hence we can state:
Theorem 2.2. An $NP - F_n$, admitting projective $N$-curvature inheritance, the relation (2.34) holds good.

Contracting the equation (2.34) with respect to indices $i$, $j$ and then using (1.7a), we obtain

$$\mathcal{L} \nabla_i \nabla_m N_{kh} = 2\alpha \nabla_i [\nabla_m] N_{kh}. \quad (2.35)$$

Hence we can state:

Corollary 2.2. An $NP - F_n$, admitting projective Ricci-like $N$-curvature inheritance, the relation (2.35) holds good.

In view of the relation (1.18) for $N_{jkh}^i$, and making use of the equation (2.24), we define

$$\mathcal{L} (\nabla_i N_{jk}^i) - \nabla_i (\mathcal{L} N_{jk}^i) = (\mathcal{L} \Pi_r^i) N_{jkr}^i - (\mathcal{L} \Pi_j^i) N_{rkh}^i - (\mathcal{L} \Pi_k^i) N_{rjh}^i - (\mathcal{L} \Pi_{lm}^i) \tilde{x}^m (\partial_r N_{jkr}^i), \quad (2.36)$$

provided the gradient vector $\nabla_i \alpha = \alpha_l$ is zero.

Since the infinitesimal transformation (1.15) is a projective motion, in view of (1.22a) and (1.6b) it becomes

$$\mathcal{L} (\nabla_i N_{jk}^i) - \alpha \nabla_i N_{jk}^i = \delta^i_l p_r N_{jkr}^i - p_j N_{rkh}^i - p_k N_{jih}^i - 2p N_{jkh}^i, \quad (2.37)$$

Transvecting (2.37) by $\dot{x}^l$, we find

$$\mathcal{L} (\nabla_i N_{jk}^i) \dot{x}^l - \alpha \nabla_i N_{jk}^i \dot{x}^l = \dot{x}^l p_r N_{jkr}^i - p_j N_{rkh}^i \dot{x}^l - p_k N_{jih}^i \dot{x}^l - 2p N_{jkh}^i, \quad (2.38)$$

in the light of (1.22a) and (1.6b).

Contracting the equation (2.38) with respect to the indices $i$, $j$ and there after using (1.7a), we get

$$\mathcal{L} (\nabla_i N_{kh}) \dot{x}^l - \alpha \nabla_i N_{kh} \dot{x}^l = -p_k N_{lkh} \dot{x}^l - 2p N_{kh}. \quad (2.39)$$

But the contraction of (2.38) with respect to the indices $i$ and $h$ and in view of (1.7c) yields

$$\mathcal{L} (\nabla_i N_{[kj]} \dot{x}^l - \alpha \nabla_i N_{[kj]} \dot{x}^l = \frac{1}{2} \dot{x}_i p_r N_{[r}^j \dot{x}^l - p_j N_{[kj] \dot{x}^l - p_k N_{[lj]} \dot{x}^l - 2p N_{[kj} \quad (2.40)$$

Hence we can state:

Theorem 2.3. An $NP - F_n$, admitting projective $N$-curvature inheritance, the relation (2.39) and (2.40) necessarily hold provided the gradient vector $\alpha_l$ is zero.
R.B. Misra and F.M. Meher [1] have defined a Recurrent $NP - F_n$ as under:

In a non flat $NP - F_n$, if there exists a non zero vector field whose components $\lambda_l$ are positively homogeneous of degree zero in directional arguments, such that the normal projective curvature tensor $N_{jkh}^i$ satisfies

$$\nabla_l N_{jkh}^i = \lambda_l N_{jkh}^i,$$

(2.41)

then such a $NP - F_n$ is called a Recurrent $NP - F_n$ or briefly $RNP - F_n$. Contracting (2.41) with respect to the indices $i$ and $j$, we have

$$\nabla_i N_{kh} = \lambda_i N_{kh},$$

(2.42)

which shows that an $NP - F_n$ of recurrent curvature is also of Ricci recurrent curvature.

The covariant derivative of (2.41) with respect to $x^m$ gives

$$\nabla_m \nabla_l N_{jkh}^i = \nabla_m \lambda_l N_{jkh}^i + \lambda_l \nabla_m N_{jkh}^i,$$

(2.43)

commuting (2.43) with respect to the indices $m$ and $l$, we get

$$\nabla_{[m} \nabla_{l]} N_{jkh}^i = \nabla_{[m} \lambda_{l]} N_{jkh}^i.$$

(2.44)

Applying Lie-derivative operator to both sides of (2.44), we find

$$\mathcal{L} \nabla_{[m} \nabla_{l]} N_{jkh}^i = (\mathcal{L} \nabla_{[m} \lambda_{l]} + \alpha \nabla_{[m} \lambda_{l]}) N_{jkh}^i.$$

(2.45)

In view of (2.24) and 2.2, the above equation reduces to

$$2\alpha \nabla_{[m} \nabla_{l]} N_{jkh}^i = (\mathcal{L} \nabla_{[m} \lambda_{l]} + \alpha \nabla_{[m} \lambda_{l]}) N_{jkh}^i.$$

(2.46)

If we assume that $\mathcal{L} \nabla_{[m} \lambda_{l]} = -\alpha \nabla_{[m} \lambda_{l]}$, the equation (2.46) takes the form

$$\nabla_{[m} \nabla_{l]} N_{jkh}^i = 0.$$

(2.47)

Contracting (2.47) with respect to the indices $i$ and $j$, we arrive at

$$\nabla_{[m} \nabla_{l]} N_{kh} = 0.$$

(2.48)

Conversely, if (2.47) is true, the equation (2.46) yields

$$(\mathcal{L} \nabla_{[m} \lambda_{l]} + \alpha \nabla_{[m} \lambda_{l]}) N_{jkh}^i = 0.$$

(2.49)

Since the recurrent $NP - F_n$ is non-flat, (2.49) provides us

$$\mathcal{L} \nabla_{[m} \nabla_{l]} = -\alpha \nabla_{[m} \lambda_{l]}.$$  

(2.50)

Accordingly we can state:

**Theorem 2.4.** An $RNP - F_n$, admitting projective $N$-curvature and projective Ricci-like $N$-curvature inheritance, the necessary and sufficient condition for $\nabla_{[m} \nabla_{l]} N_{jkh}^i = 0$ and $\nabla_{[m} \nabla_{l]} N_{kh} = 0$ to be true is that the recurrence vector $\lambda_l$ satisfies the inheritance property (2.50).
§ 3. Special cases

In this section, we wish to study three special cases of projective N-curvature inheritance in $NP - F_n$, $RNP - F_n$ and $SNP - F_n$ spaces.

(a). Contra field: In an $NP - F_n$, if the vector field $v^i(x)$ satisfies the relation

$$\nabla_j v^i = 0,$$

then the vector field $v^i(x)$ spans a contra field.

Here we consider a special infinitesimal transformation

$$\bar{x}^i = x^i + \varepsilon v^i(x),$$

which admits a projective motion in $NP - F_n$. It is assumed that the relation (2.24) is also satisfied in $NP - F_n$, then the transformation (3.52) defines a projective N-curvature inheritance. Employing (1.21) in (1.17), we obtain

$$2\delta^i_{(j|p_k)} = \nabla_j \nabla_k v^i + N^i_{hjk} v^h + \Pi_{hjk} (\nabla_l v^h) \dot{x}^l.$$  

If $v^i(x)$ spans a contra field, then (3.53) assumes the following

$$N^i_{hjk} v^h = 2\delta^i_{(j|p_k)}$$

The covariant differentiation of (3.54) with respect to $x^l$ together with (3.51) yields

$$-(\nabla_l N^i_{jhk}) v^h = \delta^i_{l} \nabla_i (p_k) + \delta^i_{k} \nabla_i (p_j),$$

where we have taken into account (1.6a). Commutating (3.55) with respect to the indices $l$ and $k$, we get

$$2\nabla[l N^i_{jhk]} v^h = 2\{\delta^i_{[k} \nabla_j (p_l) + \delta^i_{j} \nabla_l (p_k]\},$$

where the index in two parallel bars is unaffected when we consider skew-symmetric parts.

Since (3.52) defines a projective N-curvature inheritance therefore in view of (2.27), the equation (3.56) assumes the form

$$2\nabla[l N^i_{jhk]} v^h = \alpha N^i_{kl} - 2\Pi^i_{[klj]}.$$  

Contracting (3.57) with respect to indices $i$, $j$ and making use of (1.3e), (1.7a) and (1.7c), we have

$$\nabla[l N^i_{jhk}] v^h = \alpha N^i_{[kl]}$$

Hence we can state:

**Theorem 3.5.** An $NP - F_n$, admitting projective N-curvature and projective Ricci-like N-curvature inheritance, if the vector field $v^i(x)$ spans contra field, the relations (3.57) and (3.58) hold good.
Now if the space under consideration is a recurrent $NP - F_n$, then in view of (1.17), the equation (3.55) becomes
\begin{equation}
- \lambda \eta_{jk}^i v^h = \delta_j^i \nabla_l p_k + \delta^j_k \nabla_l p_j.
\end{equation}

Commutating (3.59) with respect to indices $l$ and $k$, we get
\begin{equation}
2 \lambda \eta_{jk}^i v^h = 2 \{ \delta^j_k \nabla_l p_j + \delta^j_k \nabla_l p_k \}.
\end{equation}

We now assume that the transformation (3.52) defines a projective N-curvature inheritance in $RNP - F_n$ also. Then in view of (2.27), the equation (3.60) is defined in the form
\begin{equation}
2 \lambda \eta_{jk}^i v^h = \alpha N^i_{klj} - 2 \pi \Pi^i_{[kl]j},
\end{equation}

Contracting (3.61) with respect to indices $i$ and $j$, we get
\begin{equation}
\lambda \eta_{jk}^i v^h = \alpha N^i_{[kl]},
\end{equation}

where we have taken into account (1.3e), (1.7a) and (1.7c).

Thus, we can state:

**Theorem 3.6.** An $RNP - F_n$, admitting projective N-curvature and projective Ricci-like N-curvature inheritance, if the vector field $v^i(x)$ spans a contra field then the relations (3.61) and (3.62) necessarily holds.

Next, we assume that the transformation (3.52) define a projective N-curvature inheritance in $SNP - F_n$ also. Then the equation (3.57) is define in the form
\begin{equation}
\alpha N^i_{klj} = 2 \pi \Pi^i_{[kl]j},
\end{equation}

Hence we can state:

**Theorem 3.7.** An $SNP - F_n$, admitting projective N-curvature inheritance, if the vector field $v^i(x)$ spans a contra field then the relation (3.63) necessarily holds.

Contracting (3.63) with respect to indices $i$ and $j$ and using the equations (1.3e) and (1.7c), we have
\begin{equation}
\alpha N_{ik} = \alpha N_{kl}
\end{equation}

Thus, we can state:

**Corollary 3.3.** In an $SNP - F_n$, admitting projective Ricci-like N-curvature inheritance, if the vector field $v^i(x)$ spans a contra field, the relations (3.64) necessarily symmetric.

**b. Concurrent field:** In an $NP - F_n$, if the vector field $v^i(x)$ satisfies the relation
\begin{equation}
\nabla_j v^i = c \delta^i_j,
\end{equation}
where c is non zero constant then the vector field $v^i(x)$ determines a concurrent field.

In this section, we consider the infinitesimal transformation.

\begin{equation}
\bar{x}^i = x^i + \varepsilon v^i(x), \nabla_j v^i = c \delta_j^i,
\end{equation}

which admits a projective motion and defines a projective N-curvature inheritance in $NP - F_n$. The covariant derivative of (3.54) with respect to $x^l$ and using (3.65), we get

\begin{equation}
\nabla_l N^i_{jk} v^h + c N^i_{lj} = \delta_j^i \nabla_l p_k + \delta_k^i \nabla_l p_j,
\end{equation}

where we have taken into account (1.6a).

Commutating (3.67) with respect to indices $l$ and $j$, we get

\begin{equation}
2 \nabla [l N^i_{hjk} v^h + c N^i_{lj}] = \alpha N^i_{lk} - 2 \Pi^i_{[lj]k},
\end{equation}

where we have made us of equations (1.6a) and (2.27).

Contracting (3.68) with respect to indices $i$ and $j$ and using the equations (1.3d), (1.3e), (1.7a) and (1.7b), we get

\begin{equation}
\alpha N_{ik} = -\{v^h (\nabla_l N_{hk} + \nabla_l N_{hik}) + 2c N_{ik} + \left(\frac{n-1}{n+1}\right) p G_i^j\}.
\end{equation}

Hence we can state:

**Theorem 3.8.** In an $NP - F_n$, admitting projective N-curvature and projective Ricci-like N-curvature inheritance, if the vector field $v^i(x)$ determines a concurrent field then the relations (3.68) and (3.69) necessarily hold.

Let us assume that the space under consideration is a $RNP - F_n$ and the transformation (3.66) defines a projective N-curvature inheritance in it. In this case the relation (3.67) assumes the form

\begin{equation}
\lambda_i N^i_{ljk} v^h + c N^i_{lj} = \delta_j^i \nabla_l p_k + \delta_k^i \nabla_l p_j.
\end{equation}

Commutating (3.20) with respect to indices $l$ and $j$, we get

\begin{equation}
\lambda_i N^i_{ljk} v^h + c N^i_{lj} = \delta_j^i \nabla_l p_k + \delta_k^i \nabla_l p_j,
\end{equation}

In view of (2.27), (3.71) reduces to

\begin{equation}
2 \lambda_i N^i_{ljk} v^h + 2c N^i_{lj} = \alpha N^i_{lk} - 2 \Pi^i_{[lj]k},
\end{equation}

Contracting (3.72 with respect to indices $i$ and $j$ and thereafter using the equations (1.3d), (1.7a) and (1.7b), we obtain

\begin{equation}
\alpha N_{ik} = -\{v^h (\lambda_i N_{hk} + \lambda_i N_{hik}) + 2c N_{ik}\}.
\end{equation}

Accordingly we can state:
Theorem 3.9. In an RNP $- F_n$, admitting Projective N-curvature and projective Ricci-like N-curvature inheritance, if the vector field $v^i(x)$, determines a concurrent field then the relations (3.72) and (3.73) hold good.

Next, we assume that the transformation (3.66) defines a projective N-curvature inheritance in SNP $- F_n$. Then the equation (3.68) and (3.69) can be written in the form

$$\alpha N^i_{jlk} = 2cN^i_{[j][k]} + 2\Pi^i_{[j][k]}.$$  

(3.74)

Contracting (3.74) with respect to indices $i$ and $j$ and thereafter using the equations (1.3d), (1.3e), (1.7a) and (1.7b) gives

$$\alpha N_{lk} = -\{2cN_{lk} + \frac{n-1}{n+1}pG^i_{[i]}\}.$$  

(3.75)

Thus, we can state:

Theorem 3.10. In an SNP $- F_n$, admitting projective N-curvature and projective Ricci-like N-curvature inheritance, if the vector field $v^i(x)$ determines a concurrent field then the relations (3.74) and (3.75) hold good.

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References


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