Torseforming vector field in a 3-dimensional trans-Sasakian manifold

C.S. Bagewadi and Venkatesha

Abstract. The object of the present paper is to study a torseforming vector field in a 3-dimensional trans-Sasakian manifold. Here we proved that the torseforming vector field in a 3-dimensional trans-Sasakian manifold under the condition \( \phi(\text{grad} \alpha) = \text{grad} \beta \), is a concircular vector field.

Key words: torseforming vector field, concircular vector field, trans-Sasakian structure.

Introduction. The contact manifolds are odd dimensional manifolds with specified contact structure. One can obtain different structures like Sasakian, Quasi-Sasakian, Kenmotsu and trans-Sasakian by providing additional conditions. The geometry of these manifolds is extensively studied by [2] to [11]. Now the Torseforming vector field in a Riemannian manifold has been introduced by K. Yano in 1944 [14]. In the present paper we consider a Torseforming vector field in a 3-dimensional trans-Sasakian manifold under the condition \( \phi(\text{grad} \alpha) = \text{grad} \beta \), and proved that it is of concircular type.

1 Preliminaries.

Let \( M \) be an \((2m + 1)\) dimensional almost contact metric manifold with an almost contact metric structure \((\phi, \xi, \eta, g)\), where \( \phi \) is (1,1) tensor field, \( \xi \) is a vector field, \( \eta \) is a 1-form and \( g \) is the associated Riemannian metric such that [5],

\[
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \phi = 0,
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

An almost contact metric structure \((\phi, \xi, \eta, g)\), on \( M \) is called a trans-Sasakian structure [10], if \((M \times R, J, G)\) belongs to the class \( W_4 \) [9], where \( J \) is the almost complex structure on \( M \times R \) defined by \( J(X, f d/dt) = (\phi X - f \xi, \eta(X)d/dt) \) for all

vector fields $X$ on $M$ and smooth functions $f$ on $M \times R$. This may be expressed by the condition \[6\],
\[
(1.3) \quad (\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X),
\]
for some smooth functions $\alpha$ and $\beta$ on $M$, and we say that the trans-Sasakian structure is of type $(\alpha, \beta)$.
From (1.3) it follows that
\[
(1.4) \quad \nabla_X \xi = -\alpha \phi X + \beta(X - \eta(X)X) \quad \nabla_X \eta = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).
\]
On trans-Sasakian manifolds we have the following results \[7\]:
\[
(1.5) \quad R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha \beta \eta(Y)\phi(X) - \eta(X)\phi(Y) + (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y,
\]
\[
(1.6) \quad R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi \beta)(\eta(X)\xi - X),
\]
\[
(1.7) \quad 2\alpha \beta + \xi \alpha = 0,
\]
\[
(1.8) \quad S(X, \xi) = (2m(\alpha^2 - \beta^2) - \xi \beta)\eta(X) - (2m - 1)X\beta - (\phi X)\alpha,
\]
\[
(1.9) \quad Q\xi = (2m(\alpha^2 - \beta^2) - \xi \beta)\xi - (2m - 1)\text{grad}\beta + \phi(\text{grad} \alpha).
\]
When, $\phi(\text{grad} \alpha) = (2m - 1)\text{grad} \beta$, (1.8) and (1.9) reduce to
\[
(1.10) \quad S(X, \xi) = 2m(\alpha^2 - \beta^2)\eta(X)
\]
\[
(1.11) \quad Q\xi = 2m(\alpha^2 - \beta^2)\xi.
\]
Also under the above condition $\phi(\text{grad} \alpha) = (2m - 1)\text{grad} \beta$ the expressions for Ricci tensor and the scalar curvature in a 3-dimensional trans-Sasakian manifold are given respectively by \[13\]:
\[
(1.12) \quad S(X, Y) = \left[\frac{r}{2} - (\alpha^2 - \beta^2)\right]g(X, Y) - \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right]\eta(X)\eta(Y)
\]
and
\[
(1.13) \quad R(X, Y)Z = \left[\frac{r}{2} - 2(\alpha^2 - \beta^2)\right]g(Y, Z)X - g(X, Z)Y
- \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right]g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi
- \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right]\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y.
\]
A vector field $\rho$ defined by $g(X, \rho) = w(X)$ for any vector field $X$ is said to be a torseforming vector field \[12, 14\] if
\[
(1.14) \quad (\nabla_X w)(Y) = kg(X, Y) + \pi(X)w(Y)
\]
where $k$ is a non-zero scalar and $\pi$ is a non-zero 1-form.
2 Torseforming vector field in a 3-dimensional Trans-Sasakian Manifold

We consider a unit torseforming vector field \( \tilde{\rho} \) corresponding to the vector field \( \rho \). Suppose \( g(X, \tilde{\rho}) = T(X) \). Then

\[
T(X) = \frac{w(X)}{\sqrt{w(\rho)}},
\]

From (1.14) we get

\[
\frac{(\nabla_X w)(Y)}{\sqrt{w(\rho)}} = \frac{k}{\sqrt{w(\rho)}} g(X, Y) + \frac{\pi}{\sqrt{w(\rho)}} w(Y).
\]

Using (2.1) in the above, we obtain

\[
(\nabla_X T)(Y) = \lambda g(X, Y) + \pi(X)T(Y)
\]

where \( \lambda = \frac{k}{\sqrt{w(\rho)}} \).

Putting \( Y = \tilde{\rho} \) in (2.2), we obtain

\[
(\nabla_X T)(\tilde{\rho}) = \lambda g(X, \tilde{\rho}) + \pi(X)T(\tilde{\rho})
\]

As \( T(\tilde{\rho}) = g(\tilde{\rho}, \tilde{\rho}) = 1 \), equation (2.3) reduces to

\[
\pi(x) = -\lambda T(X)
\]

and hence (2.2) can be written in the form

\[
(\nabla_X T)(Y) = \lambda [g(X, Y) - T(X)T(Y)]
\]

which implies \( T \) is closed.

Taking covariant differential of (2.5) and using Ricci identity we get

\[
-T(R(X, Y)Z) = (X\lambda) [g(Y, Z) - T(Y)T(Z)] - (Y\lambda) [g(X, Z) - T(X)T(Z)]
+ \lambda^2 [g(Y, Z)T(X) - g(X, Z)T(Y)]
\]

Putting \( Z = \xi \) in (2.6) and using (1.3), we obtain

\[
-T ((\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y)) = (X\lambda) [\eta(Y) - T(Y)T(\xi)]
- (Y\lambda) [\eta(X) - T(X)T(\xi)]
+ \lambda^2 [\eta(Y)T(X) - \eta(X)T(Y)]
\]

Since \( T(\xi) = g(\xi, \tilde{\rho}) = \eta(\tilde{\rho}) \), (2.7) reduces to

\[
[\lambda^2 + (\alpha^2 - \beta^2)] [\eta(Y)T(X) - \eta(X)T(Y)] + [X(\lambda)\eta(Y) - (Y\lambda)\eta(X)]
+ \lambda(\tilde{\rho}) [T(X)T(Y)] = 0
\]

Putting \( X = \tilde{\rho} \) in (2.8) and as \( T(\tilde{\rho}) = g(\tilde{\rho}, \tilde{\rho}) = 1 \), we get

\[
[\lambda^2 + (\alpha^2 - \beta^2) + \tilde{\rho}\lambda] [\eta(Y) - \eta(\tilde{\rho})T(Y)] = 0.
\]

Thus we have the following:
Lemma 1 If a 3-dimensional trans-Sasakian manifold admits a torseforming vector field, then the following cases occur:

\begin{align}
(2.10) & \quad \eta(Y) - \eta(\tilde{\rho})T(Y) = 0 \\
(2.11) & \quad \lambda^2 + (\alpha^2 - \beta^2) + \tilde{\rho}\lambda = 0
\end{align}

We first consider the case where (2.10) holds good. From (2.10) we get

\[ \eta(Y) = \eta(\tilde{\rho})T(Y) \]

Now \( Y = \xi \) implies

\[ 1 = (\eta(\tilde{\rho}))^2 \]

and thus

\[ \eta(\tilde{\rho}) = \pm 1. \]

So,

\[ (2.12) \quad \eta(Y) = \pm T(Y). \]

Using (2.12) in (1.4) in view of (2.5) we have

\[ -\alpha g(\phi X, Y) + \beta [g(X, Y) - T(X)T(Y)] = \pm \lambda [g(X, Y) - T(X)T(Y)] \]

This implies \( \lambda = \pm \beta. \)

Hence (2.4) reduces to

\[ (2.13) \quad \pi(X) = \pm \beta T(X) \]

Since \( T \) is closed, \( \pi \) is also closed.

Hence we can state:

Lemma 2 The equation (2.10) implies that the vector field \( \tilde{\rho} \) is a concircular vector field.

We next assume the case (2.11). Then

\[ (2.14) \quad \eta(Y) - \eta(\tilde{\rho})T(Y) \neq 0. \]

From (2.6), we get

\[ (2.15) \quad -T(QX) = 2(X\lambda) - (X\lambda) + (\tilde{\rho}\lambda)T(X) + 2\lambda^2T(X). \]

where \( g(QX, Y) = S(X, Y) \).

Put \( X = \xi \) in (2.15) and using (1.11), we obtain

\[ (2.16) \quad \xi\lambda = -\eta(\tilde{\rho})[\lambda^2 + (\alpha^2 - \beta^2)]. \]

Putting \( Y = \xi \) in (2.8), in virtue of (2.16) and \( T(\xi) = \eta(\tilde{\rho}) \) we get

\[ (2.17) \quad X\lambda = -\{\lambda^2 + (\alpha^2 - \beta^2)\} T(X). \]
From (2.4) it follows that
\[ Y\pi(X) = -[(Y\lambda)T(X) + \lambda(YT(X))] \]

Using (2.17) in the above equation, we get
\[
Y\pi(X) = - \left\{ - \left[ \lambda^2 + (\alpha^2 - \beta^2) \right] T(Y)T(X) + \lambda [YT(X)] \right\}
\]
(2.18)

Also
\[
Y\pi(Y) = - \left\{ - \left[ \lambda^2 + (\alpha^2 - \beta^2) \right] T(X)T(Y) + \lambda [XT(Y)] \right\}
\]
(2.19)

and
\[
\pi([X,Y]) = -\lambda T([X,Y]).
\]
(2.20)

From (2.18), (2.19) and (2.20), we obtain
\[
d\pi(X,Y) = -\lambda [(dT)(X,Y)].
\]
(2.21)

Since \( T \) is closed, \( \pi \) is also closed. Thus we have

**Lemma 3** The equation (2.11) implies that the vector field \( \tilde{\rho} \) is a concircular vector field. Thus from Lemma 2 and Lemma 3, we can state the following:

**Theorem 2.1** A torseforming vector field in a trans-Sasakian manifold is a concircular vector field.

From (1.4) it follows that in a trans-Sasakian manifold \( \xi \) is a torseforming vector field. Hence from Theorem 2.1, we can state the following:

**Theorem 2.2** A trans-Sasakian manifold admits a proper concircular vector field.

Conformally flat trans-Sasakian manifold has been studied by C.S.Bagewadi and Venkatesha [4]. It is known [1] that if a Conformally flat manifold \( M \) (whose dimension \( n \) is greater than 3) admits a proper Concircular vector field, then the manifold is a subprojective manifold in the sense of Kagan. Since the trans-Sasakian manifold admits a proper concircular vector field, namely the vector field \( \xi \), we can state as follows:

**Theorem 2.3** A Conformally flat trans-Sasakian manifold of dimension greater than 3, is a subprojective manifold in the sense of Kagan.

**References**


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