On torse-forming vector valued 1-forms

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Abstract

The notion of torse-forming vector valued 1-forms on a Riemannian manifold $(M, g)$ have been defined by Rosca [6]. In this paper the authors derive results regarding the wedge product of torse forming vector fields, which are applied to the case of Kenmotsu manifolds.

Key words: torse-forming vector field, Riemannian manifold, Kenmotsu manifold.

§1. Preliminaries

Let $M$ be an $m$-dimensional $C^\infty$ Riemannian manifold with metric tensor $g$, and let $TM$ be the tangent bundle over which is assumed to be not trivial. Let $\Gamma TM = \chi M$ be the set of sections of the tangent bundle and let $\flat : TM \rightarrow T^*M$ be the musical isomorphism defined by $g$ and $\sharp$ the inverse of $\flat$, i.e., $\sharp : T^*M \rightarrow TM$. Following W. A. Poor [4], we denote

$$ A^q(M, TM) = \Gamma Hom (\wedge^q TM, TM) $$

the set of vector valued $q$-forms and by

$$ d^\nabla : A^q(M, TM) \rightarrow A^{q+1}(M, TM) $$

the exterior covariant derivative with respect to $\nabla$. Notice that in general $d^\nabla^2 = d^\nabla \circ d^\nabla \neq 0$ unlike $d \circ d = 0$.

Next $dp \in A^0(M, TM)$ stands for the soldering form of $M$ [1] ($dp$ is the canonical vector valued 1-form and $d^\nabla (dp) = 0$).

A (non parallel) vector field on a Riemannian (or pseudo-Riemannian) manifold is said to the exterior concurrent (abr. EC) [5], [3], if

$$ (1.1) \quad \nabla^2 X = \pi^\xi \wedge dp $$

for some 1-form $\pi^\xi$ on $M$; this 1-form $\pi^\xi$ is called the concurrence form associated with $X$. 

This definition is a natural extension of the concept of concurrent vector fields (in this case $\xi$ is used instead of $\nabla^2$). For any E.C. vector field $X$, the Ricci tensor $R$ of $\nabla$ satisfies

$$R(X, Z) = -(n-1)fg(X, Z) \implies f = -\frac{1}{n-1}Ric(X),$$

where $Ric(X)$ is the Ricci curvature of $M$ with respect to $X$.

A vector field $T$ such that

$$\nabla T = sdp + \alpha \otimes T \quad s \in \wedge^0M, \quad \alpha \in \wedge^1M$$

is defined as torse forming (abr. TF) vector field [7].

The 1-form $\alpha$ is called the generative of $\sigma$, and one has the relation (Rosca’s lemma)

$$dT^\flat = \alpha \wedge T^\flat,$$

which proves that $T^\flat$ is exterior recurrent [1] and has $\alpha$ as recurrence form.

Let $\theta = \{e_A | A = 1, ..., n\}$ be a local field of adapted vectorial frames over $M$ and let $\vartheta^* = \{\omega^A\}$ be its associated coframe field.

Then Cartan’s structure equations written in indexless form are

$$\nabla e = \theta \otimes e$$

$$d\omega = -\theta \wedge \omega$$

$$d\Theta = -\theta \wedge \theta + \Theta.$$ 

In the above equations $\theta$ (resp $\Theta$) are the local connection forms in the tangent bundle $TM$ (resp. the curvature 2-forms on $M$).

§2. Torse forming vector valued 1-forms

Let $F$ be a vector valued 1-form in a $C^\infty$-manifold $M$ and let $d^\nabla$ be the exterior covariant derivative $\xi$ on $M$, and $dp$ the soldering form on $M$.

If $d^\nabla$ denote the covariant derivative operator i.e. if

$$A^q(M, TM) = \Gamma Hom(\wedge^q\Gamma M, TM),$$

then $d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM)$ means the exterior covariant derivative with respect to the Levi-Civita connection with respect to $g$, then if $F$ is a vector valued 1-form, then a torse-forming vector valued (abr. TFVF) 1-form is defined by

$$d^\nabla F = \omega \wedge dp + \alpha \wedge F \quad \alpha, \omega \in \wedge^1M,$$

where $\alpha$ is called the generating form and $\omega$ the associated form of $F$.

We assume in this paper that $F$ is defined by the wedge product $\wedge$ of two vector fields $U$ and $V$, that is

$$F = U \wedge V = V^\flat \otimes U - U^\flat \otimes V.$$
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Operating on $F$ by the operator $d\nabla$, one has 
\begin{equation}
(2.10) \quad d\nabla F = dV^b \otimes U - V^b \wedge \nabla U - dU^b \otimes V + U^b \wedge \nabla V.
\end{equation}

Consider now the case when the vector fields $U$ and $V$ are both torse forming vector fields. Consequently by (1.3), the covariant differentials of $U$ and $V$ satisfy 
\begin{equation}
(2.11) \quad \begin{cases}
\nabla U = adp + \alpha \otimes U \\
\nabla V = bdp + \beta \otimes V,
\end{cases}
\end{equation}
where $a, b$ are two scalar and the Pfaffians $\alpha$ and $\beta$ are the generating forms of $U$ and $V$ respectively.

Taking account of lemma (1.4) one has 
\begin{equation}
(2.12) \quad \begin{cases}
dU^b = \alpha \wedge U^b \\
dV^b = \beta \wedge V^b.
\end{cases}
\end{equation}

Then operating on (2.9) by the operator $d\nabla$, one infers after some calculations
\begin{equation}
(2.13) \quad d\nabla F = \left(bU^b - aV^b\right) \wedge dp + (\beta + \alpha) \wedge F.
\end{equation}

Consequently we derive

**Theorem 1.** The wedge product of two torse forming vector fields defines a torse forming 1-form $F$. If the sum of the torse forming vanishes, then $F$ moves to a concurrent vector valued 1-form.

§2. The f-Kenmotsu manifold case

Consider now a $f$-Kenmotsu manifold $M(\Phi, \Omega, \eta, \xi, h)$ in the sense of Z. Olszak and R. Rosca [2]. For a $f$-Kenmotsu manifold (abr. f.K), one has the following structure equations
\begin{equation}
(3.14) \quad \begin{cases}
\Phi^2 = I + \eta \otimes \xi, \quad \Phi \xi = 0, \quad \eta(\xi) = 1 \\
\nabla \xi = f(dp - \eta \otimes \xi) \\
g(Z, Z') = g(\Phi Z, \Phi Z') + \eta(Z)\eta(Z') \\
(\nabla_Z, \Phi)Z = -f[\eta(Z)\Phi Z' + g(\Phi Z, Z')\xi] \\
\Omega(Z, Z') = g(\Phi Z, Z').
\end{cases}
\end{equation}

By the second equation (3.14) it is seen that the structure vector field $\xi$ is a TF vector field.

Assume now that $M^\xi$ carries a second TF, say $U$ such that
\begin{equation}
(3.15) \quad \nabla U = adp + \alpha \otimes U.
\end{equation}

Since in the case under discussion one may write
\begin{equation}
(3.16) \quad dp = \omega^a e_a + \eta \xi, \quad a, b \in \{1, \ldots, 2m\}
\end{equation}

and making use of (1.5) one derives the relations
\begin{equation}
(3.17) \quad dU^a + U^b \theta_b^a = a\omega^a + fU^a\eta,
\end{equation}
where the index "o" corresponds to \( \xi \), i.e., \( \theta^a_\xi = f \omega^a \). From (3.18) we infer

\[
dU^o = fU^\flat + (a + fU^o) \eta
\]

and since by (2.12) one has \( dU^\flat = \alpha \land U^\flat \).

By (3.14) we get \( d\eta = 0 \), and derive by exterior differentiation

\[
d\alpha \land U^\flat + \eta \land \alpha \land U^\flat = 0
\]

By the above equation we can see that the existence of TF vector field \( U \) on a \( f \)-Kenmotsu manifold is determined by a closed differential system.

Hence we yield

**Theorem 2.** On any \( f \)-Kenmotsu manifold with structure vector field \( \xi \), there exists an infinity of TF vector fields \( U \) such that the wedge product \( U \land \xi \) defines a torse forming 1-form on \( M \).

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**References**


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