Spin 2 particle with anomalous magnetic moment in Riemann space-time, restriction to massless case, gauge symmetry

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The theory of massive and massless fields of spin 2, after the fundamental investigations by W. Pauli and M. Fierz, always attracted attention. The most of investigations were performed with the use of the 2-nd ordered equations. Probably F.I. Fedorov did the first study within the 1-st order equations. It turned out that the spin 2 particle requires for its description a 30-component set of tensors. Besides, F.I. Fedorov proposed a more general theory, which is based on 50-component set of tensors. It turned out that this theory describes the spin 2 particle with anomalous magnetic moment. In the present work, we consider this theory in presence of arbitrary electromagnetic field and any Riemannian space-time background. First we study the 50-component theory for a massive spin 2 particle. In such a generalized framework, there arises non-minimal interaction with the curved space-time background through Ricci and Riemann tensors. It is important that the theory under consideration allows for a new and generalized massless limit for spin 2 field. This fact is of special interest, because as known the conventional Pauli - Fierz theory for massless field does not possess gauge symmetry in curved space-time; in particular, in models with zero Ricci tensor. We have shown that the a generalized theory possesses the gauge symmetry in all space - time models for which the Ricci tensor vanishes, this case is the most interesting in physical applications of General relativity. We study the 50-component theory for a massive spin 2 particle in presence of electromagnetic fields and any Riemannian space-time background. Such a generalized theory describes the particle with anomalous magnetic moment; in addition, there arises a non-minimal interaction with the curved space-time background through Ricci and Riemann tensors.

Keywords: spin 2 particle,anomalous magnetic moment, external electromagnetic fields, Riemannian space-time, non-minimal interaction, massless limit, gauge symmetry, non-minimal interaction.

1 Introduction

The theory of massive and massless fields of spin 2 after fundamental investigations by Pauli and Fierz [1], [2], always attracted attention: De Broglie [3], Fedorov et al. [4], Regge [7], Hagen [8], Cox [9], Bogush et al [10] – [12]. The most of investigations were performed in the frames of 2-nd order equations. Probably, Fedorov [4] did the first study within the 1-st order equations. It turned out that the spin 2 particle requires for its description a 30-component set of tensors. Besides, Fedorov proposed more general theory, which is based on 50-component set of tensors. It turned out that this theory describes the spin 2 particle with anomalous magnetic moment [5,6].

In the present work, we consider this theory in presence of arbitrary electromagnetic field and any Riemannian space-time background. First we study the 50-component theory for the massive particle. In such a generalized theory there arises non-minimal interaction with the curved space-time background through the Ricci and Riemann tensors. It is important that the theory under consideration allows for a new and generalized massless limit for spin 2 field. This fact is of special interest, because as known the conventional Pauli – Fierz theory for massless field does not possess gauge symmetry in curved space-time, in particular, in all models with the zero Ricci tensor. We have shown that the generalized theory possesses the gauge symmetry in all space – time models for which the Ricci tensor vanishes, this case is the most interesting in physical applications of General relativity.

2 The massive case

We start with the known 50-component system Fedorov et al [4]–[11] for a massive spin 2 particle with anomalous magnetic moment[1]. The extension to the generally covariant case is done by simple changing the ordinary Minkowski tensors and derivatives to covariant ones:

$$2\,\lambda_1 D^a \Psi^{(1)}_a + 2\,\lambda_2 D^a \Psi^{(2)}_a + iM\;\Psi = 0\;, \tag{2.1}$$

$$\lambda_3\; D_a \Psi + 2\lambda_4\; D^b \Psi_{(ba)} + iM\;\Psi^{(1)}_a = 0\;, \tag{2.2}$$

$$\lambda_5\; D_a \Psi + 2\lambda_6\; D^b \Psi_{(ba)} + iM\;\Psi^{(2)}_a = 0\;, \tag{2.3}$$

$$\begin{gathered}\frac{\lambda_7}{2}\;(D_a \Psi^{(1)}_b + D_b \Psi^{(1)}_a - \frac{1}{2}\;g_{ab} D^c \Psi^{(1)}_c) + \frac{\lambda_8}{2}\;(D_a \Psi^{(2)}_b + D_b \Psi^{(2)}_a - \frac{1}{2}\;g_{ab} D^c \Psi^{(2)}_c) \\ +2\lambda_9\; D^c \Psi_{(abc)} - 2\lambda_{10}\;(\;D^c \Psi_{a[bc]} + D^c \Psi_{b[ac]}\;) + iM\;\Psi_{(ab)} = 0\;,\end{gathered} \tag{2.4}$$

$$\frac{\lambda_{11}}{2}\;(\;D_c \Psi_{(ab)} - D_b \Psi_{(ac)} - \frac{1}{3} g_{ca} D^m \Psi_{(mb)} + \frac{1}{3} g_{ba} D^m \Psi_{(mc)}\;) + iM\;\Psi_{a[bc]} = 0\;, \tag{2.5}$$

$$\begin{gathered}\frac{\lambda_{12}}{3}\;(\;D_a \Psi_{(bc)} + D_b \Psi_{(ca)} + D_c \Psi_{(ab)} \\ -\frac{1}{3}\;g_{ac} D^m \Psi_{(mb)} - \frac{1}{3}\;g_{cb} D^m \Psi_{(ma)} - \frac{1}{3}\;g_{ba} D^m \Psi_{(mc)}\;) + iM\Psi_{(abc)} = 0\;,\end{gathered} \tag{2.6}$$

where $D_a = \nabla_a + ieA_a$; ∇_a stands for the covariant derivative, A_a is an electromagnetic 4-potential, e stands for a particle charge, $\lambda_1, ..., \lambda_{12}$ are some numerical parameters obeying the following constraints

$$\begin{gathered}2\lambda_{10}\lambda_{11} - \frac{2}{3}\lambda_9\lambda_{12} = 1\;, \quad \lambda_4\lambda_7 + \lambda_6\lambda_8 + \frac{8}{9}\lambda_9\lambda_{12} = \frac{1}{3}\;, \\ \lambda_1\lambda_3 + \lambda_2\lambda_5 = -\frac{1}{4}\;, \quad (\lambda_1\lambda_4 + \lambda_2\lambda_6)\;(\lambda_3\lambda_7 + \lambda_5\lambda_8) = -\frac{1}{12}\;.\end{gathered} \tag{2.7}$$

The complete field function consists of the following tensor set

$$\Psi\;, \quad B_a\;, \quad C_a\;, \quad \Psi_{(ab)}\;, \quad \Psi_{a[bc]}\;, \quad \Psi_{(abc)}\;. \tag{2.8}$$

In total, it contains 52 components: $1 + 4 + 4 + 10 + 24 + 8$.

Recall that in the minimal 30-component theory, the set of involved tensors consists of

$$\bar{\Phi}\;, \quad \bar{\Phi}_a\;, \quad \bar{\Phi}_{(ab)}\;, \quad \bar{\Phi}_{[ab]c}\;; \tag{2.9}$$

[1] In Minkowski space the metric with signature $+1, -1, -, -1)$ is applied.

The corresponding minimal system is as follows

$$
\begin{gathered}
D^a \bar{\Phi}_a - M\ \bar{\Phi} = 0\ , \qquad \frac{1}{2}\ D_a \bar{\Phi} - \frac{1}{3}\ D^b \bar{\Phi}_{(ab)} - M\ \bar{\Phi}_a = 0\ , \\
D_a \bar{\Phi}_b\ +\ D_b \bar{\Phi}_a - \frac{1}{2} g_{ab}\ D^k \bar{\Phi}_k + \\
+\frac{1}{2}\ (D^k \bar{\Phi}_{[ka]b} + D^k \bar{\Phi}_{[kb]a} - \frac{1}{2} g_{ab} D^k \bar{\Phi}_{[kn]}^{\ \ \ n}\ ) - M\ \bar{\Phi}_{(ab)} = 0\ , \\
D_a \bar{\Phi}_{(bc)} - D_b \bar{\Phi}_{(ac)} + \frac{1}{3}\ (g_{bc} D^k \bar{\Phi}_{(ak)} - g_{ac}\ D^k \bar{\Phi}_{(bk)}) - M\ \bar{\Phi}_{[ab]c} = 0\ .
\end{gathered} \tag{2.10}
$$

Here the number of independent components equals to 30:

$$
\Phi(x) \Longrightarrow 1, \quad \Phi_a \Longrightarrow 4, \quad \Phi_{(ab)} \Longrightarrow (10-1) = 9, \quad \Phi_{[ab]c} \Longrightarrow 6 \times 4 - 4 - 4 = 16.
$$

Let us prove that excluding from the system (2.1) – (2.6) one vector and 3-rank tensor $\Psi_{(abc)}$, we may reduce the resulting system to minimal form, which contains additional interaction terms with external electromagnetic and gravitational fields.

To this end, instead of vectors $\Psi_a^{(1)}$ and $\Psi_a^{(2)}$, we introduce the new variables

$$
\left| \begin{array}{c} B_a \\ C_a \end{array} \right| = \left| \begin{array}{cc} \lambda_1 & \lambda_2 \\ \lambda_7 & \lambda_8 \end{array} \right| \left| \begin{array}{c} \Psi_a^{(1)} \\ \Psi_a^{(2)} \end{array} \right| , \qquad \left| \begin{array}{c} \Psi_a^{(1)} \\ \Psi_a^{(2)} \end{array} \right| = \frac{1}{\lambda_1\lambda_8 - \lambda_2\lambda_7} \left| \begin{array}{cc} \lambda_8 & -\lambda_2 \\ -\lambda_7 & \lambda_1 \end{array} \right| \left| \begin{array}{c} B_a \\ C_a \end{array} \right| . \tag{2.11}
$$

Then the system (2.1)–(2.6) can be presented as follows

$$
2\ D^a B_a + im\ \Psi = 0\ , \tag{2.12}
$$

$$
-\frac{1}{4}\ D_a \Psi + 2\,(\lambda_1\lambda_4 + \lambda_2\lambda_6)\ D^b \Psi_{(ba)} + iM\ B_a = 0\ , \tag{2.13}
$$

$$
(\lambda_7\lambda_3 + \lambda_8\lambda_5)\ D_a \Psi + 2(\lambda_7\lambda_4 + \lambda_8\lambda_6)\ D^b \Psi_{(ba)} + iM\ C_a = 0\ , \tag{2.14}
$$

$$
\begin{gathered}
\frac{1}{2}\ (D_a C_b + D_b C_a - \frac{1}{2}\ g_{ab} D^c C_c) \\
+2\lambda_9\ D^c \Psi_{(abc)} - 2\lambda_{10}\ (\ D^c \Psi_{a[bc]} + D^c \Psi_{b[ac]}) + iM\ \Psi_{(ab)} = 0\ ,
\end{gathered} \tag{2.15}
$$

$$
\frac{\lambda_{11}}{2}\ (D_c \Psi_{(ab)} - D_b \Psi_{(ac)} - \frac{1}{3} g_{ca} D^m \Psi_{(mb)} + \frac{1}{3} g_{ba} D^m \Psi_{(mc)}) + iM\ \Psi_{a[bc]} = 0\ , \tag{2.16}
$$

$$
\begin{gathered}
\frac{\lambda_{12}}{3} [D_a \Psi_{(bc)} + D_b \Psi_{(ca)} + D_c \Psi_{(ab)} - \frac{1}{3} g_{ac} D^m \Psi_{(mb)} \\
-\frac{1}{3}\ g_{cb} D^m \Psi_{(ma)} - \frac{1}{3} g_{ba} D^m \Psi_{(mc)}] + +iM \Psi_{(abc)} = 0\ .
\end{gathered} \tag{2.17}
$$

Let us multiply eq. (2.14) by $(\lambda_1\lambda_4 + \lambda_2\lambda_6)$. Taking in mind (2.7). We get

$$
-\frac{1}{12} D_a \Psi + 2\,(\lambda_1\lambda_4 + \lambda_2\lambda_6)\,(\lambda_7\lambda_4 + \lambda_8\lambda_6)\ D^b \Psi_{(ba)} + iM\ \ (\lambda_1\lambda_4 + \lambda_2\lambda_6)\ C_a = 0\ .
$$

By substituting the expression of $D_a \Psi$ from (2.13), we obtain

$$
\begin{gathered}
-\frac{2}{3} (\lambda_1\lambda_4 + \lambda_2\lambda_6)\ D^b \Psi_{(ba)} - \frac{iM}{3}\ B_a \\
+2(\lambda_1\lambda_4 + \lambda_2\lambda_6)(\lambda_7\lambda_4 + \lambda_8\lambda_6)\ D^b \Psi_{(ba)} + iM\ (\lambda_1\lambda_4 + \lambda_2\lambda_6) C_a = 0\ ,
\end{gathered}
$$

whence it follows

$$
C_a = \frac{1}{3(\lambda_1\lambda_4 + \lambda_2\lambda_6)}\ B_a - \frac{2}{iM}\ [(\lambda_7\lambda_4 + \lambda_8\lambda_6) - \frac{1}{3}] D^n \Psi_{(na)}\ .
$$

This relation permits us to exclude the vector C_a. In particular, by substituting the above expression for C_a in eq. (2.15), we derive

$$\frac{1}{6(\lambda_1\lambda_4+\lambda_2\lambda_6)}\ (D_aB_b + D_bB_a - \frac{1}{2}\ g_{ab}D^cB_c)$$
$$-\frac{1}{iM}\ \left[(\lambda_7\lambda_4+\lambda_8\lambda_6)-\frac{1}{3}\right]\ \left(D_aD^n\Psi_{(nb)} + D_bD^n\Psi_{(na)} - \frac{1}{2}\ g_{ab}D^cD^n\Psi_{(nc)}\right)$$
$$+2\lambda_9\ D^c\Psi_{(abc)} - 2\lambda_{10}\ (\ D^c\Psi_{a[bc]} + D^c\Psi_{b[ac]}) + iM\ \Psi_{(ab)} = 0\ .$$

Thus, instead of the equations (2.12)–(2.17), we may consider the equivalent equations

$$2\ D^aB_a + iM\ \Psi = 0\ , \tag{2.18}$$

$$-\frac{1}{4}\ D_a\Psi + 2(\lambda_1\lambda_4+\lambda_2\lambda_6)\ D^b\Psi_{(ba)} + iM\ B_a = 0\ , \tag{2.19}$$

$$C_a = \frac{1}{3(\lambda_1\lambda_4+\lambda_2\lambda_6)}\ B_a - \frac{2}{iM}\ [(\lambda_7\lambda_4+\lambda_8\lambda_6)-\frac{1}{3}]D^n\Psi_{(na)}\ , \tag{2.20}$$

$$\begin{gathered}\frac{1}{6(\lambda_1\lambda_4+\lambda_2\lambda_6)}\ (D_aB_b + D_bB_a - \frac{1}{2}\ g_{ab}D^cB_c)\\ -\frac{1}{iM}\ [(\lambda_7\lambda_4+\lambda_8\lambda_6)-\frac{1}{3}]\ (D_aD^n\Psi_{(nb)} + D_bD^n\Psi_{(na)} - \frac{1}{2}\ g_{ab}D^cD^n\Psi_{(nc)})\\ +2\lambda_9\ D^c\Psi_{(abc)} - 2\lambda_{10}\ (\ D^c\Psi_{a[bc]} + D^c\Psi_{b[ac]}) + iM\ \Psi_{(ab)} = 0\ ,\end{gathered} \tag{2.21}$$

$$\frac{\lambda_{11}}{2}\ (D_c\Psi_{(ab)} - D_b\Psi_{(ac)} - \frac{1}{3}g_{ca}D^m\Psi_{(mb)} + \frac{1}{3}g_{ba}D^m\Psi_{(mc)}) + iM\ \Psi_{a[bc]} = 0\ , \tag{2.22}$$

$$\begin{gathered}\frac{\lambda_{12}}{3}\ [D_a\Psi_{(bc)} + D_b\Psi_{(ca)} + D_c\Psi_{(ab)}\\ -\frac{1}{3}\ g_{ac}D^m\Psi_{(mb)} - \frac{1}{3}\ g_{cb}D^m\Psi_{(ma)} - \frac{1}{3}\ g_{ba}D^m\Psi_{(mc)}] + iM\Psi_{(abc)} = 0\ .\end{gathered} \tag{2.23}$$

Now, from eqs. (2.22) and (2.23), let us express the 3-rank tensors $\Psi_{a[bc]}$ and $\Psi_{(abc)}$, in terms of the 2-rank tensor:

$$\Psi_{a[bc]} = \frac{i\lambda_{11}}{2M}\ (\ D_c\Psi_{(ab)} - D_b\Psi_{(ac)} - \frac{1}{3}g_{ca}D^m\Psi_{(mb)} + \frac{1}{3}g_{ba}D^m\Psi_{(mc)}\ )\ , \tag{2.24}$$

$$\begin{gathered}\Psi_{(abc)} = \frac{i\lambda_{12}}{3M}(D_a\Psi_{(bc)} + D_b\Psi_{(ca)} + D_c\Psi_{(ab)}\\ -\frac{1}{3}g_{ac}D^m\Psi_{(mb)} - \frac{1}{3}g_{cb}D^m\Psi_{(ma)} - \frac{1}{3}g_{ba}D^m\Psi_{(mc)}).\end{gathered} \tag{2.25}$$

Now we substitute these expressions into eq. (2.21), which yields

$$\begin{gathered}\frac{1}{6(\lambda_1\lambda_4+\lambda_2\lambda_6)}\ (D_aB_b + D_bB_a - \frac{1}{2}\ g_{ab}D^cB_c)\\ +\frac{i}{M}\ [(\lambda_7\lambda_4+\lambda_8\lambda_6)-\frac{1}{3}]\ (D_aD^c\Psi_{(cb)} + D_bD^c\Psi_{(ca)} - \frac{1}{2}\ g_{ab}D^cD^n\Psi_{(nc)})\\ +i\frac{2\lambda_9\lambda_{12}}{3M}D^c(D_a\Psi_{(bc)} + D_b\Psi_{(ca)} + D_c\Psi_{(ab)}\\ -\frac{1}{3}g_{ac}D^m\Psi_{(mb)} - \frac{1}{3}g_{cb}D^m\Psi_{(ma)} - \frac{1}{3}g_{ba}D^m\Psi_{(mc)})\\ -i\frac{\lambda_{10}\lambda_{11}}{M}[D^c(D_c\Psi_{(ab)} - D_b\Psi_{(ac)} - \frac{1}{3}g_{ca}D^m\Psi_{(mb)} + \frac{1}{3}g_{ba}D^m\Psi_{(mc)})\\ +D^c(D_c\Psi_{(ba)} - D_a\Psi_{(bc)} - \frac{1}{3}g_{cb}D^m\Psi_{(ma)} + \frac{1}{3}g_{ab}D^m\Psi_{(mc)})\ ] + iM\ \Psi_{(ab)} = 0\ .\end{gathered}$$

In the last equation, we are to take into account the following constraints (see (2.7))

$$\lambda_{10}\lambda_{11} = \frac{1}{2} + \frac{1}{3}\lambda_9\lambda_{12}\,, \quad \lambda_4\lambda_7 + \lambda_6\lambda_8 - \frac{1}{3} = -\frac{8}{9}\lambda_9\lambda_{12}\,.$$

We also multiply the result by $-iM$ (for brevity let us introduce the notation $\lambda_9\lambda_{12} = \mu$). In this way, from the last equation, we derive the following one

$$\begin{aligned}
&\frac{M}{6i(\lambda_1\lambda_4+\lambda_2\lambda_6)}(D_aB_b + D_bB_a - \frac{1}{2}\, g_{ab}D^cB_c) \\
&-\mu\frac{8}{9}D_aD^c\Psi_{(cb)} - \mu\frac{8}{9}D_bD^c\Psi_{(ca)} + \mu\frac{4}{9}\, g_{ab}D^cD^n\Psi_{(nc)} \\
&+\mu\frac{2}{3}D^cD_a\Psi_{(bc)} + \mu\frac{2}{3}D^cD_b\Psi_{(ca)} + \mu\frac{2}{3}D^cD_c\Psi_{(ab)} - \mu\frac{2}{9}g_{ac}D^cD^m\Psi_{(mb)} \\
&-\mu\frac{2}{9}g_{cb}D^cD^m\Psi_{(ma)} - \mu\frac{2}{9}g_{ba}D^cD^m\Psi_{(mc)} \\
&-\frac{1}{2}D^cD_c\Psi_{(ab)} + \frac{1}{2}D^cD_b\Psi_{(ac)} + \frac{1}{6}g_{ca}D^cD^m\Psi_{(mb)} - \frac{1}{6}g_{ba}D^cD^m\Psi_{(mc)} \\
&-\frac{1}{2}D^cD_c\Psi_{(ba)} + \frac{1}{2}D^cD_a\Psi_{(bc)} + \frac{1}{6}g_{cb}D^cD^m\Psi_{(ma)} - \frac{1}{6}g_{ab}D^cD^m\Psi_{(mc)} \\
&-\mu\frac{1}{3}D^cD_c\Psi_{(ab)} + \mu\frac{1}{3}D^cD_b\Psi_{(ac)} + \mu\frac{1}{9}g_{ca}D^cD^m\Psi_{(mb)} - \mu\frac{1}{9}g_{ba}D^cD^m\Psi_{(mc)} \\
&-\frac{\mu}{3}D^cD_c\Psi_{(ba)} + \frac{\mu}{3}D^cD_a\Psi_{(bc)} + \frac{\mu}{9}g_{cb}D^cD^m\Psi_{(ma)} - \frac{\mu}{9}g_{ab}D^cD^m\Psi_{(mc)} + M^2\Psi_{(ab)} = 0.
\end{aligned} \tag{2.26}$$

Then, after regrouping the terms, we derive the equation

$$\begin{aligned}
&\frac{1}{6i\ (\lambda_1\lambda_4+\lambda_2\lambda_6)}\ (D_aB_b + D_bB_a - \frac{1}{2}\ g_{ab}D^cB_c) \\
&-\frac{1}{M}\Big[D^cD_c\Psi_{(ba)} - \frac{1}{2}\ (\ D^cD_b\Psi_{(ac)} + D^cD_a\Psi_{(bc)}\ ) \\
&+\frac{1}{3}\ g_{ab}\ D^nD^m\Psi_{(nm)} - \frac{1}{6}\ (\ D_aD^m\Psi_{(mb)} + D_bD^m\Psi_{(ma)}\ )\Big] \\
&+\frac{\mu}{M}\Big([D^c,D_a]_-\Psi_{(bc)} + [D^c,D_b]_-\Psi_{(ac)}\Big) + M\ \Psi_{(ab)} = 0.
\end{aligned} \tag{2.27}$$

Instead of definition (2.24, let us introduce a slightly new tensor variable (the numerical parameter γ will be fixed later):

$$\Phi'_{[bc]a} = -\frac{1}{M}\frac{\gamma}{2}\Big(D_c\Psi_{(ab)} - D_b\Psi_{(ac)} + \frac{1}{3}g_{ab}D^m\Psi_{(mc)} - \frac{1}{3}g_{ac}D^m\Psi_{(mb)}\Big). \tag{2.28}$$

Then we readily derive the identity

$$\begin{aligned}
&\frac{1}{\gamma}\ (D^c\Phi'_{[bc]a} + D^c\Phi'_{[ac]b}) \\
&= -\frac{1}{M}\Big[\frac{1}{2}(D^cD_c\Psi_{(ab)} - D^cD_b\Psi_{(ac)} + \frac{g_{ab}}{3}D^cD^m\Psi_{(mc)} - \frac{g_{ac}}{3}D^cD^m\Psi_{(mb)}) \\
&+\frac{1}{2}(D^cD_c\Psi_{(ba)} - D^cD_a\Psi_{(bc)} + \frac{g_{ba}}{3}D^cD^m\Psi_{(mc)} - \frac{g_{bc}}{3}D^cD^m\Psi_{(ma)})\Big] = \\
&= -\frac{1}{M}(D^cD_c\Psi_{(ab)} - \frac{1}{2}D^cD_b\Psi_{(ac)} - \frac{1}{3}D^cD_a\Psi_{(bc)} + \frac{g_{ab}}{3}D^cD^m\Psi_{(mc)} \\
&-\frac{g_{ac}}{6}D^cD^m\Psi_{(mb)} - \frac{g_{bc}}{6}D^cD^m\Psi_{(ma)}),
\end{aligned}$$

which coincides with the expression in square brackets from eq. (2.27). Therefore, eq. (2.27) may be re-written as follows (here, we consider $\gamma = \sqrt{2}$):

$$\frac{1}{6i\Big(\lambda_1\lambda_4 + \lambda_2\lambda_6\Big)}\Big(D_a B_b + D_b B_a - \frac{1}{2}\, g_{ab} D^c B_c\Big) + \frac{1}{\sqrt{2}}\Big(D^c \Phi'_{[bc]a} + D^c \Phi'_{[ac]b}\Big)$$
$$+ \frac{\mu}{M}\Big(\, [D^c, D_a]_- \Psi_{(bc)} + [D^c, D_b]_- \Psi_{(ac)}\,\Big) + M\ \Psi_{(ab)} = 0\ . \tag{2.29}$$

Hence, the system takes the form

$$2\ D^a B_a + iM\ \Psi = 0\ ,$$

$$-\frac{1}{4}\ D_a \Psi + 2(\lambda_1\lambda_4 + \lambda_2\lambda_6)\ D^b \Psi_{(ba)} + iM\ B_a = 0\ ,$$

$$\frac{1}{6i\ (\lambda_1\lambda_4 + \lambda_2\lambda_6)}\ (D_a B_b + D_b B_a - \frac{1}{2}\ g_{ab} D^c B_c)\ + \frac{1}{\sqrt{2}}\ (D^c \Phi'_{[bc]a} + D^c \Phi'_{[ac]b})$$
$$+ \frac{\mu}{M}(\ [D^c, D_a]_- \Psi_{(bc)} + [D^c, D_b]_- \Psi_{(ac)}\ ) + M\ \Psi_{(ab)} = 0\ ,$$
$$\frac{1}{\sqrt{2}}(D_c \Psi_{(ab)} - D_b \Psi_{(ac)} + \frac{1}{3} g_{ab} D^m \Psi_{(mc)} - \frac{1}{3} g_{ac} D^m \Psi_{(mb)}) + M \Phi'_{[bc]a} = 0.$$

Instead of the variables B_a and Ψ, we will consider the new ones

$$\Phi' = -\frac{1}{4\sqrt{3}(\lambda_1\lambda_4 + \lambda_2\lambda_6)}\Psi, \quad \Phi'_a = \frac{i}{\sqrt{6}(\lambda_1\lambda_4 + \lambda_2\lambda_6)} B_a, \quad (\text{let } \Phi'_{(ab)} = \Phi_{(ab)})\ . \tag{2.30}$$

Thus, we arrive at a modified 30-component first order system

$$\frac{1}{\sqrt{2}} D^a \Phi'_a + M\ \Phi' = 0\ ,$$

$$\frac{1}{\sqrt{2}}\ D_a \Phi' + \sqrt{\frac{2}{3}}\ D^b \Psi'_{(ba)} + M\ \Phi'_a = 0\ ,$$

$$-\frac{1}{\sqrt{6}}\ \Big(\ D_a \Phi'_b + D_b \Phi'_a - \frac{1}{2}\ g_{ab} D^c \Phi'_c\ \Big)\ + \frac{1}{\sqrt{2}}\ \Big(\ D^c \Phi'_{[bc]a} + D^c \Phi'_{[ac]b}\ \Big)$$
$$+ \frac{\mu}{M}\ \Big\{\ [D^c, D_a]_- \Psi_{(bc)} + [D^c, D_b]_- \Psi'_{(ac)}\ \Big\} + M\ \Psi_{(ab)} = 0\ ,$$
$$\frac{1}{\sqrt{2}}\ (\ D_c \Psi_{(ab)} - D_b \Psi_{(ac)} + \frac{1}{3} g_{ab} D^m \Psi_{(lc)} - \frac{1}{3} g_{ac} D^m \Psi_{(lb)}\ ) + M\ \Phi'_{a[bc]} = 0\ ;$$

Through the simple re-designation of the terms

$$\Phi' = -\bar{\Phi}\ , \qquad \Psi'_a = \sqrt{2}\ \bar{\Phi}_a\ , \qquad \Phi'_{(ab)} = \frac{1}{\sqrt{3}}\ \bar{\Phi}_{(ab)}, \qquad \Phi'_{[bc]a} = \frac{1}{\sqrt{6}}\ \bar{\Phi}_{[bc]a}\ . \tag{2.31}$$

The above systems gets the form:

$$D^a \bar{\Phi}_a - M\ \bar{\Phi} = 0\ , \quad \frac{1}{2}\ D_a \bar{\Phi} - \frac{1}{3}\ D^b \bar{\Psi}_{(ba)} - M\ \bar{\Phi}_a = 0\ ,$$
$$\Big(\ D_a \bar{\Phi}_b + D_b \bar{\Phi}_a - \frac{1}{2}\ g_{ab} D^c \bar{\Phi}_c\ \Big)\ + \frac{1}{2}\ \Big(\ D^c \bar{\Phi}_{[ca]b} + D^c \bar{\Phi}_{[cb]a}\ \Big)$$
$$- \frac{\mu}{M}\Big([D^c, D_a]_-\ \bar{\Phi}_{(bc)} + [D^c, D_b]_-\ \bar{\Phi}_{(ac)}\Big) - M\ \bar{\Phi}_{(ab)} = 0\ , \tag{2.32}$$
$$D_c \bar{\Phi}_{(ba)} - D_b \bar{\Phi}_{(ca)} + \frac{1}{3} g_{ba} D^m \bar{\Psi}_{(mc)} - \frac{1}{3} g_{ca} D^m \bar{\Psi}_{(mb)} - M\ \bar{\Phi}_{[cb]a} = 0\ .$$

Hence, by setting $\mu = 0$, we obtain the following 30-component system

$$
\begin{aligned}
D^a \bar{\Phi}_a - M\, \bar{\Phi} = 0\,, \quad \frac{1}{2}\, D_a \bar{\Phi} - \frac{1}{3}\, D^b \bar{\Psi}_{(ba)} - M\, \bar{\Phi}_a = 0\,, \\
(\, D_a \bar{\Phi}_b + D_b \bar{\Phi}_a - \frac{1}{2}\, g_{ab} D^c \bar{\Phi}_c\,)\; + \frac{1}{2}\,(\, D^c \bar{\Phi}_{[ca]b} + D^c \bar{\Phi}_{[cb]a}\,) - M\, \bar{\Phi}_{(ab)} = 0\,, \\
D_c \bar{\Phi}_{(ba)} - D_b \bar{\Phi}_{(ca)} + \frac{1}{3} g_{ba} D^m \bar{\Psi}_{(mc)} - \frac{1}{3} g_{ca} D^m \bar{\Psi}_{(mb)} - M\, \bar{\Phi}_{[cb]a} = 0\,.
\end{aligned}
\tag{2.33}
$$

Comparing this with the system (2.10), we can see that they differ only in the third equation. It may be readily shown that in fact the systems are equivalent. Indeed, in the fourth equation of the system (2.10), let us perform the convolution in indices a and c, whence we obtain

$$
g^{ac}\, D_a\, \bar{\Phi}_{(bc)} - g^{ac}\, D_b\, \bar{\Phi}_{(ac)} + \frac{1}{3}(g^{ac}\, g_{bc}\, D^k\, \bar{\Phi}_{(ak)} - g^{ac}\, g_{ac}\, D^k\, \bar{\Phi}_{(bk)}) - M\, g^{ac}\, \bar{\Phi}_{[ab]c} = 0\,.
$$

Because $g^{ac}\, \Phi_{(ac)} = 0,\; g^{ac}\, g_{ac} = 4$, we get $\bar{\Phi}^b_{[ab]} = 0$, and therefore the term $\frac{1}{2}\, g_{ab}\, D^c\, \bar{\Phi}^n_{[cn]}$ from the third equation of the system (2.10) vanishes identically. Thus, the systems (2.10) and (2.32) are equivalent.

Let us detail the additional interaction term (see the fourth equation in (2.32))

$$
\Delta_{ab} = \frac{\mu}{M}\Big([D^c, D_a]_- \bar{\Phi}_{(bc)} + [D^c, D_b]_- \bar{\Phi}_{(ac)}\Big). \tag{2.34}
$$

It suffices to consider the first term

$$
[D^c, D_a]_- \bar{\Phi}_{(bc)} = [\nabla_c + ieA_c, \nabla_a + ieA_a]_- \bar{\Phi}_b{}^c = (\nabla_c \nabla_a - \nabla_a \nabla_c)\, \bar{\Phi}_b{}^c + ieF_{ca}\, \bar{\Phi}_b{}^c.
$$

Taking in mind the rule

$$
(\nabla_c \nabla_a - \nabla_a \nabla_c) A_{bk} = -A_{nk}\, R^n{}_{b\; ca} - A_{bn} R^n{}_{k\; ca} \quad \Longrightarrow
$$

$$
(\nabla_c \nabla_a - \nabla_a \nabla_c) A_b{}^c = -A_n{}^c R^n{}_{b\; ca} - A_{bn} R^{nc}{}_{ca},
$$

we find (remembering the symmetry of the curvature tensor)

$$
(\nabla^c \nabla_a - \nabla_a \nabla^c) A_{bc} = R_{ca\; bn} A^{nc} + A_b{}^n R_{na}\,,
$$

and hence the needed commutator takes the form

$$
(\nabla^c \nabla_a - \nabla_a \nabla^c) \bar{\Phi}_{bc} = R_{ca\; bn} \bar{\Phi}^{cn} + R_{ac} \bar{\Phi}^c{}_b\,. \tag{2.35}
$$

Therefore, we have

$$
[D^c, D_a]_- \bar{\Phi}_{(bc)} = ieF_{ca} \bar{\Phi}_b{}^c + R_{ca\; bn} \bar{\Phi}^{cn} + R_{ac} \bar{\Phi}^c{}_b\,.
$$

Adding the symmetric term, we find the needed expression for the above additional interaction term

$$
\begin{aligned}
\Delta_{ab} = \frac{\mu}{M}\Big([D^c, D_a]_- \bar{\Phi}_{(bc)} + [D^c, D_b]_- \bar{\Phi}_{(ac)}\Big) \\
= \frac{\mu}{M}\Big\{ie\,(\bar{\Phi}_a{}^c F_{cb} + \bar{\Phi}_b{}^c F_{ca}) + (R_{ca\; bn} \bar{\Phi}^{cn} + R_{cb\; an} \bar{\Phi}^{cn}) + (R_{ac} \bar{\Phi}^c{}_b + R_{bc} \bar{\Phi}^c{}_a\Big\}\,.
\end{aligned}
\tag{2.36}
$$

This relation means that the parameter μ, which in absence of a curved space time background, was interpreted as related to anomalous magnetic moment of the spin 2 particle, and also determines an additional interaction with the space-time geometry through the Ricci and the Riemann curvature tensors.

3 Massless limit, gauge solutions

The theory under consideration allows for the massless limit the new form (in comparison with the ordinary Pauli – Fierz theory)

$$\begin{gathered} D^a \bar{\Phi}_a = 0 \, , \quad \frac{1}{2} \, D_a \bar{\Phi} - \frac{1}{3} \, D^b \bar{\Psi}_{(ba)} = \bar{\Phi}_a \, , \\ \Big(D_a \bar{\Phi}_b + D_b \bar{\Phi}_a - \frac{1}{2} \, g_{ab} D^c \bar{\Phi}_c \Big) \; + \frac{1}{2} \Big(D^c \bar{\Phi}_{[ca]b} + D^c \bar{\Phi}_{[cb]a} \Big) - \mu \Delta_{ab} = 0 \, , \\ D_c \bar{\Phi}_{(ba)} - D_b \bar{\Phi}_{(ca)} + \frac{1}{3} g_{ba} D^m \bar{\Psi}_{(mc)} - \frac{1}{3} g_{ca} D^m \bar{\Psi}_{(mb)} = \bar{\Phi}_{[cb]a} \, . \end{gathered} \tag{3.1}$$

where

$$\Delta_{ab} = R_{cabn} \bar{\Phi}^{cn} + R_{cban} \bar{\Phi}^{cn} + R_{ac} \bar{\Phi}^c{}_b + R_{bc} \bar{\Phi}^c{}_a \, ; \tag{3.2}$$

dimensions of the involved quantities are as follows

$$[\bar{\Phi}_a] = [\frac{\bar{\Phi}}{L}], \quad [\bar{\Phi}_{ab}] = [\bar{\Phi}], \quad [\bar{\Phi}_{[ab]c}] = [\frac{\bar{\Phi}}{L}], \quad [\mu] = 1. \tag{3.3}$$

These modified equations (3.1) may be of special interest because, as shown in [10], the conventional Pauli – Fierz massless theory does not possess gauge symmetry in the curved space-time, and requires some modification by introducing non-minimal interaction terms by hands. The problem of existence or non-existence of the gauge symmetry in the generalized massless theory still needs additional study. We shall further investigate it again, now starting with the system (3.1).

Because different authors use different conventions on definitions of Riemann and Ricci tensor, we recall the definition applied in the present paper [12]:

$$R^{\alpha}{}_{\beta\rho\sigma} = \frac{\partial \Gamma^{\alpha}_{\beta\sigma}}{\partial x^{\rho}} - \frac{\partial \Gamma^{\alpha}_{\beta\rho}}{\partial x^{\sigma}} + \Gamma^{\alpha}_{\gamma\rho} \Gamma^{\gamma}_{\beta\sigma} - \Gamma^{\alpha}_{\gamma\sigma} \Gamma^{\gamma}_{\beta\sigma}; \tag{3.4}$$

the commutators of covariant derivatives, acting on tensors of 1-st and 2-nd ranks:

$$[\nabla_\rho \nabla_\beta - \nabla_\beta, \nabla_\rho] \, A_\alpha = [\nabla_\rho, \nabla_\beta] \, A_\alpha = R^{\sigma}{}_{\alpha\beta\rho} A_\sigma, \tag{3.5}$$

$$[\nabla_\sigma \nabla_\rho - \nabla_\rho \nabla_\sigma] \, A_{\alpha\beta} = [\nabla_\sigma, \nabla_\rho] \, A_{\alpha\beta} = R^{\gamma}{}_{\beta\rho\sigma} A_{\alpha\gamma} + R^{\gamma}{}_{\alpha\rho\sigma} A_{\gamma\beta}. \tag{3.6}$$

We turn to the minimal system

$$\nabla^\alpha \bar{\Phi}_\alpha = 0, \tag{3.7}$$

$$\frac{1}{2} \nabla_\alpha \bar{\Phi} - \frac{1}{3} \nabla^\beta \bar{\Phi}_{\alpha\beta} = \bar{\Phi}_\alpha, \tag{3.8}$$

$$\frac{1}{2} \Big(\nabla^\mu \bar{\Phi}_{\mu\alpha\beta} + \nabla^\mu \bar{\Phi}_{\mu\beta\alpha} - \frac{1}{2} g_{\alpha\beta} \nabla^\mu \bar{\Phi}_{\mu\sigma}{}^{\sigma} \Big) + \Big(\nabla_\alpha \bar{\Phi}_\beta + \nabla_\beta \bar{\Phi}_\alpha - \frac{1}{2} g_{\alpha\beta} \nabla^\mu \bar{\Phi}_\mu \Big) = 0, \tag{3.9}$$

$$\nabla_\alpha \bar{\Phi}_{\beta\sigma} - \nabla_\beta \bar{\Phi}_{\alpha\sigma} + \frac{1}{3} \Big(g_{\beta\sigma} \nabla^\mu \bar{\Phi}_{\alpha\mu} - g_{\alpha\sigma} \nabla^\mu \bar{\Phi}_{\beta\mu} \Big) = \bar{\Phi}_{\alpha\beta\sigma}. \tag{3.10}$$

As known, the gauge solutions for main constituents are determined by the formulas (which are obtained from those in Minkowski space by the evident covariant extension)

$$\bar{\Phi}^G = \nabla^\beta A_\beta, \quad \bar{\Phi}^G_{\alpha\beta} = \nabla_\alpha A_\beta + \nabla_\beta A_\alpha - \frac{1}{2} g_{\alpha\beta} \nabla^\sigma A_\sigma \tag{3.11}$$

where $A_\beta(x)$ is an arbitrary vector field.

By simple calculations for the concomitant vector gauge component, we obtain the expression

$$\bar{\Phi}^G_\alpha = \frac{1}{2}\Big(\nabla_\alpha \nabla^\beta A_\beta - \nabla^\beta \nabla_\beta A_\alpha - R^\sigma_\alpha A_\sigma\Big). \tag{3.12}$$

The equation $\nabla^\alpha \bar{\Phi}^G_\alpha = 0$ leads to

$$-\frac{2}{3}\nabla^\alpha \left(R_\alpha{}^\sigma A_\sigma\right) = 0. \tag{3.13}$$

This relationship becomes identity $0 \equiv 0$, only if $R_{\alpha\beta}(x) = 0$.

From equation (3.10) we obtain the following expression for the gauge 3-rank tensor

$$\begin{aligned}\bar{\Phi}^G_{\alpha\beta\sigma} &= \nabla_\sigma \left(\nabla_\alpha A_\beta - \nabla_\beta A_\alpha\right) \\ &-\frac{1}{3}\left(g_{\beta\sigma}\nabla_\alpha - g_{\alpha\sigma}\nabla_\beta\right)\nabla^\mu A_\mu \\ &+\frac{1}{3}\left(g_{\beta\sigma}\nabla^\mu \nabla_\mu A_\alpha - g_{\alpha\sigma}\nabla^\mu \nabla_\mu A_\beta\right) \\ &+R^\mu{}_{\sigma\beta\alpha}A_\mu + R^\mu{}_{\beta\sigma\alpha}A_\mu - R^\mu{}_{\alpha\sigma\beta}A_\mu + \\ &+\frac{1}{3}\Big(g_{\beta\sigma}R^\mu_\alpha A_\mu - g_{\alpha\sigma}R^\mu_\beta A_\mu\Big).\end{aligned} \tag{3.14}$$

Now we substitute these expressions (3.12) and (3.14) in eq. (3.9). At this, we will consider separately the part $\bar{\psi}^{\rm geom}_{\alpha\beta\sigma}$ (which contains the terms with Riemann and Ricci tensors), and the part $\bar{\varphi}^{\rm cov}_{\alpha\beta\sigma}$ (which contains the terms with covariant derivatives):

$$\bar{\Phi}^G_{[\alpha\beta]\sigma} = \bar{\varphi}^{\rm cov}_{[\alpha\beta]\sigma} + \bar{\psi}^{\rm geom}_{[\alpha\beta]\sigma}, \tag{3.15}$$

where

$$\begin{aligned}\bar{\varphi}^{\rm cov}_{[\alpha\beta]\sigma} &\equiv \nabla_\sigma \left(\nabla_\alpha A_\beta - \nabla_\beta A_\alpha\right) \\ -\frac{1}{3}\left(g_{\beta\sigma}\nabla_\alpha - g_{\alpha\sigma}\nabla_\beta\right)\nabla^\mu A_\mu &+ \frac{1}{3}\left(g_{\beta\sigma}\nabla^\mu\nabla_\mu A_\alpha - g_{\alpha\sigma}\nabla^\mu\nabla_\mu A_\beta\right), \\ \bar{\psi}^{\rm geom}_{[\alpha\beta]\sigma} &\equiv R^\mu{}_{\sigma\beta\alpha}A_\mu + R^\mu{}_{\beta\sigma\alpha}A_\mu - R^\mu{}_{\alpha\sigma\beta}A_\mu + \frac{1}{3}\left(g_{\beta\sigma}R^\mu_\alpha A_\mu - g_{\alpha\sigma}R^\mu_\beta A_\mu\right).\end{aligned} \tag{3.16}$$

Then eq. (3.9) may be written formally as follows

$$\bar{\varphi}^{\rm cov}_{\alpha\beta} + \bar{\psi}^{\rm geom}_{\alpha\beta} = 0. \tag{3.17}$$

First, let us derive expression for the term $\varphi^{\rm cov}_{\alpha\beta}$. We find (see eq. (3.9))

$$\begin{aligned}\frac{1}{2}\nabla^\mu \bar{\varphi}^{\rm cov}_{\mu\alpha\beta} = \frac{1}{2}\Big[\nabla^\mu\nabla_\beta\Big(\nabla_\mu A_\alpha - \nabla_\mu\nabla_\alpha A_\mu\Big) - \frac{1}{3}g_{\alpha\beta}\Big[\nabla^\sigma\nabla_\sigma, \nabla^\mu\Big]A_\mu + \\ +\frac{1}{3}\Big(\nabla_\beta\nabla_\alpha\nabla^\sigma A_\sigma - \nabla_\beta\nabla^\sigma\nabla_\sigma A_\alpha\big)\Big],\end{aligned} \tag{3.18}$$

$$\begin{aligned}\frac{1}{2}\nabla^\mu \bar{\varphi}^{\rm cov}_{\mu\beta\alpha} = \frac{1}{2}\Big[\nabla^\mu\nabla_\alpha\Big(\nabla_\mu A_\beta - \nabla_\mu\nabla_\beta A_\mu\Big) - \frac{1}{3}g_{\alpha\beta}\Big[\nabla^\sigma\nabla_\sigma, \nabla^\mu\Big]A_\mu \\ +\frac{1}{3}\Big(\nabla_\alpha\nabla_\beta\nabla^\sigma A_\sigma - \nabla_\alpha\nabla^\sigma\nabla_\sigma A_\beta\Big)\Big]\end{aligned} \tag{3.19}$$

By applying a similar decomposition for the vector (3.12), we get

$$\begin{aligned}\left(\nabla_\alpha\bar\varphi^{\rm cov}_\beta+\nabla_\beta\bar\varphi^{\rm cov}_\alpha\ -\frac{1}{2}g_{\alpha\beta}\nabla^\mu\bar\varphi^{\rm cov}_\mu\right)\\ =\frac{1}{3}\Big[\nabla_\alpha\nabla_\beta\nabla^\sigma A_\sigma-\nabla_\alpha\nabla^\sigma\nabla_\sigma A_\beta+\nabla_\beta\nabla_\alpha\nabla^\sigma A_\sigma-\nabla_\beta\nabla^\sigma\nabla_\sigma A_\alpha\\ -\frac{1}{2}g_{\alpha\beta}\Big(\nabla^\mu\nabla_\mu\nabla^\sigma A_\sigma-\nabla^\mu\nabla^\sigma\nabla_\sigma A_\mu\Big)\Big].\end{aligned} \tag{3.20}$$

By summing up (3.18), (3.19) and (3.20), after regrouping the similar terms, for the covariant part in eq. (3.9), we derive the following result

$$\begin{aligned}\frac{1}{2}\Big[\,(\nabla^\sigma\nabla_\beta\nabla_\sigma A_\alpha-\nabla_\beta\nabla^\sigma\nabla_\sigma A_\alpha)+(\nabla^\sigma\nabla_\alpha\nabla_\sigma A_\beta-\nabla_\alpha\nabla^\sigma\nabla_\sigma A_\beta)+\\ +(\nabla_\alpha\nabla_\beta\nabla^\sigma A_\sigma-\nabla^\sigma\nabla_\alpha\nabla_\beta A_\sigma)+(\nabla_\beta\nabla_\alpha\nabla^\sigma A_\sigma-\nabla^\sigma\nabla_\beta\nabla_\alpha A_\sigma)-\\ -g_{\alpha\beta}\,(\nabla^\sigma\nabla_\sigma\nabla^\mu A_\mu-\nabla^\sigma\nabla^\mu\nabla_\mu A_\sigma)\Big]+\bar\psi^{\rm geom}_{\alpha\beta}=0,\end{aligned} \tag{3.21}$$

which transforms to

$$\begin{aligned}\frac{1}{2}\Big(\,[\nabla^\sigma,\nabla_\beta]\,\nabla_\sigma A_\alpha+[\nabla^\sigma,\nabla_\alpha]\,\nabla_\sigma A_\beta+[\nabla_\alpha\nabla_\beta,\nabla^\sigma]\,A_\sigma+[\nabla_\beta\nabla_\alpha,\nabla^\sigma]\,A_\sigma\\ -g_{\alpha\beta}\,[\nabla^\sigma\nabla_\sigma,\nabla^\mu]\,A_\mu\Big)+\bar\psi^{\rm geom}_{\alpha\beta}=0.\end{aligned} \tag{3.22}$$

Further, with the use of the identities

$$\begin{aligned}[\nabla_\alpha\nabla_\beta,\nabla^\sigma]\,A_\sigma&=\nabla_\alpha\,[\nabla_\beta,\nabla^\sigma]\,A_\sigma+[\nabla_\alpha,\nabla^\sigma]\,\nabla_\beta A_\sigma,\\ [\nabla_\beta\nabla_\alpha,\nabla^\sigma]\,A_\sigma&=\nabla_\beta\,[\nabla_\alpha,\nabla^\sigma]\,A_\sigma+[\nabla_\beta,\nabla^\sigma]\,\nabla_\alpha A_\sigma,\\ [\nabla^\sigma\nabla_\sigma,\nabla^\mu]\,A_\mu&=\nabla^\mu\,[\nabla_\mu,\nabla^\sigma]\,A_\sigma+[\nabla^\mu,\nabla^\sigma]\,\nabla_\mu A_\sigma,\end{aligned}$$

we obtain as result of eq. (3.9), the following

$$\begin{aligned}\Big\{\frac{1}{2}\Big(\,[\nabla^\sigma,\nabla_\beta]\,\nabla_\sigma A_\alpha+[\nabla^\sigma,\nabla_\alpha]\,\nabla_\sigma A_\beta\\ +\nabla_\alpha\,[\nabla_\beta,\nabla^\sigma]\,A_\sigma+[\nabla_\alpha,\nabla^\sigma]\,\nabla_\beta A_\sigma+\nabla_\beta\,[\nabla_\alpha,\nabla^\sigma]\,A_\sigma\\ +[\nabla_\beta,\nabla^\sigma]\,\nabla_\alpha A_\sigma+\nabla^\mu\,[\nabla_\mu,\nabla^\sigma]\,A_\sigma+[\nabla^\mu,\nabla^\sigma]\,\nabla_\mu A_\sigma\Big)\Big\}+\bar\psi^{\rm geom}_{\alpha\beta}=0.\end{aligned} \tag{3.23}$$

Taking into account the formulas (3.5), we can present the terms in brackets {...} as geometrical quantities,

$$\begin{aligned}[\nabla^\sigma,\nabla_\beta]\,\nabla_\sigma A_\alpha&=R^\gamma{}_{\alpha\beta}{}^\sigma\nabla_\sigma A_\gamma+R^\gamma{}_{\sigma\beta}{}^\sigma\nabla_\gamma A_\alpha,\\ [\nabla^\sigma,\nabla_\alpha]\,\nabla_\sigma A_\beta&=R^\gamma{}_{\beta\alpha}{}^\sigma\nabla_\sigma A_\gamma+R^\gamma{}_{\sigma\alpha}{}^\sigma\nabla_\gamma A_\beta,\\ \nabla_\beta\,[\nabla_\alpha,\nabla^\sigma]\,A_\sigma&=\nabla_\beta\,(R^\gamma{}_\sigma{}^\sigma{}_\alpha A_\gamma)\,,\\ [\nabla_\beta,\nabla^\sigma]\,\nabla_\alpha A_\sigma&=R^\gamma{}_\sigma{}^\sigma{}_\beta\nabla_\alpha A_\gamma+R^\gamma{}_\alpha{}^\sigma{}_\beta\nabla_\gamma A_\sigma,\\ \nabla_\alpha\,[\nabla_\beta,\nabla^\sigma]\,A_\sigma&=\nabla_\alpha\left(R^\gamma{}_\sigma{}^\sigma{}_\beta A_\gamma\right),\\ [\nabla_\alpha,\nabla^\sigma]\,\nabla_\beta A_\sigma&=R^\gamma{}_\sigma{}^\sigma{}_\alpha\nabla_\beta A_\gamma+R^\gamma{}_\beta{}^\sigma{}_\alpha\nabla_\gamma A_\sigma,\\ \nabla^\mu\,[\nabla_\mu,\nabla^\sigma]\,A_\sigma&=\nabla^\mu\left(R^\gamma{}_\sigma{}^\sigma{}_\mu A_\gamma\right),\\ [\nabla^\mu,\nabla^\sigma]\,\nabla_\mu A_\sigma&=R^\gamma{}_\sigma{}^{\sigma\mu}\nabla_\mu A_\gamma+R^\gamma{}_\mu{}^{\sigma\mu}\nabla\gamma A_\sigma.\end{aligned} \tag{3.24}$$

Then, allowing for the symmetry properties of Riemann and Ricci tensors, we arrive at

$$\begin{aligned}\frac{1}{2}\Big[g_{\alpha\beta}\nabla^\mu\left(R^\sigma_\mu A_\sigma\right)-A_\sigma\left(\nabla_\beta R^\sigma_\alpha+\nabla_\alpha R^\sigma_\beta\right)-R^\sigma_\alpha\left(2\nabla_\beta A_\sigma+\nabla_\mu A_\beta\right)\\ -R^\sigma_\beta\left(2\nabla_\alpha A_\sigma-\nabla_\sigma A_\alpha\right)\Big]+\psi^{geom}_{\alpha\beta}=0.\end{aligned} \tag{3.25}$$

Now, we are to perform similar calculations for the geometric part $\bar{\psi}^{\rm geom}_{\alpha\beta}$.
The summands in the first brackets of (3.9) take the form

$$\frac{1}{2}\nabla^\mu \psi^{geom}_{\mu\alpha\beta} = \frac{1}{2}\Big[\nabla^\mu \left(R^\gamma{}_{\beta\alpha\mu}A_\gamma\right) + \nabla^\mu \left(R^\gamma{}_{\alpha\beta\mu}A_\gamma\right) - \nabla^\mu \left(R^\gamma{}_{\mu\beta\alpha}A_\gamma\right) + \frac{1}{3}\left(g_{\alpha\beta}\nabla^\mu \left(R^\gamma_\mu A_\gamma\right) - \nabla_\beta \left(R^\gamma_\alpha A_\gamma\right)\right)\Big],$$
$$\frac{1}{2}\nabla^\mu \psi^{geom}_{\mu\beta\alpha} = \frac{1}{2}\Big[\nabla^\mu \left(R^\gamma{}_{\alpha\beta\mu}A_\gamma\right) + \nabla^\mu \left(R^\gamma{}_{\beta\alpha\mu}A_\gamma\right) - \nabla^\mu \left(R^\gamma{}_{\mu\alpha\beta}A_\gamma\right) + \frac{1}{3}\left(g_{\alpha\beta}\nabla^\mu \left(R^\gamma_\mu A_\gamma\right) - \nabla_\alpha \left(R^\gamma_\beta A_\gamma\right)\right)\Big]. \tag{3.26}$$

The terms in the second brackets of eq. (3.9) yield

$$\left(\nabla_\alpha\bar{\psi}_\beta + \nabla_\beta\bar{\psi}_\alpha - \frac{1}{2}g_{\alpha\beta}\nabla^\mu\bar{\psi}_\mu\right) = \frac{1}{3}\Big[\frac{1}{2}g_{\alpha\beta}\nabla^\mu \left(R^\sigma_\mu A_\sigma\right) - \nabla_\alpha \left(R^\sigma_\beta A_\sigma\right) - \nabla_\beta \left(R^\sigma_\alpha A_\sigma\right)\Big]. \tag{3.27}$$

Summing the last three expressions, for $\bar{\psi}^{\rm geom}_{\alpha\beta}$, we obtain

$$\bar{\psi}^{\rm geom}_{\alpha\beta} = \nabla^\mu \left(R^\gamma{}_{\beta\alpha\mu}A_\gamma\right) + \nabla^\mu \left(R^\gamma{}_{\alpha\beta\mu}A_\gamma\right) + \frac{1}{2}g_{\alpha\beta}\nabla^\mu \left(R^\gamma_\mu A_\gamma\right) - -\frac{1}{2}\left[\nabla_\alpha \left(R^\gamma_\beta A^\gamma\right) + \nabla_\beta \left(R^\gamma_\alpha A_\gamma\right)\right]. \tag{3.28}$$

Thus, we find the form of (3.9), written with the use of Riemann and Ricci tensors:

$$g_{\alpha\beta}\nabla^\mu \left(R^\sigma_\mu A_\sigma\right) - A_\sigma \left(\nabla_\beta R^\sigma_\alpha + \nabla_\alpha R^\sigma_\beta\right) + \nabla^\mu \left(R^\sigma{}_{\beta\alpha\mu}A_\sigma + R^\sigma{}_{\alpha\beta\mu}A_\sigma\right) -\frac{3}{2}\left(R^\sigma_\alpha\nabla_\beta A_\sigma + R^\sigma - \beta\nabla_\alpha A_\sigma\right) + \frac{1}{2}\left(R^\sigma_\alpha\nabla_\sigma A_\beta + R^\sigma_\beta\nabla_\sigma A_\alpha\right) = 0, \tag{3.29}$$

whence it follows

$$\nabla^\mu \left(R^\sigma{}_{\beta\alpha\mu}A_\sigma + R^\sigma{}_{\alpha\beta\mu}A_\sigma\right) = \nabla^\mu A_\sigma \left(R^\sigma{}_{\beta\alpha\mu} + R^\sigma{}_{\alpha\beta\mu}\right) + A_\sigma \left(\nabla^\mu R^\sigma{}_{\beta\alpha\mu} + \nabla^\mu R^\sigma{}_{\alpha\beta\mu}\right) = \nabla^\mu A_\sigma \left(R^\sigma{}_{\beta\alpha\mu} + R^\sigma{}_{\alpha\beta\mu}\right) + \left\{A^\sigma \left(\nabla_\mu R^\mu{}_{\alpha\beta\sigma} + \nabla_\mu R^\mu{}_{\beta\alpha\sigma}\right)\right\}. \tag{3.30}$$

With the use of the differential Bianchi identity [12]

$$\nabla_\mu R^\mu{}_{\alpha\beta\sigma} + \nabla_\mu R^\mu{}_{\beta\alpha\sigma} = \nabla_\beta R_{\alpha\sigma} + \nabla_\alpha R_{\beta\sigma} - 2\nabla_\sigma R_{\alpha\beta}, \tag{3.31}$$

the previous equation transforms to

$$g_{\alpha\beta}\nabla^\mu \left(R^\sigma_\mu A_\sigma\right) - 2A^\sigma\nabla_\sigma R_{\alpha\beta} + \nabla^\mu A_\sigma \left(R^\sigma{}_{\beta\alpha\mu} + R^\sigma{}_{\alpha\beta\mu}\right) -\frac{3}{2}\left(R^\sigma_\alpha\nabla_\beta A_\sigma + R^\sigma - \beta\nabla_\alpha A_\sigma\right) + \frac{1}{2}\left(R^\sigma_\alpha\nabla_\sigma A_\beta + R^\sigma_\beta\nabla_\sigma A_\alpha\right) = 0; \tag{3.32}$$

this is the result of eq. (3.9) for the gauge solution.
Recall that the modified system (3.1)

$$\nabla^\alpha\bar{\Phi}_\alpha = 0, \qquad \frac{1}{2}\nabla_\alpha\bar{\Phi} - \frac{1}{3}\nabla^\beta\bar{\Phi}_{\alpha\beta} = \bar{\Phi}_\alpha,$$

$$\frac{1}{2}\left(\nabla^\mu\bar{\Phi}_{\mu\alpha\beta} + \nabla^\mu\bar{\Phi}_{\mu\beta\alpha}\right)$$

$$+\left(\nabla_\alpha\bar{\Phi}_\beta+\nabla_\beta\bar{\Phi}_\alpha-\frac{1}{2}g_{\alpha\beta}\nabla^\mu\bar{\Phi}_\mu\right)-\mu\left([\nabla^\mu,\nabla_\alpha]\,\bar{\Phi}_{\beta\mu}+[\nabla^\mu,\nabla_\beta]\,\bar{\Phi}_{\alpha\mu}\right)=0,$$

$$\nabla_\alpha\bar{\Phi}_{\beta\sigma}-\nabla_\beta\bar{\Phi}_{\alpha\sigma}+\frac{1}{3}\left(g_{\beta\sigma}\nabla^\mu\bar{\Phi}_{\alpha\mu}-g_{\alpha\sigma}\nabla^\mu\bar{\Phi}_{\beta\mu}\right)=\bar{\Phi}_{\alpha\beta\sigma}$$

being specified for the gauge solutions differ from the initial one only by the term

$$-\mu\left([\nabla^\mu,\nabla_\alpha]\,\Phi_{\beta\mu}+[\nabla^\mu,\nabla_\beta]\,\Phi_{\alpha\mu}\right),$$

taking in mind that the term $-\frac{1}{2}g_{\alpha\beta}\nabla^\mu\Phi_{\mu\sigma}{}^\sigma$ identically vanishes.

Let us try to find the coefficient μ, in order to get the combination of Ricci and Riemann tensor which cancel the terms with the Riemann tensor in the expression (3.32). Taking into account identities

$$[\nabla^\mu,\nabla_\alpha]\,\Phi_{\beta\mu}+[\nabla^\mu,\nabla_\beta]\,\Phi_{\alpha\mu}=R^\gamma_\alpha\Phi_{\beta\gamma}+R^\gamma_\beta\Phi_{\alpha\gamma}+\left(R^\gamma{}_{\beta\alpha}{}^\mu+R^\gamma{}_{\alpha\beta}{}^\mu\right)\Phi_{\gamma\mu}, \tag{3.33}$$

and

$$\begin{aligned}&R^\gamma_\alpha\Phi_{\beta\gamma}+R^\gamma_\beta\Phi_{\alpha\gamma}\\&=R^\gamma_\alpha\left(\nabla_\beta A_\gamma+\nabla_\gamma A_\beta\right)+R^\gamma_\beta\left(\nabla_\alpha A_\gamma+\nabla_\gamma A_\alpha\right)-R_{\alpha\beta}\nabla^\sigma A_\sigma,\end{aligned} \tag{3.34}$$

$$\left(R^\gamma{}_{\beta\alpha}{}^\mu+R^\gamma{}_{\alpha\beta}{}^\mu\right)\Phi_{\gamma\mu}=2\nabla_\mu A_\sigma\left(R^\mu{}_{\beta\alpha}{}^\sigma+R^\mu{}_{\alpha\beta}{}^\sigma\right)+R_{\alpha\beta}\nabla^\sigma A_\sigma, \tag{3.35}$$

we obtain

$$\begin{aligned}[\nabla^\mu,\nabla_\alpha]\,\Phi_{\beta\mu}+[\nabla^\mu,\nabla_\beta]\,\Phi_{\alpha\mu}=2\nabla_\mu A_\sigma\left(R^\mu{}_{\beta\alpha}{}^\sigma+R^\mu{}_{\alpha\beta}{}^\sigma\right)+R_{\alpha\beta}\nabla^\sigma A_\sigma\\+R^\gamma_\alpha\left(\nabla_\beta A_\gamma+\nabla_\gamma A_\beta\right)+R^\gamma_\beta\left(\nabla_\alpha A_\gamma+\nabla_\gamma A_\alpha\right)-R_{\alpha\beta}\nabla^\sigma A_\sigma\end{aligned} \tag{3.36}$$

Comparing (3.32), we conclude that the coefficient μ should be taken as $\mu=1/2$. With such a choice of μ, the equation (3.9) for gauge solutions takes the form

$$g_{\alpha\beta}\nabla^\mu\left(R^\sigma_\mu A_\sigma\right)-2A^\sigma\nabla_\sigma R_{\alpha\beta}-2\left(R^\sigma_\alpha\nabla_\beta A_\sigma+R^\sigma_\beta\nabla_\alpha A_\sigma\right)=0. \tag{3.37}$$

We immediately conclude that for the whole space-time model with vanishing Ricci tensor, the system of massless equations (3.1) permits existence of the gauge solution - in other words, the system possesses the gauge symmetry.

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