Nonlinear connections in Lagrange spaces with \((\alpha, \beta)\)-metrics

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Abstract. Within the framework of Geometry of Lagrange spaces endowed with \((\alpha, \beta)\)-metrics, is continued the study developed in [5], by examining the canonic nonlinear connection and investigating the properties of torsion and curvature.


Key words: Lagrange space, \((\alpha, \beta)\)-metric, Euler-Lagrange equations.

1 Introduction

In the Ph.D. Thesis [5] has been studied the theory of Lagrange spaces with \((\alpha, \beta)\)-metrics \(L^n = (M, \tilde{L}(\alpha, \beta))\). This class of spaces includes the well known categories of Randers spaces \((RF^n = (M, (\alpha, \beta)^2))\) and the Lagrange spaces \(L^n\) of electrodynamics with have the fundamental function

\[
L(x, y) = mc\gamma_{ij}(x)y^iy^j + \frac{2e}{mc} A_i(x)y^i + U(x).
\]

In this paper, we introduce the concept of canonical nonlinear connection \(N\) with coefficients \(N^i_j(x, y)\) in (3.1), which derives from the canonic semispray \(S\) described in (2.4) and (2.5). We study then the torsion \(t^i_{jk}\) and the curvature \(R^i_{jk}\) of \(N\), and show that \(t^i_{jk} = 0\). The condition \(R^i_{jk} = 0\) gives the case when the nonlinear connection \(N\) is integrable. Throughout the paper, we use the results regarding the space \(L^n = (M, \tilde{L})\) obtained in [5].

2 Lagrange spaces with \((\alpha, \beta)\)-metrics

This class of Lagrange spaces was studied by the author in his Ph.D. thesis [5] in 2003, presented at University of Craiova. Now we study problems related to the notion of nonlinear connection of a Lagrange space with \((\alpha, \beta)\)-metrics.

Let \(\alpha = \sqrt{\gamma_{ij}(x)y^iy^j}\) and \(\beta = A_i(x)y^i\) be a Riemannian norm produced by an underlying Riemannian metric \(\gamma_{ij}(x)\) on \(M\) and respectively a transvected linear term.
produced by an electromagnetic field $A_i(x)$. Then the following Lagrangians:

1) $L(x, y) = \alpha^2(x, y) + \beta(x, y)$;
2) $L(x, y) = a\alpha(x, y) + b\beta(x, y) + c\beta^2(x, y), \ a, b, c \in \mathbb{R}, a \neq 0$;
3) $L(x, y) = (\alpha(x, y) + \beta(x, y))^2$,

are $(\alpha, \beta)$-type Lagrangians, being expressed in terms of $\alpha$ and $\beta$. We further focus on a Lagrangian of the type

$$L(x, y) = \tilde{L}(\alpha(x, y) + \beta(x, y)).$$

(2.1)

Then the fundamental metric tensor field of the space is given by

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 \tilde{L}}{\partial y^i \partial y^j} = \rho \gamma_{ij} + \rho_0 A_i A_j + \rho_{-1} (y_i A_j + y_j A_i) + \rho_{-2} y_i y_j,$$

(2.2)

where $y_i = \gamma_{ij} y^j$ and $\rho, \rho_0, \rho_{-1}, \rho_{-2}$ are the corresponding coefficients in the detailed expression of $g_{ij}$,

$$g_{ij} = \frac{1}{2} \{ \alpha^{-2} \tilde{L}_{\alpha\alpha} y_i y_j + \alpha^{-1} \tilde{L}_{\alpha\beta} (y_i A_j + y_j A_i) + \tilde{L}_{\beta\beta} A_i A_j + \tilde{L}_{\alpha\alpha}^{-1} (\gamma_{ij}(x) - \alpha^{-2} y_i y_j) \}.$$

(2.3)

The canonic semispray $S$ of the space $L^n = (M, \tilde{L}(\alpha, \beta))$ obtained in [5] is

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

(2.4)

where the coefficients $G^i$ are given by

$$2G^i(x, y) = \{ i_{jk} \} y^j y^k - \lambda(x, y) F^i_{j} y^j,$$

(2.5)

with

$$\lambda(x, y) = \frac{\tilde{L}_{\beta}}{\tilde{L}_{\alpha}}, \ F^i_{j} = \gamma^{ih} F_{hj}, \ F_{hj} = \frac{1}{2} \frac{\partial A_j}{\partial x^h} - \frac{\partial A_h}{\partial x^j}.$$  

(2.6)

Obviously, here $F_{ij}(x)$ is the electromagnetic tensor of the space, and $\{ i_{jk} \}(x)$ are the Christoffel symbols of $\gamma_{ij}(x)$. The semispray $S$ is called canonic, since it depends only on the fundamental function $\tilde{L}(\alpha, \beta)$.

3 Nonlinear connections of the space $L^n = (M, \tilde{L})$

The canonic semispray $S$ allows us to determine the nonlinear connection $N$ of the Lagrange space with $(\alpha, \beta)$-metrics $L^n = (M, \tilde{L}(\alpha, \beta))$, which is canonical, since it depends on $\tilde{L}$ only ([1]). Following the general theory in [1], it follows that the coefficients of $N$ are

$$N^i_j = \frac{\partial G^i}{\partial y^j}.$$

(3.1)
From (2.5) and (2.6), we obtain

\[ N^i_j = \{^i_{jk}\} y^k - \frac{1}{2} \left\{ \lambda F^i_j + \frac{\partial \lambda}{\partial y^i} F^i_j y^j \right\}. \]  

(3.2)

where

\[ \frac{\partial \lambda}{\partial y^i} = \frac{1}{L_\alpha} \left\{ L_\alpha \frac{\partial L_\beta}{\partial y^i} - L_\beta \frac{\partial L_\alpha}{\partial y^j} \right\}. \]  

(3.3)

A convenient form for \( N^i_j \) is

\[ N^i_j = \{^i_{jk}\}(x) y^k - \frac{1}{2} \lambda^k_j F^i_j, \text{ where } \lambda^k_j = \lambda \delta^k_j + \frac{\partial \lambda}{\partial y^j}. \]  

(3.4)

The horizontal distribution determined by \( N \) has the property

\[ T_p(TM) = N_p \oplus V_p, \forall p \in TM. \]  

(3.5)

The associated adapted dual basis is \( (dx^i, \delta y^i)_p \), \( \delta y^i = dy^i + N^i_j dx^j \). The weak torsion of \( N \) is

\[ t^i_{jk} = \frac{\partial N^i_j}{\partial y^k} - \frac{\partial N^i_k}{\partial y^j}, \]  

and the curvature of \( N \) is

\[ R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}. \]

Obviously, \( N \) depends on the fundamental function \( \hat{L}(\alpha, \beta) \) only. Hence, we have the following

**Theorem 1.** The non-linear connection \( N \) of the space \( L^n \) produced by the canonical spray \( S \) has the local coefficients (3.4). Moreover, \( N \) depends only on the fundamental function \( \hat{L}(\alpha, \beta) \) and its weak curvature \( t^i_{jk} \) identically vanishes.

Indeed, the coefficients \( N^i_j \) defined by (3.1) have the form (3.4), and depend only on \( \hat{L}(\alpha, \beta) \); as well, the weak torsion is \( t^i_{jk} = -\frac{1}{2} \left\{ \frac{\partial \lambda^i_j}{\partial y^j} - \frac{\partial \lambda^i_k}{\partial y^k} \right\} \), q.e.d..

The connection \( N \) is called the canonical nonlinear connection of the space \( L^n \). It is straightforward to prove

**Theorem 2.** The autoparallel curves of the canonical nonlinear connection \( N \) are locally given by

\[ \frac{dx^i}{dt} = y^i, \quad \frac{\delta y^i}{\delta t} = \frac{dy^i}{dt} + \{^i_{jk}\} y^j y^k - \frac{1}{2} \lambda^k_j F^i_j y^j = 0. \]

**Example.** For the Randers spaces \( (RF^n = (M, \alpha + \beta) \), we have \( L = (\alpha + \beta)^2 \), hence \( \lambda = 1 \). The canonical nonlinear connection has the coefficients

\[ N^i_j = \{^i_{jk}\}(x) y^k - \frac{1}{2} F^i_j. \]

The autoparallel curves of \( N \) are in fact the Lorentz equations corresponding to the potentials \( A_i(x) \),

\[ \frac{dx^i}{dt} = y^i, \quad \frac{d^2 x^i}{dt^2} + \{^i_{jk}\} \frac{dx^j}{dt} \frac{dx^k}{dt} = \frac{1}{2} F^i_j \frac{dx^j}{dt}. \]
Returning to the general case, we shall consider further the curvature $R^i_{jk}$ of the canonical nonlinear connection $N$. We obtain from (3.4) that
\[
R^i_{jk} = y^s \rho^i_{s,jk} + \frac{1}{2} (\lambda^s_j F^i_{s|k} - \lambda^s_k F^i_{s|j}) - \frac{1}{2} \left( \frac{\delta \lambda^s_i}{\delta x^k} - \frac{\delta \lambda^s_k}{\delta x^j} \right) F^i_s,
\]
where $\rho^i_{s,jk}$ is the tensor of curvature of the Riemannian metric $\gamma_{ij}$ and $|_k$ is the covariant derivative w.r.t. $\frac{\partial}{\partial x^k}$ corresponding to the Levi-Civita connection associated to $\gamma_{ij}$.

In the case of Randers spaces $\tilde{L} = (\alpha + \beta)^2$, the tensor $R^i_{jk}$ has the expression
\[
R^i_{jk} = y^s \rho^i_{s,jk} + \frac{1}{2} (F^i_{j|k} - F^i_{k|j}).
\]
In this case, the condition $R^i_{jk} = 0$ is equivalent to $\rho^i_{s,jk} = 0, F^i_{j|k} = 0$. In general, the space $L^n = (M, \tilde{L})$ has the canonical nonlinear connection integrable iff the curvature tensor $R^i_{jk}$ of $N$ identically vanishes.

Analogously, one can study the equation $R^i_{jk} = 0$ in the notable cases considered in [5], this being subject of further concern.

References


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