On weak symmetries of almost $r-$paracontact Riemannian manifold of $P-$Sasakian type

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Abstract. In this paper, we consider weakly symmetric and weakly Ricci-symmetric almost $r-$paracontact Riemannian manifolds of $P-$Sasakian type. We find necessary conditions in order that an almost $r-$paracontact Riemannian manifold of $P-$Sasakian type be weakly symmetric and weakly Ricci-symmetric.

Key words: $r-$paracontact Riemannian manifold, weakly symmetric, weakly Ricci-symmetric.

§ 1. Introduction

Weakly symmetric Riemannian manifolds are generalizations of locally symmetric manifolds and pseudo-symmetric manifolds. These are manifolds in which the covariant derivative $DR$ of the curvature tensor $R$ is a linear expression in $R$. The appearing coefficients of this expression are called associated 1−forms. They satisfy in the specified types of manifolds gradually weaker conditions.

Firstly, the notions of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by L. Tamásy and T.Q. Binh in 1992 (see [4],[5]). In [4], the authors considered weakly symmetric and weakly projective-symmetric Riemannian manifolds. In 1993, the authors considered weakly symmetric and weakly Ricci-symmetric Einstein and Sasakian manifolds [5]. In 2000, U. C. De, T.Q. Binh and A.A. Shaikh gave necessary conditions for the compatibility of several $K-$contact structures with weak symmetry and weakly Ricci-symmetry [1]. In 2002, C. Özgür, considered weakly symmetric and weakly Ricci-symmetric Lorentzian para-Sasakian manifolds [6]. Recently in [7], C. Özgür studied weakly symmetric Kenmotsu manifolds.

In this study, we consider weakly symmetric and weakly Ricci-symmetric almost $r-$paracontact Riemannian manifold of $P-$Sasakian type.

§ 2. Preliminaries

Let $(M^n, g)$ be an $n-$dimensional Riemann manifold. We denote by $D$ the covariant differentiation with respect to the Riemann metric $g$. Then we have

$$R(X, Y)Z = DXDYZ - DYDXZ - D_{[X,Y]}Z.$$
The Riemannian curvature tensor is defined by
\[ R(X, Y, Z, W) = g(R(X, Y)Z, W). \]

The Ricci tensor of \( M \) is defined by
\[ \text{Ric}(X, Y) = \text{trace} \{ Z \rightarrow R(X, Z)Y \}. \]

Locally, \( \text{Ric} \) is given by
\[ \text{Ric}(X, Y) = \sum_{i=1}^{n} R(X, E_i, Y, E_i), \]
where \( \{E_1, E_2, ..., E_n\} \) is a local orthonormal frames field on \( M \) and \( X, Y, Z, W \) are vector fields on \( M \).

The Ricci operator \( Q \) is a tensor field of type \((1, 1)\) on \( M \) defined by
\[ g(QX, Y) = \text{Ric}(X, Y), \]
for all vector fields on \( M \).

A non-flat differentiable manifold \((M^n, g), \ (n > 2)\), is called weakly symmetric if there exist 1–forms \( \alpha, \beta, \gamma, \delta \) and \( \sigma \) on \( M \) such that
\[ (D_X R)(Y, Z, U, V) = \alpha(X)R(Y, Z, U, V) + \beta(Y)R(X, Z, U, V) + \gamma(Z)R(Y, X, U, V) + \delta(U)R(Y, Z, X, V) + \sigma(V)R(Y, Z, U, X) \]
holds for any vector fields \( X, Y, Z, U, V \) on \( M \). A weakly symmetric manifold is said to be proper if \( \alpha = \beta = \gamma = \delta = \sigma = 0 \) is not the case \([1]\).

A differentiable manifold \((M^n, g), \ (n > 2)\), is called weakly Ricci-symmetric if there exist 1–forms \( \rho, \mu, \nu \) such that the relation
\[ (D_X \text{Ric})(Y, Z) = \rho(X)\text{Ric}(Y, Z) + \mu(Y)\text{Ric}(X, Z) + \nu(Z)\text{Ric}(X, Y) \]
holds for all vector fields \( X, Y, Z, U, V \) on \( M \). A weakly Ricci-symmetric manifold is said to be proper if \( \rho = \mu = \nu = 0 \) is not the case \([1]\).

From (2.1), an easy calculation shows that if \( M \) is weakly symmetric then we obtain (see \([4], [5]\))
\[ (D_X \text{Ric})(Z, U) = \alpha(X)\text{Ric}(Z, U) + \beta(Z)\text{Ric}(X, U) + \delta(U)\text{Ric}(Z, X) + \beta(R(X, Z)U) + \delta(R(X, U)Z). \]

\[ § \ 3. \text{ Almost } r-\text{paracontact Riemannian manifolds} \]

We need the following definition \([3]\)

Let \((M, g)\) be a Riemannian manifold with \( \dim(M) = 2n + r \) and denote by \( T(M) \) the tangent space of \( M \). Then \( M \) is said to be an almost \( r-\text{paracontact} \) Riemannian manifold if there exist on \( M \) a tensor field \( \phi \) of type \((1, 1)\) and \( r \) global vector fields \( \xi_1, ..., \xi_s \) (called structure vector fields) such that
(i) If $\eta_1, ..., \eta_r$ are dual 1-forms of $\xi_1, ..., \xi_r$, then:

\[ \eta_i(\xi_j) = \delta_i^j, \quad g(\xi_i, X) = \eta_i(X) \quad \phi^2 = I - \sum_{i=1}^r \xi_i \otimes \eta_i \]

(ii)

\[ g(\phi X, \phi Y) = g(X, Y) - \sum_{i=1}^r \eta_i(X) \eta_i(Y) , \]

for any $X, Y \in T(M)$, $i = 1, ..., r$.

In an almost $r$–paracontact Riemannian manifold $M$, besides the relations (3.4) and (3.5) the following also hold

\[ \phi \xi = 0 \]  
\[ \eta \circ \phi = 0. \]

An almost $r$–paracontact Riemannian manifold $M$ is said to be of $P$–Sasakian type if

\[ D_X \xi_i = \phi X \]
\[ (D_X \phi) Y = - \sum_{i=1}^r \left[ g(\phi X, \phi Y) \xi_i + \eta_i(Y) \phi^2 X \right] \]

for all $X, Y \in T(M)$.

In an almost $r$–paracontact Riemannian manifold of $P$–Sasakian type $M$, the following hold

\[ \text{Ric}(\xi_i, X) = -2n \sum_{\beta=1}^r \eta_{\beta}(X) \]
\[ R(\xi_i, X) \xi_\beta = X - \sum_{\gamma=1}^r \eta_\gamma(X) \xi_\gamma \]
\[ g(R(\xi_i, X)Y, \xi_\beta) = -g(X, Y) + \sum_{\gamma=1}^r \eta_\gamma(X) \eta_\gamma(Y) \]

for any vector fields $X, Y \in T(M)$.

Since $\phi$ and the Ricci operator $Q$ are symmetric in an almost $r$–paracontact Riemannian manifold of $P$–Sasakian type $M$, $Q\phi + \phi Q = 0$ and the Lie derivative of $\text{Ric}$ vanishes, i.e.

\[ L_{\xi_\alpha} \text{Ric} = 0, \]

for any $\alpha = 1, ..., r$. 
4. Weakly symmetric almost $r$–paracontact Riemannian manifolds of $P$–Sasakian type

In this chapter we investigate weakly symmetric almost $r$–paracontact Riemannian manifold of $P$–Sasakian type. We assume that the weakly symmetric manifold is almost $r$–paracontact Riemannian manifold of $P$–Sasakian type. Then we have,

**Theorem 1.** Any weakly symmetric almost $r$–paracontact Riemannian manifold of $P$–Sasakian type $M$ satisfies $\alpha + \delta + \beta = 0$.

**Proof.** Since the manifold is weakly symmetric, from (2.3), by putting $X = \xi$, yields

$$ (D_{\xi}\text{Ric})(Z, U) = \alpha(\xi)\text{Ric}(Z, U) + \beta(Z)\text{Ric}(\xi, U) + \delta(U)\text{Ric}(Z, \xi) + \beta(R(\xi, Z)U) + \delta(R(\xi, U)Z) $$

From (3.13), it follows that

$$ (D_{\xi}\text{Ric})(Z, U) = -\text{Ric}(D_{Z}\xi, U) - \text{Ric}(Z, D_{U}\xi). $$

By virtue of (3.8), we get from

$$ (D_{\xi}\text{Ric})(Z, U) = -\text{Ric}(\phi Z, U) - \text{Ric}(Z, \phi U). $$

Now, since $\phi$ is skew symmetric and Ricci operator is symmetric, we obtain

$$ (D_{\xi}\text{Ric})(Z, U) = 0. $$

From (4.14) and (4.15), we have

$$ \alpha(\xi)\text{Ric}(Z, U) + \beta(Z)\text{Ric}(\xi, U) + \delta(U)\text{Ric}(Z, \xi) + \beta(R(\xi, Z)U) + \delta(R(\xi, U)Z) = 0 $$

Putting $Z = U = \xi$, in (4.16), we get

$$ \alpha(\xi)\text{Ric}(\xi, \xi) + \beta(\xi)\text{Ric}(\xi, \xi) + \delta(\xi)\text{Ric}(\xi, \xi) + \beta(R(\xi, \xi)\xi) + \delta(R(\xi, \xi)\xi) = 0. $$

And using (3.11), we have

$$ (\alpha(\xi) + \beta(\xi) + \delta(\xi))\text{Ric}(\xi, \xi) = 0. $$

which gives us

$$ \alpha(\xi) + \beta(\xi) + \delta(\xi) = 0. $$

So the vanishing of the 1-form $\alpha + \beta + \delta$ over the vector field $\xi$ is necessary in order that $M$ be a almost $r$–paracontact Riemannian manifold of $P$–Sasakian type.

Now we will show that $\alpha + \beta + \delta = 0$ holds for all vector fields on $M$.

In (2.3), taking $X = Z = \xi$, we get

$$ (D_{\xi}\text{Ric})(\xi, U) = \alpha(\xi)\text{Ric}(\xi, U) + \beta(\xi)\text{Ric}(\xi, U) + \delta(U)\text{Ric}(\xi, \xi) + \beta(R(\xi, \xi)U) + \delta(R(\xi, U)\xi). $$


From (4.19) and (3.11), we get
\[\alpha(\xi_\alpha)Ric(\xi_\alpha, U) + \beta(\xi_\alpha)Ric(\xi_\alpha, U) + \delta(U)Ric(\xi_\alpha, \xi_\alpha) + \delta(R(\xi_\alpha, U)\xi_\alpha) = 0\]
Replacing \(U\) by \(X\) in (4.19) we have
\[\alpha(\xi_\alpha)Ric(\xi_\alpha, X) + \beta(\xi_\alpha)Ric(\xi_\alpha, X) + \delta(X)Ric(\xi_\alpha, \xi_\alpha) + \delta(R(\xi_\alpha, X)\xi_\alpha) = 0.\]
In (2.3), taking \(X = U = \xi_\alpha\), we get
\[(D_{\xi_\alpha}Ric)(Z, \xi_\alpha) = \alpha(\xi_\alpha)Ric(Z, \xi_\alpha) + \beta(Z)Ric(\xi_\alpha, \xi_\alpha) + \delta(\xi_\alpha)Ric(Z, \xi_\alpha) + \beta(R(\xi_\alpha, Z)\xi_\alpha) + \delta(R(\xi_\alpha, \xi_\alpha)Z) = 0.\]
From (4.15) and (3.11), we get
\[\alpha(\xi_\alpha)Ric(Z, \xi_\alpha) + \beta(Z)Ric(\xi_\alpha, \xi_\alpha) + \delta(\xi_\alpha)Ric(Z, \xi_\alpha) + \beta(R(\xi_\alpha, Z)\xi_\alpha) = 0.\]
Replacing \(Z\) by \(X\) in (4.21) we have
\[\alpha(\xi_\alpha)Ric(X, \xi_\alpha) + \beta(X)Ric(\xi_\alpha, \xi_\alpha) + \delta(\xi_\alpha)Ric(X, \xi_\alpha) + \beta(R(\xi_\alpha, X)\xi_\alpha) = 0.\]
In (2.3), taking \(Z = U = \xi_\alpha\), we get
\[(D_XRic)(\xi_\alpha, \xi_\alpha) = \alpha(X)Ric(\xi_\alpha, \xi_\alpha) + \beta(X)Ric(\xi_\alpha, \xi_\alpha) + \delta(\xi_\alpha)Ric(\xi_\alpha, X) + \beta(R(X, \xi_\alpha)\xi_\alpha) + \delta(R(X, \xi_\alpha)\xi_\alpha).\]
We also have
\[(D_XRic)(\xi_\alpha, \xi_\alpha) = 0\]
Using (4.24) in (4.23), we have
\[\alpha(X)Ric(\xi_\alpha, \xi_\alpha) + \beta(X)Ric(\xi_\alpha, \xi_\alpha) + \delta(\xi_\alpha)Ric(\xi_\alpha, X) + \beta(R(X, \xi_\alpha)\xi_\alpha) + \delta(R(X, \xi_\alpha)\xi_\alpha) = 0.\]
Adding (4.20), (4.22) and (4.25), we obtain
\[2(\alpha(\xi_\alpha) + \beta(\xi_\alpha) + \delta(\xi_\alpha))Ric(\xi_\alpha, X) + (\alpha(X) + \delta(X) + \beta(X))Ric(\xi_\alpha, \xi_\alpha) + \delta(R(\xi_\alpha, X)\xi_\alpha) + \beta(R(\xi_\alpha, X)\xi_\alpha) + \delta(R(X, \xi_\alpha)\xi_\alpha) = 0.\]
Using (3.11) and (4.18) in (4.26) we have
\[(\alpha(X) + \delta(X) + \beta(X))Ric(\xi_\alpha, \xi_\alpha) = 0\]
Hence from (4.27), we obtain
\[(4.28)\quad \alpha(X) + \delta(X) + \beta(X) = 0 \quad \text{for all } X.\]

Thus
\[\alpha + \delta + \beta = 0.\]

Our theorem is thus proved.

§ 5. Weakly Ricci-symmetric almost \(\tau\)-paracontact Riemannian manifolds of \(P\)-Sasakian type

In this chapter we investigate weakly Ricci-symmetric almost \(\tau\)-paracontact Riemannian manifolds of \(P\)-Sasakian type. We suppose that the considered weakly Ricci-symmetric manifold is almost \(\tau\)-paracontact Riemannian manifold of \(P\)-Sasakian type. We have,

**Theorem 2.** Any weakly Ricci-symmetric almost \(\tau\)-paracontact Riemannian manifold of \(P\)-Sasakian type satisfies
\[(5.29)\quad (D_{\xi_{\alpha}}\text{Ric})(Y, Z) = \rho(\xi_{\alpha})\text{Ric}(Y, Z) + \mu(Y)\text{Ric}(\xi_{\alpha}, Z) + v(Z)\text{Ric}(\xi_{\alpha}, Y).\]

By virtue of (4.15) and (5.29), we have
\[(5.30)\quad \rho(\xi_{\alpha})\text{Ric}(Y, Z) + \mu(Y)\text{Ric}(\xi_{\alpha}, Z) + v(Z)\text{Ric}(\xi_{\alpha}, Y) = 0.\]

Putting \(Y = Z = \xi_{\alpha}\) in (5.30), we get
\[(5.31)\quad (\rho(\xi_{\alpha}) + \mu(\xi_{\alpha}) + v(\xi_{\alpha}))\text{Ric}(\xi_{\alpha}, \xi_{\alpha}) = 0,\]

which gives
\[(5.32)\quad \rho(\xi_{\alpha}) + \mu(\xi_{\alpha}) + v(\xi_{\alpha}) = 0.\]

In (2.2), taking \(X = Y = \xi_{\alpha}\), and using (4.15), we get
\[(5.33)\quad (D_{\xi_{\alpha}}\text{Ric})(\xi_{\alpha}, Z) = \rho(\xi_{\alpha})\text{Ric}(\xi_{\alpha}, Z) + \mu(\xi_{\alpha})\text{Ric}(\xi_{\alpha}, Z) + v(Z)\text{Ric}(\xi_{\alpha}, \xi_{\alpha}) = 0.\]

Replacing \(Z\) by \(X\) in (5.33) we have
\[(5.34)\quad (D_{\xi_{\alpha}}\text{Ric})(\xi_{\alpha}, X) = \rho(\xi_{\alpha})\text{Ric}(\xi_{\alpha}, X) + \mu(\xi_{\alpha})\text{Ric}(\xi_{\alpha}, X) + v(X)\text{Ric}(\xi_{\alpha}, \xi_{\alpha}) = 0.\]

In (2.2), taking \(X = Z = \xi_{\alpha}\), and using (4.15), we get
\[(5.35)\quad (D_{\xi_{\alpha}}\text{Ric})(Y, \xi_{\alpha}) = \rho(\xi_{\alpha})\text{Ric}(Y, \xi_{\alpha}) + \mu(\xi_{\alpha})\text{Ric}(\xi_{\alpha}, \xi_{\alpha}) + v(\xi_{\alpha})\text{Ric}(\xi_{\alpha}, Y) = 0.\]
Replacing $Y$ by $X$ in (5.35) we have

$$
(D_{\xi} Ric)(X, \xi) = \rho(\xi) Ric(X, \xi) + \mu(\xi)Ric(\xi, X) + v(\xi)Ric(\xi, X) = 0.
$$

(5.36)

In (2.2), taking $Y = Z = \xi$, and using (4.24), we get

$$
(D_{\xi} Ric)(\xi, \xi) = \rho(X)Ric(\xi, \xi) + \mu(\xi)Ric(X, \xi) + v(\xi)Ric(X, \xi) = 0.
$$

(5.37)

Adding (5.34), (5.36) and (5.37) and then using (4.28), we obtain

$$
2(\rho(\xi) + \mu(\xi) + v(\xi))Ric(\xi, X) + (\rho(X) + \mu(X) + v(X))Ric(\xi, \xi) = 0
$$

From this, it follows that

$$
(\rho(X) + \mu(X) + v(X))Ric(\xi, \xi) = 0.
$$

(5.38)

Hence from (5.38), we have,

$$
\rho(X) + \mu(X) + v(X) = 0, \text{ for all } X.
$$

Thus

$$
\rho + \mu + v = 0.
$$

Hence our theorem is proved.

References


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