$L^\infty$-uniqueness of Schrödinger operators on a Riemannian manifold

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Abstract. The main purpose of this paper is to study $L^\infty$-uniqueness of Schrödinger operators and generalized Schrödinger operators on a complete Riemannian manifold. Also, we prove the $L^1(E, d\mu)$-uniqueness of weak solutions for the Fokker-Planck equation associated with this pre-generators.

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§ 1. Preliminaries

Let $E$ be a Polish space equipped with a $\sigma$-finite measure $\mu$ on its Borel $\sigma$-field $B$. It is well known that, for a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on $L^1(E, d\mu)$, its adjoint semigroup $\{T^\ast(t)\}_{t \geq 0}$ is no longer strongly continuous on the dual topological space $L^\infty(E, d\mu)$ of $L^1(E, d\mu)$ with respect to the strong dual topology of $L^\infty(E, d\mu)$. In [15] Wu and Zhang introduce a new topology on $L^\infty(E, d\mu)$ for which the usual semigroups in the literature becomes $C_0$-semigroups. That is the topology of uniform convergence on compact subsets of $(L^1(E, d\mu), \| \cdot \|_1)$, denoted by $C(L^\infty, L^1)$. More precisely, for an arbitrary point $y_0 \in L^\infty(E, d\mu)$, a basis of neighborhoods with respect to $C(L^\infty, L^1)$ is given by:

$$N(y_0; K, \varepsilon) := \left\{ y \in L^\infty(E, d\mu) \left| \sup_{x \in K} |\langle x, y \rangle - \langle x, y_0 \rangle| < \varepsilon \right. \right\} \tag{1.1}$$

where $K$ runs over all compact subsets of $(L^1(E, d\mu), \| \cdot \|_1)$ and $\varepsilon > 0$.

If $\{T(t)\}_{t \geq 0}$ is a $C_0$-semigroup on $L^1(E, \mu)$ with generator $L$, then $\{T^\ast(t)\}_{t \geq 0}$ is a $C_0$-semigroup on $(L^\infty(E, d\mu), C(L^\infty, L^1))$ with generator $L^\ast$. Moreover, one can prove that $(L^\infty(E, d\mu), C(L^\infty, L^1))$ is complete and that the topological dual of $(L^\infty(E, d\mu), C(L^\infty, L^1))$ is $(L^1(E, d\mu), \| \cdot \|_1)$.

Let $A : D \rightarrow L^\infty(E, d\mu)$ be a linear operator with domain $D$ dense in $(L^\infty(E, d\mu))$, with respect to the topology $C(L^\infty, L^1)$. $A$ is said to be a pre-generator in $(L^\infty(E, d\mu))$, if there exists some $C_0$-semigroup on $(L^\infty(E, d\mu), C(L^\infty, L^1))$ such that its generator $L$ extends $A$. We say that $A$ is an essential generator in $(L^\infty(E, d\mu))$ (or...
\( L^\infty(\mathbb{E}, d\mu), C (L^\infty, L^1) \)-unique), if \( \mathcal{A} \) is closable and its closure \( \overline{\mathcal{A}} \) with respect to \( C (L^\infty, L^1) \) is the generator of some \( C_0 \)-semigroup on \( (L^\infty, d\mu), C (L^\infty, L^1) \). This uniqueness notion was studied by Arendt [1], Eberle [5], Djellout [4], Röckner [10], Wu [13] and [14] and others in the Banach spaces setting and Wu and Zhang [15], Lemle and Wu [6] in the case of locally convex spaces. The main result concerning \( (L^\infty(\mathbb{E}, d\mu), C (L^\infty, L^1)) \)-uniqueness of pre-generators is (see [15] or [6] for much more general results)

**Theorem 1.1.** Let \( \mathcal{A} \) be a linear operator on \( (L^\infty(\mathbb{E}, d\mu), C (L^\infty, L^1)) \) with domain \( \mathcal{D} \) (the test-function space) which is assumed to be dense in \( (L^\infty(\mathbb{E}, d\mu), C (L^\infty, L^1)) \). Assume that there is a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) on \( (L^\infty(\mathbb{E}, d\mu), C (L^\infty, L^1)) \) such that its generator \( \mathcal{L} \) is an extension of \( \mathcal{A} \) (i.e., \( \mathcal{A} \) is a pre-generator in \( (L^\infty(\mathbb{E}, d\mu), C (L^\infty, L^1)) \)). The following assertions are equivalent:

(i) \( \mathcal{A} \) is a \( (L^\infty(\mathbb{E}, d\mu), C (L^\infty, L^1)) \)-essential generator (or \( (L^\infty(\mathbb{E}, d\mu), C (L^\infty, L^1)) \)-unique);

(ii) the closure of \( \mathcal{A} \) in \( (L^\infty(\mathbb{E}, d\mu), C (L^\infty, L^1)) \) is exactly \( \mathcal{L} \) (i.e., \( \mathcal{D} \) is a core for \( \mathcal{L} \));

(iii) \( \mathcal{A}^* = \mathcal{L}^* \) which is the generator of the dual \( C_0 \)-semigroup \( \{T^*(t)\}_{t \geq 0} \) on \( (L^1(\mathbb{E}, d\mu)) \);

(iv) for some \( \lambda > \omega \) (\( \omega \geq 0 \) is the constant in definition of \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \)), the range \( (\lambda I - \mathcal{L})(\mathcal{D}) \) is dense in \( (L^\infty(\mathbb{E}, d\mu), C (L^\infty, L^1)) \);

(v) (Liouville property) for some \( \lambda > \omega \), \( \ker (\lambda I - \mathcal{A}^*) = \{0\} \) (i.e., if \( y \in \mathcal{D}(\mathcal{A}^*) \) satisfies \( (\lambda I - \mathcal{A}^*)y = 0 \), then \( y = 0 \));

(vi) (uniqueness of solutions for the resolvent equation) for all \( \lambda > \omega \) and all \( y \in L^1(\mathbb{E}, d\mu) \), the resolvent equation of \( \mathcal{A}^* \)

\[
(\lambda I - \mathcal{A}^*)z = y
\]

has the unique solution \( z = ((\lambda I - \mathcal{L})^{-1}y = (\lambda I - \mathcal{L}^*)^{-1}y) \);

(vii) (uniqueness of strong solutions for the Cauchy problem) for each \( x \in \mathcal{D}(\overline{\mathcal{A}}) \), there is a \( (L^\infty(\mathbb{E}, d\mu), C (L^\infty, L^1)) \)-unique strong solution \( v(t) = T(t)x \) of the Cauchy problem (or the Kolmogorov backward equation)

\[
\begin{align*}
\partial_t v(t) &= \overline{\mathcal{A}} v(t) \\
v(0) &= x
\end{align*}
\]

i.e., \( t \mapsto v(t) \) is differentiable from \( \mathbb{R}^+ \) to \( (L^\infty(\mathbb{E}, d\mu), C (L^\infty, L^1)) \) and its derivative \( \partial_t v(t) \) coincides with \( \overline{\mathcal{A}} v(t) \);

(viii) (uniqueness of weak solutions for the dual Cauchy problem) for every \( y \in L^1(\mathbb{E}, d\mu) \), the dual Cauchy problem (the Kolmogorov forward equation or the Fokker-Planck equation)

\[
\begin{align*}
\partial_t u(t) &= \mathcal{A}^* u(t) \\
u(0) &= y
\end{align*}
\]

has a \( (L^1(\mathbb{E}, d\mu)) \)-unique weak solution \( u(t) = T^*(t)y \). More precisely, there is a unique function \( \mathbb{R}^+ \ni t \mapsto u(t) = T^*(t)y \) which is continuous from \( \mathbb{R}^+ \) to \( (L^1(\mathbb{E}, d\mu), \| \cdot \|_1) \) such that

\[
\langle x, u(t) - y \rangle = \int_0^t \langle A x, u(s) \rangle \, ds , \quad \forall x \in \mathcal{D};
\]
(ix) there is only one \( C_0 \)-semigroup on \((L^\infty(E, d\mu), \mathcal{C}(L^\infty, L^1))\) such that its generator extends \( A \).

(x) there is only one \( C_0 \)-semigroupe on \((L^1(E, d\mu), \| \|_1)\) such that its generator is contained in \( A^* \).

\section{\( L^\infty \)-uniqueness of Schrödinger operators on a complete Riemannian manifold}

Let \( M \) be a complete Riemannian manifold with volume measure \( dx \). Denote by \( \Delta \) the Laplace-Beltrami operator with domain \( C_\infty^0(M) \). The \( L^p(M, dx) \)-uniqueness of \( C_0 \)-semigroup generated by \((\Delta, C_\infty^0(M))\) is proven by Strichartz \[12\] for \( 1 < p < \infty \), by Davies \[3\] for \( p = 1 \) and by Li \[7\] for \( p = \infty \).

In fact, Li \[7\] showed that if \( M \) is complete and has bounded geometry, then \( M \) satisfies \( L^1 \)-Liouville theorem: if the Ricci curvature of \( M \) has a negative quadratic lower bound, i.e., if there exists a fixed point \( 0 \in M \) and a constant \( C > 0 \) such that

\[
\text{Ric}(x) \geq -C \left(1 + d(x,0)^2\right), \quad \forall x \in M
\]

where \( d(x,0) \) denotes distance from \( x \) to \( o \), then every non-negative \( L^1 \)-integrable subharmonic function must be constant. Under the same condition, Li also proved that for each \( h \in L^1(M, dx) \), the heat diffusion equation

\[
\begin{cases}
\partial_t u(t) = \Delta u(t) \\
u(0) = h
\end{cases}
\]

has one \( L^1(M, dx) \)-unique weak solution.

Consider the Schrödinger operator \((A, C_\infty^0(M))\)

\[
A = \frac{\Delta}{2} - V
\]

where \( V \geq 0 \). In that case, we have

\textbf{Theorem 2.2.} Let \( M \) be a complete Riemannian manifold such that its Ricci curvature has a negative quadratic lower bound. Then \((A, C_\infty^0(M))\) is \( L^\infty(M, dx) \)-unique with respect to the topology \( \mathcal{C}(L^\infty, L^1) \).

\textbf{Proof.} We prove the theorem by two steps.

\textbf{step 1} \((A, C_\infty^0(M))\) is a pre-generator on \((L^\infty(M, dx), \mathcal{C}(L^\infty, L^1))\).

Let \((B_t)_{t \geq 0}\) be the Brownian Motion with values in the one-point compactification space \( M \cup \partial \) defined on some filtered probabilities space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (B_t)_{t \geq 0})\) with \( \mathbb{P}_x(B_0 = x) = 1 \) for any initial point \( x \in M \). Let denote by

\[
\tau_c = \inf \{ t \geq 0 \mid B_t = \partial \}
\]

the explosion time. Consider the Feynman-Kac semigroup

\[
P^V_t f(x) = \mathbb{E}^x 1_{[t < \tau_c]} f(B_t) e^{-\frac{1}{2} \int_0^t V(B_s) ds}
\]
Since $\{P^V_t\}_{t \geq 0}$ is a $C_0$-semigroup on $L^1(M, dx)$, it is also a $C_0$-semigroup on $L^\infty(M, dx)$ with respect to the topology $C(L^\infty, L^1)$. By Ito’s formula, for every $f \in C^\infty_0(M)$ it follows that

\begin{equation}
 f(B_t)e^{-\int_0^t V(B_s) \, ds} - f(B_0) - \int_0^t \left( \frac{\Delta}{2} - V \right) f(B_s)e^{-\int_0^s V(B_r) \, dr} \, ds
\end{equation}

is a local martingale. As it is bounded over bounded time intervals, it is a true martingale. Thus taking the expectation under $P_x$ in the above formula, we find that

\begin{equation}
 P^V_t f(x) - f(x) = \int_0^t P^V_s \left( \frac{\Delta}{2} - V \right) f(x) \, ds , \quad \forall t \geq 0
\end{equation}

which means that $f$ belongs to the domain of the generator $L^V_{(\infty)}$ of $\{P^V_t\}_{t \geq 0}$, i.e. $L^V_{(\infty)}$ extend $\mathcal{A}$.

**step 2** $(\mathcal{A}, C^\infty_0(M))$ is $(L^\infty(M, dx), C(L^\infty, L^1))$-unique.

By Theorem 1.1, it follows that the operator $(\mathcal{A}, C^\infty_0(M))$ is $L^\infty(M, dx)$-unique if and only if for some $\lambda > \omega$, the range $(\lambda I - \mathcal{A})(C^\infty_0(M))$ is dense in $(L^\infty, C(L^\infty, L^1))$.

It is enough to show that if $h \in L^1(M, dx)$ satisfies $(\lambda I - \mathcal{A})h = 0$ in the sense of distribution, then $h = 0$.

Let $h \in L^1(M, dx)$ such that for some $\lambda > \omega$

\begin{equation}
 (\lambda I - \mathcal{A})h = 0
\end{equation}

in the distribution sense. Then, by Kato’s inequality, we have

\begin{equation}
 \Delta |h| \geq sgn(h) \Delta h = 2 sgn(h)(\lambda + V)(h) = 2(\lambda + V)|h| \geq 0 .
\end{equation}

Thus $|h|$ is a subharmonic function. By [7, Theorem 1, p. 447] it follows that $|h|$ is constant, therefore $h$ is constant. Consequently $h = 0$.

**Corollary 2.3.** If $M$ is a complete Riemannian manifold such that its Ricci curvature has a negative quadratic lower bound, then for every $h \in L^1(M, dx)$, the Fokker-Planck equation

\begin{equation}
 \begin{cases}
 \partial_t u(t) = \left( \frac{\Delta}{2} - V \right) u(t) \\
 u(0) = h 
\end{cases}
\end{equation}

has one $L^1(M, dx)$-unique weak solution.

**Proof.** By the Theorem 1.1 the $L^1(M, dx)$-uniqueness of weak solutions for the heat diffusion equation

\begin{equation}
 \begin{cases}
 \partial_t u(t) = \left( \frac{\Delta}{2} - V \right) u(t) \\
 u(0) = h 
\end{cases}
\end{equation}

is equivalent to the $(L^\infty(M, dx), C(L^\infty, L^1))$-uniqueness of the Schrödinger operator $(\mathcal{A}, C^\infty_0(M))$ which follows by the Theorem 2.2. \qed
§ 3. $L^\infty$-uniqueness of generalized Schrödinger operators on a complete Riemannian manifold

Consider the generalized Schrödinger operator (or the Nelson’s diffusion operator in stochastic mechanics see [14])

$$ (3.17) \quad \mathcal{A} = \Delta - \nabla \phi \cdot \nabla $$

with domain $C_0^\infty(M)$ and an invariant measure $d\mu_\phi = e^{-\phi(x)}dx$, where $\Delta$ is the Laplace-Beltrami operator and $\nabla$ is the gradient operator on $M$ and $\phi \in C^\infty(M)$. The study of this operator is very important because it is well known that $\mathcal{A} = \Delta - \nabla \phi \cdot \nabla$ considered as a symmetric operator on $L^2(M, e^{-\phi(x)}dx)$ is unitary equivalent to the Schrödinger operator $H = \Delta - V$, with $V = \frac{1}{2}|\nabla \phi|^2 - \frac{1}{2}\Delta \phi$ considered on $L^2(M, dx)$.

This unitary isomorphism $U : L^2(M, e^{-\phi(x)}dx) \to L^2(M, dx)$ is given by $Uf = e^{-\frac{x}{2}}f$. In the case where $M = \mathbb{R}^d$ or $M = D$ is an open domain in $\mathbb{R}^d$ for the diffusion operator $\mathcal{A} = \Delta + 2\nabla \phi \cdot \nabla$, $L^1(M, \phi^2dx)$-uniqueness has been studied by Liskevitch [9], Stannat [11] and Wu [14].

The purpose of this section is to study $L^\infty(M, e^{-\phi(x)}dx)$-uniqueness of the operator $(\mathcal{A}, C_0^\infty(M))$ using some very recent result of Li [8] concerning the Liouville theorem on a complete Riemannian manifold. Recall that for any constant $m \geq n = \dim(M)$ the symmetric 2-tensor

$$ (3.18) \quad Ric_{m,n}(\mathcal{A})(x) = Ric(x) + \nabla^2 \phi(x) - \frac{\nabla \phi(x) \otimes \nabla \phi(x)}{m-n}, \quad \forall x \in M $$

is called the Bakry-Emery Ricci curvature of the operator $\mathcal{A}$ (see [2]), with the convention that $m = n$ iff $\mathcal{A} = \Delta$.

The main result of this section is

**Theorem 3.4.** Let $M$ be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exists two constants $m > n$ and $C > 0$ such that

$$ (3.19) \quad Ric_{m,n}(\mathcal{A})(x) \geq -C \left(1 + d^2(x,0)\right), \quad \forall x \in M $$

where $d(x,0)$ is the distance from $x$ to a fixed point $0 \in M$. Then $(\mathcal{A}, C_0^\infty(M))$ is $L^\infty(M, e^{-\phi(x)}dx)$-unique with respect to the topology $C(L^\infty, L^1)$.

**Proof.** We prove the theorem by two steps.

**step 1** $(\mathcal{A}, C_0^\infty(M))$ is a pre-generator on $(L^\infty(M, e^{-\phi(x)}dx), C(L^\infty, L^1))$.

Let $(W_t)_{t \geq 0}$ be the Riemannian Brownian motion on $M$. The corresponding stochastic differential equation

$$ (3.20) \quad dX_t = \sqrt{2}dW_t - \nabla(X_t)dt $$

admits one unique martingale solution $(X_t)_{0 \leq t < \tau_e}$, where $\tau_e$ is the explosion time. The transition semigroup $\{P_t\}_{t \geq 0}$ is given by the Feynman-Kac formula

$$ (3.21) \quad P_tf(x) = \mathbb{E}^xf(X_t)1_{[t < \tau_e]}, \quad f \in C_0^\infty(M) $$
Since \( \{P_t\}_{t \geq 0} \) is a \( C_0 \)-semigroup on \( L^1(M, e^{-\phi(x)}dx), \| \cdot \|_1 \) it is also a \( C_0 \)-semigroup on \( L^\infty(M, e^{-\phi(x)}dx) \) with respect to the topology \( C(L^\infty, L^1) \). By Ito’s formula, for every \( f \in C_0^\infty(M) \)

\[
(3.22) \quad f(X_t) - f(x) = \int_0^t Af(X_s)ds
\]

is a local martingale. As it is bounded over bounded time intervals, it is a true martingale. By taking the expectation under \( P_x \) in the above formula, we find that

\[
(3.23) \quad P_t f(x) - f(x) = \int_0^t P_s Af(x)ds \quad \forall t \geq 0
\]

which means that \( f \) belongs to the domain of the generator \( \mathcal{L}_{(\infty)} \) of \( \{P_t\}_{t \geq 0} \) i.e., \( \mathcal{L}_{(\infty)} \) extend \( A \).

**step 2** \( (A, C_0^\infty(M)) \) is \( L^\infty(M, e^{-\phi(x)}dx) \)-unique with respect to the topology \( C(L^\infty, L^1) \). By Theorem 1.1, it follows that the operator \( (A, C_0^\infty(M)) \) is \( L^\infty(M, e^{-\phi(x)}dx) \)-unique if and only if for some \( \lambda > \omega \), the range \( (A - \lambda I)(C_0^\infty(M)) \) is dense in \( (L^\infty, C(L^\infty, L^1)) \). It is enough to show that if \( u \in L^1(M, e^{-\phi(x)}dx) \) satisfies \( (A - \lambda I)u = 0 \) in the sense of distribution, then \( u = 0 \).

Let \( u \in L^1(M, e^{-\phi(x)}dx) \) such that

\[
(3.24) \quad (A - I)u = 0
\]

in the distribution sense, i.e.

\[
(3.25) \quad \langle u, (A - I)f \rangle = 0 \quad \forall f \in C_0^\infty(M).
\]

Then from

\[
(3.26) \quad \int_M u \Delta f e^{-\phi(x)}dx - \int_M u \nabla \phi \nabla f e^{-\phi(x)}dx = \int_M uf e^{-\phi(x)}dx
\]

we will obtain the following equality

\[
(3.27) \quad -\int_M \nabla u \nabla f e^{-\phi(x)}dx - \int_M u \nabla \phi \nabla f e^{-\phi(x)}dx = \int_M uf e^{-\phi(x)}dx
\]

for all positive function \( f \in H^{1,2}(M, e^{-\phi(x)}dx) \) with compact support. Now on can follow Eberle [5, proof of Theorem 2.5, Step 2] to show that (an inequality of Kato’s type)

\[
(3.28) \quad -\int_M \nabla |u| \nabla f e^{-\phi(x)}dx - \int_M |u| \nabla \phi \nabla f e^{-\phi(x)}dx \geq \int_M |u| f e^{-\phi(x)}dx
\]

for all positive function \( f \in H^{1,2}(M, e^{-\phi(x)}dx) \) with compact support. But this is equivalent to

\[
(3.29) \quad \int_M |u| Af e^{-\phi(x)}dx \geq \int_M |u| f e^{-\phi(x)}dx \geq 0.
\]
That means $|u|$ is a $A$-subharmonic function. By the Theorem [8, Theorem 7.1], it follows that $|u|$ must be a constant function and then $u = 0$. By the Theorem 1.1 it follows that $(A, C^\infty_0(M))$ is $L^\infty(M, e^{-\phi(x)}dx)$-unique.

\textbf{Corollary 3.5.} In the hypothesis of Theorem 3.4, for every $h \in L^1(M, dx)$, the Fokker-Planck equation

\begin{equation}
\begin{aligned}
\partial_t u(t) &= A^* u(t) \\
u(0) &= h
\end{aligned}
\end{equation}

has one $L^1(M, dx)$-unique weak solution.

\textbf{Proof.} The assertion follows by Theorem 1.1 and the Theorem 3.4. \hfill \Box

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