A criterion for dispersive dynamical systems on a
topological manifold

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Abstract
In this paper we show that Auslander-Bhatia’s-theorem [2], relative to a
dynamical system on a metric space, can be extended to dynamical systems
defined on topological manifolds.

Key words: dispersive dynamical system, periodic orbit.

1 Preliminaries
Let $X$ be a topological manifold over $\mathbb{R}^n$, and let $\mathcal{A}$ be the atlas which gives the
manifold structure, $\dim(X) = n$,

$$\mathcal{A} = \{ h_a(\chi_a, U_a) \}_{a \in J},$$

where $U_a \subset X$, and $\chi_a : U_a \to \chi_a(U_a) \subset \mathbb{R}^n$ are homeomorphisms for all $a \in J$.

Definition 1.1. [2] A continuous dynamical system on a topological manifold $X$ is
defined by a triplet $(X, R, \Phi)$, where $\Phi$ is a transformation $\Phi : X \times R \to X$ satisfying
the following properties:

i) $\Phi(x, 0) = x$, $(\forall) x \in X$;

ii) $\Phi(\Phi(x, t), s) = \Phi(x, t + s)$ $(\forall) x \in X$, $(\forall) t, s \in R$;

iii) $\Phi$ is continuous.

In the following we use the notation: $\Phi(x, t) \equiv xt$. The properties (i-ii) can be
respectively rewritten:

i') $x0 = x$, $(\forall) x \in X$;

ii') $xt(s) = x(t + s)$ $(\forall) x \in X$, $(\forall) t, s \in R$.

We note that the property of continuity iii) is equivalent to

iii') If $(x_n)$ and $(t_n)$ are sequences in $X$, respectively in $R$, such that $x_n \to x, t_n \to t$,
then $x_nt_n \to xt$.

In concordance with the above notation, if $M \subset X$ and $A \subset R$ we will write

$$MA = \{ xt | x \in M, \ t \in A \}.$$
the set \( \gamma(x) \triangleq xR = \{ xt \mid t \in R \} \). The set \( \gamma^+(x) = xR^+ \), respectively \( \gamma^-(x) = xR^- \),
is called the positive semi-trajectory, respectively the negative semi-trajectory through \( x \). A subset \( M \subset X \) with the property that \( xR \subset M \), \( (\forall) x \in M \), is called invariant set.

The trajectory through \( x \in X \) with the property that there exists some \( \tau \in R \) such that \( x(t + \tau) = xt \) for any \( t \in R \), is called periodic trajectory of period \( \tau \).

A point \( x \in X \) such that \( xt = x \) for any \( t \in R \) is called rest point or equilibrium point.

For any fixed \( x \in X \), the application \( \Phi_x : R \to X \) defined by \( \Phi_x(t) = xt \) is called the motion of \( x \).

2 Omega limit prolongation and prolongational limit set

**Definition 2.1.** \( y \in X \) is a positive (negative) limit point of some \( x \in X \) if there exists a sequence \( (t_n), t_n \to \infty(-\infty) \), such that \( xt_n \to y \).

The \( \omega \)-limit set of a point \( x \in X \) is denoted by \( \omega(x) = \{ y \in X | (\exists) (t_n) \subset R^+, t_n \to \infty \text{ and } xt_n \to y \} \).

**Definition 2.2** [2] The positive prolongation limit set of a point \( x \), respectively the negative prolongation limit set is the set defined by:

\[
\begin{align*}
D^+(x) &= \{ y \in X | (\exists) (x_n) \subset X \text{ and } (t_n) \subset R^+, \text{ such that } x_n \to x \text{ and } x_n t_n \to y \} \\
D^-(x) &= \{ y \in X | (\exists) (x_n) \subset X \text{ and } (t_n) \subset R^-, \text{ such that } x_n \to x \text{ and } x_n t_n \to y \}
\end{align*}
\]

**Definition 2.3** [2] The positive prolongational limit set, and the negative prolongational limit set of any \( x \in X \) are the sets defined respectively by:

\[
\begin{align*}
J^+(x) &= \{ y \in X | (\exists) (x_n) \subset X \text{ and } (t_n) \subset R, \text{ such that } x_n \to x, y_n \to \infty \text{ and } x_n t_n \to y \} \\
J^-(x) &= \{ y \in X | (\exists) (x_n) \subset X \text{ and } (t_n) \subset R, \text{ such that } x_n \to x, y_n \to -\infty \text{ and } x_n t_n \to y \}
\end{align*}
\]

From these definitions it follows immediately:

**Proposition 2.4.** \( \gamma^+(x) \subset D^+(x) \) and \( D^+(x) = \gamma^+(x) \cup J^+(x) \).

3 Dispersive dynamical systems

**Definition 3.1** [2] A dynamical system \((X, R, \Phi)\) is called dispersive if for any \( x, y \in X \) there exist two neighborhoods \( U_x \) and \( U_y \) and a constant \( T > 0 \) such that \( \Phi_t(U_x) \cap U_y = \emptyset \) for all \( t \in R, \; |t| \geq T \).

**Theorem 3.2** \( J^+(x) \) is a closed invariant set for all \( x \in X \).

**Proof.** We show that \( J^+(x) \) is closed. For a sequence \( (y_k) \subset J^+(x) \) such that \( y_k \to y \), it follows that \( y \in J^+(x) \). Indeed, for each \( k \in N^* \) there exists the sequence \( (t_n^k) \subset R; t_n^k \to \infty \) and \( (x_n^k) \subset X; x_n^k \to x \) with \( \lim_{n \to \infty} x_n^k t_n^k = y_k \).

Consider \( y \in U \), where \( U \) is the geometric zone of the chart \( h(U, \chi) \in \mathcal{A} \). Then
there exists an integer $n_0 \in \mathbb{N}^*$ such that for each $k \geq n_0, y_k \in U$. For each $y_k \in U, k \geq n_0$ there exists $n_k^t$ such that for $n_k \in \mathbb{N}^*, n_k \geq n_k^t$ we have that $t_{n_k}^k > k$, $x_n^k t_{n_k} \in U$, and

$$||\chi(x_{n_k}^k t_{n_k}) - \chi(y_k)|| < \frac{1}{k}$$

Consider now the sequences:

$$(t_k : t_k = t_{n_k}^k), \quad (x_k : x_k = x_{n_k}^k).$$

We observe that $t_k \to \infty, x_k \to x$, and $x_k t_k \to y$ because

$$||\chi(x_k t_k) - \chi(y)|| \leq ||\chi(x_k t_k) - \chi(y_k)|| + ||\chi(y_k) - \chi(y)|| \leq \frac{1}{k} + ||\chi(y_k) - \chi(y)||$$

Hence $y \in J^+(x)$.

In order to show that $J^+(x)$ is an invariant set we prove that for every $y \in J^+(x)$ we have $yR \subset J^+(x)$. For this, we take the sequences $(t_n) \subset R, t_n \to \infty$ and $(x_n) \subset X, x_n \to x, x_n t_n \to y \in J^+(x)$, and $\tau \in R$. We have $x_n(t_n + \tau) = (x_n t_n) \tau \to y\tau$ and this implies $y\tau \in J^+(x)$. q.e.d.

Recently we have proved [4] for a dynamical system on a topological space the following property:

**Theorem 3.3** A dynamical system $(X, R, \Phi)$ is dispersive if and only if for each $x \in X, J^+(x) = \emptyset$.

By means of this result we extend the Auslander-Bhatia’s theorem [2] which states that a continuous dynamical system on a metric space is dispersive if and only if the positive prolongation limit set and the positive semi-trajectory of each point are identically and the system does not exhibit rest points or periodic trajectories.

**Theorem 3.4** The dynamical system $(X, R, \Phi)$ is dispersive if and only if for each $x \in X$ the positive prolongation limit set of $x$ coincides with the positive semi-trajectory through $x$, i.e. $D^+(x) = \gamma^+(x)$, and the system does not exhibit rest points or periodic trajectories.

**Proof.** If $(X, R, \Phi)$ is dispersive then by Theorem 3.3, $J^+(x) = \emptyset \quad (\forall) x \in X$. Using Proposition 2.4 we obtain $D^+(x) = \gamma^+(x), (\forall) x \in X$. If $x$ is a rest point or $\gamma(x)$ is a periodic trajectory, then $\gamma(x) \equiv \omega(x) \subset J^+(x)$, and this contradicts $J^+(x) = \emptyset$.

Conversely, suppose that $D^+(x) = \gamma^+(x)$ and that there are no rest points or periodic orbits. We must prove that $J^+(x) = \emptyset$.

From Proposition 2.4 we have $D^+(x) = \gamma^+(x) \cup J^+(x) = \gamma^+(x)$. This implies that $J^+(x) \subset \gamma^+(x)$; By Theorem 3.2, $J^+(x)$ is a closed invariant set and if $J^+(x)$ is not empty, we have that $\gamma(x) \subset J^+(x) \subset \gamma^+(x)$, i.e. $\gamma(x) = \gamma^+(x)$. This shows that if $t' < 0$, then there exists $t \geq 0$ such that $xt = xt$, i.e.

$$x = x(t - t').$$

Since $t - t' > 0$, we conclude that the trajectory $\gamma(x)$ is closed and has the period $t - t'$, which is a contradiction. q.e.d.
References


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