Some inequalities for warped products in cosymplectic space forms

Dae Won Yoon

Abstract
In this article, we investigate the inequality between the warping function of a warped product submanifold isometrically immersed in a cosymplectic space form of constant ϕ-sectionai curvature and the squared mean curvature. Furthermore, some applications are derived.

Key words: warped product, mean curvature, cosymplectic space form, totally real submanifold.

§1. Introduction
Let $M_1$ and $M_2$ be Riemannian manifolds of positive dimension $n_1$ and $n_2$, equipped with Riemannian metrics $g_1$ and $g_2$, respectively. Let $f$ be a positive function on $M_1$. The warped product $M_1 \times f M_2$ is defined to be the product manifold $M_1 \times M_2$ with the warped metric: $g = g_1 + f^2 g_2$ (see, for instance [3]).

It is well-known that the notion of warped products plays some important role in differential geometry as well as in physics. For a recent survey on warped products as Riemannian submanifolds, we refer to [3].

Let $\phi : M_1 \times f M_2 \rightarrow \tilde{M}(c)$ be an isometric immersion of a warped product $M_1 \times f M_2$ into a Riemannian manifold $\tilde{M}(c)$ with constant sectional curvature $c$. We denote by $h$ the second fundamental form of $\phi$ and $H_i = \frac{1}{n_i} \text{trace} h_i$, where trace $h_i$ is the trace of $h$ restricted to $M_i$. We call $H_i$ ($i = 1, 2$) the partial mean curvature vectors. The immersion $\phi$ is said to be mixed totally geodesic if $h(X, Z) = 0$, for any vector fields $X$ and $Z$ tangent to $M_1$ and $M_2$ respectively.

Recently, in [6] B. Y. Chen established the following sharp relationship between the warping function $f$ of a warped product $M_1 \times f M_2$ isometrically immersed in a real space form $\tilde{M}(c)$ and the squared mean curvature $\|H\|^2$.

**Theorem 1.1** ([6]). Let $\phi : M_1 \times f M_2 \rightarrow \tilde{M}(c)$ be an isometric immersion of a warped product into a Riemannian $m$-manifold of constant sectional curvature $c$. Then, we have

$$\frac{\Delta f}{f} \leq \frac{(n_1 + n_2)^2}{4n_2} \|H\|^2 + n_1 c,$$
where $\Delta$ is the Laplacian operator of $M_1$.

As an immediate application, he obtained necessary conditions for a warped product to admit a minimal isometric immersion in a Euclidean space or in a real space form.

On the other hand, for the above related researches B. Y. Chen investigated the inequality (1.1) of a warped product submanifold into complex hyperbolic space ([5]) to admit a minimal isometric immersion in a Euclidean space or in a real space form.

In this paper, we prove similar inequality for warped product submanifolds of cosymplectic space forms of constant $\varphi$-sectional curvature $c$.

§2. Preliminaries

Let $\tilde{M}$ be a $(2m+1)$-dimensional almost contact manifold endowed with an almost contact structure $(\varphi, \xi, \eta)$, that is, $\varphi$ is a $(1,1)$ tensor field, $\xi$ is a vector field and $\eta$ is a 1-form such that

$$\varphi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1.$$  

Then, $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$.

Let $g$ be a compatible Riemannian metric with $(\varphi, \xi, \eta)$, that is, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ or equivalent, $g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$ for all $X, Y \in \tilde{M}$. Then, $\tilde{M}$ becomes an almost contact metric manifold equipped with an almost contact metric structure $(\varphi, \xi, \eta, g)$. An almost contact metric manifold is cosymplectic ([1]) if $\nabla_X \varphi = 0$, where $\nabla$ is the Levi-Civita connection of the Riemannian metric $g$. From the formula $\nabla_X \varphi = 0$ it follows that $\nabla_X \xi = 0$.

A plane section $\pi$ in $T_p \tilde{M}$ of an almost contact metric manifold $\tilde{M}$ is called a $\varphi$-section if $\pi \perp \xi$ and $\varphi(\pi) = \pi$. $\tilde{M}$ is of constant $\varphi$-sectional curvature $c$ if and only if its curvature tensor $\tilde{R}$ is of the form ([7])

$$4\tilde{R}(X, Y, Z, W) = c\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W) + g(X, \varphi W)g(Y, \varphi Z) - g(X, \varphi Z)g(Y, \varphi W) - 2g(X, \varphi Y)g(Z, \varphi W) - g(X, W)\eta(Y)\eta(Z) + g(X, Z)\eta(Y)\eta(W) - g(Y, Z)\eta(X)\eta(W) + g(Y, W)\eta(X)\eta(Z)\}.$$  

(2.1)

Let $M$ be an $n$-dimensional submanifold of a manifold $\tilde{M}$ equipped with a Riemannian metric $g$. The Gauss and Wiengarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad \text{and} \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N$$

for all $X, Y \in TM$ and $N \in T^\perp M$, where $\tilde{\nabla}, \nabla$ and $\nabla^\perp$ are the Riemannian, induced Riemannian and induced normal connections in $\tilde{M}, M$ and the normal bundle $T^\perp M$ of $M$ respectively, and $h$ is the second fundamental form related to the shape operator $A$ by $g(h(X, Y), N) = g(A_N X, Y)$. 
For any vector $X$ tangent to $M$ we put $\varphi X = PX + FX$, where $PX$ and $FX$ are the tangential and the normal components of $\varphi X$, respectively. Given an orthonormal basis $\{e_1, \ldots, e_n\}$ of $M$, we define the squared norm of $P$ by

$$||P||^2 = \sum_{i,j=1}^{n} g^2(\varphi e_i, e_j)$$

and the mean curvature vector $H(p)$ at $p \in M$ is given by $H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)$.

We put $h_{ij}^r = g(h(e_i, e_j), e_r)$ and $||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j))$ where $\{e_{n+1}, \ldots, e_{2m+1}\}$ is an orthonormal basis of $T_p^\perp M$ and $r = n + 1, \ldots, 2m + 1$. A submanifold $M$ is totally geodesic in $\tilde{M}$ if $h = 0$, and minimal if $H = 0$.

On the other hand, $M$ is said to be a totally real submanifold if $P$ is identically zero, that is, $\varphi X \in T_p^\perp M$ for any $X \in T_p M, p \in M$.

For an $n$-dimensional Riemannian manifold $M$, we denote by $K(\pi)$ the sectional curvature of $M$ associated with a plane section $\pi \subset T_p M, p \in M$. For any orthonormal basis $e_1, \ldots, e_n$ of the tangent space $T_p M$, the scalar curvature $\tau$ at $p$ is defined by to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

§3. Some inequality for warped product submanifolds

We give the following lemma for later use.

**Lemma 3.1** ([4]). Let $a_1, \ldots, a_n, a_{n+1}$ be $n + 1$ ($n \geq 2$) real numbers such that

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1) \left(\sum_{i=1}^{n} a_i^2 + a_{n+1}\right).$$

Then, $2a_1a_2 \geq a_{n+1}$, with the equality holding if and only if $a_1 = a_2 = a_3 = \cdots = a_n$.

We investigate warped product submanifolds tangent to the structure vector field $\xi$ in a cosymplectic space form $\tilde{M}(c)$.

**Theorem 3.2.** Let $\phi : M_1 \times_f M_2 \rightarrow \tilde{M}(c)$ be an isometric immersion of an $n$-dimensional warped product into a $(2m + 1)$-dimensional cosymplectic space form of constant $\varphi$-sectional curvature $c$ whose structure vector field $\xi$ is tangent to $M_1$. Then, we have

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} ||H||^2 + \frac{c}{4}(n_1 + 2),$$

where $n_i = \dim M_i, i = 1, 2$, and $\Delta$ is the Laplacian operator of $M_1$. 
Proof. Let $M_1 \times_f M_2$ be a warped product submanifold of a cosymplectic space form $\tilde{M}(c)$ with constant $\varphi$-sectional curvature $c$ whose structure vector field $\xi$ is tangent to $M_1$. Since $M_1 \times_f M_2$ is a warped product, it is easily seen that

\begin{equation}
\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z,
\end{equation}

for any vector fields $X, Z$ tangent to $M_1, M_2$, respectively. If $X$ and $Z$ are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by $X$ and $Z$ is given by

\begin{equation}
K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\{(\nabla_X X)f - X^2f\}.
\end{equation}

We choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m+1}\}$ such that $e_1, \ldots, e_{n_1} = \xi$ are tangent to $M_1$, $e_{n_1+1}, \ldots, e_n$ are tangent to $M_2$ and $e_{n+1}$ is parallel to $H$. Then, using (3.3) we obtain

\begin{equation}
\Delta f = \sum_{j=1}^{n_1} K(e_j \wedge e_s),
\end{equation}

for each $s \in \{n_1 + 1, \ldots, n\}$. From the equation of Gauss, we obtain

\begin{equation}
2\tau = \{n(n-1)+3||P||^2 - 2n + 2\}c_4 + n^2||H||^2 - ||h||^2.
\end{equation}

We denote

\begin{equation}
\delta = 2\tau - \{n(n-1)+3||P||^2 - 2n + 2\}c_4 - \frac{n^2}{2}||H||^2.
\end{equation}

Substituting (3.5) in (3.6), we have

\begin{equation}
n^2||H||^2 = 2(\delta + ||h||^2).\end{equation}

With respect to the above orthonormal basis, (3.7) takes the following form:

\begin{equation}
\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left(\delta + \sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{1 \leq i \neq j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 \right),
\end{equation}

which implies

\begin{equation}
\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left\{\delta + \sum_{i=1}^{n} a_i^2 + \sum_{1 \leq i \neq j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 \right. \\
\left. - \sum_{2 \leq j \neq k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1 \leq s \neq t \leq n} h_{ss}^{n+1} h_{tt}^{n+1}\right\},
\end{equation}

where $a_1 = h_{11}^{n+1}, a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{i=n_1+1}^{n} h_{ii}^{n+1}$. 

Dae Won Yoon
Applying Lemma 3.1 to (3.8) yields

\[ \sum_{1 \leq j < k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} + \sum_{n_1 + 1 \leq s < t \leq n} h_{ss}^{n+1} h_{tt}^{n+1} \]

(3.9)

\[ \geq \frac{\delta}{2} + \sum_{1 \leq \alpha < \beta \leq n} (h_{\alpha\beta}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2m+1} \sum_{\alpha, \beta = 1}^{n} (h_{\alpha\beta}^r)^2, \]

with equality holding if and only if we have

(3.10) \[ \sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \]

On the other hand, (2.5) and (3.4) imply

\[ n_2 \frac{\Delta f}{f} = \tau - \sum_{1 \leq j < k \leq n_1} K(e_j \wedge e_k) - \sum_{n_1 + 1 \leq s < t \leq n} K(e_s \wedge e_t) \]

\[ = \tau - \frac{n_1(n_1 - 1)c}{8} - \frac{3c}{4} \sum_{1 \leq j < k \leq n_1} g^2(Pe_j, e_k) - \frac{c}{4}(1 - n_1) \]

(3.11)

\[ - \frac{2m+1}{8} \sum_{r=n+1}^{2m+1} \sum_{1 \leq j < k \leq n_1} (h_{jj}^r h_{kk}^r - (h_{jk}^r)^2) - \frac{n_2(n_2 - 1)c}{8} \]

\[ - \frac{3c}{4} \sum_{n_1 + 1 \leq s < t \leq n} g^2(Pe_s, e_t) - \frac{3c}{4} \sum_{n_1 + 1 \leq s < t \leq n} (h_{ss}^r h_{tt}^r - (h_{st}^r)^2). \]

Combining (3.9) and (3.11) and taking account of (3.4), we have

\[ n_2 \frac{\Delta f}{f} \leq \tau - \frac{n(n-1)c}{8} + \frac{c}{4} \sum_{1 \leq j < k \leq n_1} g^2(Pe_j, e_k) - \frac{3c}{4} \sum_{n_1 + 1 \leq s < t \leq n} g^2(Pe_s, e_t). \]

(3.12)

By (3.6), the inequality (3.12) reduces to

\[ \frac{\Delta f}{f} \leq \frac{n^2}{4n_2} ||H||^2 + (n_1 - 1)\frac{c}{4} + \frac{3c}{4} \sum_{1 \leq j < k \leq n_1} g^2(Pe_j, e_k) \]

(3.13)

\[ \leq \frac{n^2}{4n_2} ||H||^2 + (n_1 - 1)\frac{c}{4} + \frac{3c}{4} \min \left\{ \frac{n_1}{n_2}, 1 \right\}. \]

We distinguish two cases:

(a) \( n_1 \leq n_2 \), in this case the inequality (3.13) implies (3.1).

(b) \( n_1 > n_2 \), in this case (3.13) also becomes (3.1). It completes the proof. □

**Corollary 3.3.** Let \( \phi : M = M_1 \times f M_2 \longrightarrow \tilde{M}(c) \) be an isometric immersion of an \( n \)-dimensional totally real warped product into a \( (2m+1) \)-dimensional cosymplectic space form whose the structure vector field \( \xi \) is tangent to \( M_1 \). Then, we have

\[ \frac{\Delta f}{f} \leq \frac{n^2}{4n_2} ||H||^2 + (n_1 - 1)\frac{c}{4}. \]

(3.14)
where, \( n_i = \dim M_i, i = 1, 2 \), and \( \Delta \) is the Laplacian operator of \( M_1 \).

Moreover, the equality case of (3.14) holds if and only if \( \phi \) is a mixed totally geodesic immersion and \( n_i H_1 = n_2 H_2 \), where, \( H_i, i = 1, 2 \) are the partial mean curvatures.

**Proof.** By (3.13), we can easily obtain the inequality (3.14). Also, we see that the equality sign of (3.13) holds if and only if

\[
(3.15) \quad h''_{ij} = 0, \quad 1 \leq j \leq n_1, \quad n_1 + 1 \leq t \leq n, \quad n + 1 \leq r \leq 2m + 1,
\]

and

\[
(3.16) \quad \sum_{i=1}^{n_1} h''_{ii} = \sum_{t=n_1+1}^{n} h''_{tt} = 0, \quad n + 2 \leq r \leq 2m + 1.
\]

Obviously (3.15) is equivalent to the mixed totally geodesic of the warped product \( M \) and (3.10) and (3.16) imply \( n_i H_1 = n_2 H_2 \). The converse statement is straightforward. \( \square \)

**Corollary 3.4.** Let \( M_1 \times_f M_2 \) be a totally real warped product in a cosymplectic space form \( \tilde{M}(c) \) whose the structure vector \( \xi \) is tangent to \( M_1(n_1 > 1) \) and a warping function \( f \) is a harmonic. Then, \( M_1 \times_f M_2 \) admits no minimal totally real immersion into a cosymplectic space form \( \tilde{M}(c) \) with \( c < 0 \).

**Proof.** Assume \( f \) is a harmonic function on \( M_1 \) and \( M_1 \times_f M_2 \) admits a minimal totally real immersion in a cosymplectic space form \( \tilde{M}(c) \). Then, the inequality (3.14) becomes \( c \geq 0 \). \( \square \)

**Corollary 3.5.** Let \( M_1 \times_f M_2 \) be a totally real warped product in a cosymplectic space form \( \tilde{M}(c) \) whose the structure vector \( \xi \) is tangent to \( M_1(n_1 > 0) \). If the warping function \( f \) of a warped product \( M_1 \times_f M_2 \) is an eigenfunction of the Laplacian on \( M_1 \) with corresponding eigenvalue \( \lambda > 0 \), then \( M_1 \times_f M_2 \) dose not admit a minimal totally real immersion into a cosymplectic space form \( \tilde{M}(c) \) with \( c \leq 0 \).

**Proof.** If \( f \) is an eigenfunction of the Laplacian on \( M_1 \) with eigenvalue \( \lambda > 0 \). Then inequality (3.14) implies that \((n_1 - 1)\frac{\varsigma}{4} \geq \lambda > 0 \). Therefore, we have Corollary 3.5. \( \square \)

**Corollary 3.6.** Let \( M_1 \times_f M_2 \) be a compact totally real warped product in a cosymplectic space form \( \tilde{M}(c) \) such that the structure vector \( \xi \) is tangent to \( M_1(n_1 > 1) \) and \( c < 0 \). Then \( M_1 \times_f M_2 \) is a Riemannian product.

**Theorem 3.7.** Let \( \phi : M_1 \times_f M_2 \longrightarrow \tilde{M}(c) \) be an isometric immersion of an \( n \)-dimensional warped product into a \((2m + 1)\)-dimensional cosymplectic space form of constant \( \varsigma \)-sectional curvature \( c \) whose structure vector field \( \xi \) is tangent to \( M_2 \). Then, we have

\[
\frac{\Delta f}{f} \leq \frac{n_i^2}{4n_2} \|H\|^2 + \left(3 + n_1 - \frac{n_1}{n_2}\right) \frac{c}{4},
\]

where \( n_i = \dim M_i \), \( i = 1, 2 \), and \( \Delta \) is the Laplacian operator of \( M_1 \).

**Corollary 3.8.** Let \( \phi : M = M_1 \times_f M_2 \longrightarrow \tilde{M}(c) \) be an isometric immersion of an \( n \)-dimensional totally real warped product into a \((2m + 1)\)-dimensional a cosymplectic
space form whose the structure vector field $\xi$ is tangent to $M_2$. Then, we have

$$\Delta f \leq \frac{n_i^2}{4n_2}||H||^2 + \left( n_1 - \frac{n_1}{n_2} \right) \frac{c}{4},$$

where, $n_i = \dim M_i$, $i = 1, 2$, and $\Delta$ is the Laplacian operator of $M_1$.

Moreover, the equality case of (3.17) holds if and only if $\phi$ is a mixed totally geodesic immersion and $n_1H_1 = n_2H_2$, where, $H_i, i = 1, 2$ are the partial mean curvatures.

**Corollary 3.9.** Let $M_1 \times_f M_2$ be a totally real warped product in a cosymplectic space form $\tilde{M}(c)$ whose the structure vector $\xi$ is tangent to $M_2(n_2 > 1)$ and a warping function $f$ is a harmonic. Then, $M_1 \times_f M_2$ admits no minimal totally real immersion into a cosymplectic space form $\tilde{M}(c)$ with $c < 0$.

**Corollary 3.10.** Let $M_1 \times_f M_2$ be a totally real warped product in a cosymplectic space form $\tilde{M}(c)$ whose the structure vector $\xi$ is tangent to $M_2(n_2 > 0)$. If the warping function $f$ of a warped product $M_1 \times_f M_2$ is an eigenfunction of the Laplacian on $M_1$ with corresponding eigenvalue $\lambda > 0$, then $M_1 \times_f M_2$ dose not admit a minimal totally real immersion into a cosymplectic space form $\tilde{M}(c)$ with $c \leq 0$.

**Corollary 3.11.** Let $M_1 \times_f M_2$ be a compact totally real warped product in a cosymplectic space form $\tilde{M}(c)$ such that the structure vector $\xi$ is tangent to $M_2(n_2 > 1)$ and $c \leq 0$. Then $M_1 \times_f M_2$ is a Riemannian product.

**References**


Author’s address:
Dae Won Yoon
Department of Mathematics Education and RINS,
Gyeongsang National University,
Chinju 660-701, South Korea
e-mail: dwyon@gsnu.ac.kr