HYDROGEN ATOM IN SPHERICAL SPACE, DIRAC THEORY, EXACT SOLUTIONS AND ENERGY SPECTRUM

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The known systems of radial equations describing relativistic hydrogen atom on the base of Dirac equation in spherical Riemann space is investigated. The relevant 2-nd order differential equation has six regular singular points, its solutions of Frobenius type are constructed. To produce the quantization rule for energy values we use the known condition separating transcendental Frobenius solutions. Convergence of the involved series is studied analytically. Squared integrability of solutions is demonstrated numerically.

The Hydrogen atom is spherical Riemann space

In Riemannian spherical space S parameterized by the coordinates

$$dS^2 = dt^2 - dr^2 - \sin^2 r(d\theta^2 + \sin^2 d\phi^2),\ \ r \in (0, \pi), \tag{1}$$

we have the following system of differential equations:

$$\begin{aligned} \left(\frac{d}{dr} + \frac{\nu}{\sin r}\right) f + \left(E + \frac{e}{\tan r} + m\right) g = 0, \\ \left(\frac{d}{dr} - \frac{\nu}{\sin r}\right) g - \left(E + \frac{e}{\tan r} - m\right) f = 0; \end{aligned} \tag{2}$$

the dimensionless radial coordinate varies in the interval r∈ [0,π].

In the other variables

$$z = i \tan\frac{r}{2}, \quad \cos r = \frac{1+z^2}{1-z^2}, \quad \sin r = \frac{-2iz}{1-z^2}, \quad z \in [0, +i\infty) \tag{3}$$

the above system takes the form

$$\begin{aligned} \frac{df}{dz} + \frac{\nu}{z} f + \left(\frac{e}{z} + \frac{iE - e + im}{z-1} + \frac{-iE - e - im}{z+1}\right) g = 0\,, \\ \frac{dg}{dz} - \frac{\nu}{z} g + \left(-\frac{e}{z} + \frac{-iE + e + im}{z-1} + \frac{iE + e - im}{z+1}\right) f = 0\,, \end{aligned} \tag{4}$$

whence there follows the 2-nd order equation for f(z):

$$\frac{d^2f}{dz^2}+\left[\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z+1}+2\,\frac{-ez+iE+im}{ez^2-2i\left(E+m\right)z+e}\right]\frac{df}{dz}+$$
$$+\left[-2i\,\frac{2\,Ee^2-\left(E+m\right)\nu}{ez}+\frac{e^2-\nu^2}{z^2}+\frac{\left(E+ie\right)^2-m^2+\nu}{z-1}+\right.$$
$$+\frac{-\left(E+ie\right)^2+m^2}{\left(z-1\right)^2}+\frac{-\left(E-ie\right)^2+m^2-\nu}{z+1}+\frac{-\left(E-ie\right)^2+m^2}{\left(z+1\right)^2}+$$
$$\left.+\frac{2\nu\left[iez(E+m)+2(E+m)^2+e^2\right]}{e\left[-ez^2+2i\left(E+m\right)z-e\right]}\right]f=0. \tag{5}$$

Equation (5) has 6 singular points (let $\frac{E+m}{e}=\sigma>0$**)**

$$0\;,\;\;\infty\;,\;\;\pm1,\;\;z_{1,\,2}=i\left(\sigma\pm\sqrt{\sigma^2+1}\right)\;; \tag{6}$$

the physical region for the variable z is the interval z ∈ [0,+i∞). Equation (5) may be written differently

$$\frac{d^2f}{dz^2}+\left[\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z+1}-\frac{1}{z-z_1}-\frac{1}{z-z_2}\right]\frac{df}{dz}+$$
$$+\left[\frac{-4\,iEe+2\,i\sigma\,\nu}{z}+\frac{e^2-\nu^2}{z^2}+\frac{\left(E+ie\right)^2-m^2+\nu}{z-1}+\frac{-\left(E+ie\right)^2+m^2}{\left(z-1\right)^2}+\right.$$
$$\left.+\frac{-\left(E-ie\right)^2+m^2-\nu}{z+1}+\frac{-\left(E-ie\right)^2+m^2}{\left(z+1\right)^2}+\frac{A}{z-z_1}+\frac{B}{z-z_2}\right]f=0\,, \tag{7}$$

where

$$A = -\frac{2\nu\,\left(iz_1\,\sigma + 1 + 2\,\sigma^2\right)}{z_1 - z_2}, \qquad B = -\frac{2\nu\,\left(iz_2\,\sigma + 1 + 2\,\sigma^2\right)}{z_2 - z_1},$$

$$C = -\left(E + ie\right)^2 + m^2, \qquad D = -\left(E - ie\right)^2 + m^2 \,.$$

Then eq. (7) takes the form

$$\frac{d^2 f}{dz^2} + \left[\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z+1} - \frac{1}{z-z_1} - \frac{1}{z-z_2}\right]\frac{df}{dz} +$$
$$+ \left[\frac{C - D + 2\,i\sigma\,\nu}{z} + \frac{e^2 - \nu^2}{z^2} - \frac{C-\nu}{z-1} + \frac{C}{\left(z-1\right)^2} +\right.$$
$$\left. + \frac{D-\nu}{z+1} + \frac{D}{\left(z+1\right)^2} + \frac{A}{z-z_1} + \frac{B}{z-z_2}\right] f = 0\,.$$

The Frobenius solutions in the vicinity of the point z =0 are searched in the form

$$f(z) = z^M (z-1)^\alpha (z+1)^\beta \; g(z) = \varphi(z)\; g(z)\,.$$

We impose the following restrictions:

$$M = \pm\sqrt{\nu^2 - e^2}\,,$$
$$\alpha = \pm\sqrt{-C} = \pm\sqrt{\left(E + ie\right)^2 - m^2} = \pm\sqrt{E^2 - m^2 - e^2 + 2ieE}\,,$$
$$\beta = \pm\sqrt{-D} = \pm\sqrt{\left(E - ie\right)^2 - m^2} = \pm\sqrt{E^2 - m^2 - e^2 - 2ieE}\,. \qquad (8)$$

To have solutions vanishing at the point $z = 0$ $(r = 0)$, **we must use positive value for** M: $M = +\sqrt{\nu^2 - e^2}$; **near the point** $z = +\infty$ $(r = \pi)$ **the multiplier** φ **before** $g(z)$ **behaves as follows**

$$\varphi = z^M (z-1)^\alpha (z+1)^\beta \sim x^{\sqrt{\nu^2 - e^2} + (\alpha+\beta)}; \tag{9}$$

depending on signs at α, β, there exist 4 possibilities:

$$(-,-) \qquad \alpha + \beta = -\sqrt{E^2 - m^2 - e^2 + 2ieE} - \sqrt{E^2 - m^2 - e^2 - 2ieE} < 0 \ ;$$

$$(+,+) \qquad \alpha + \beta = \sqrt{E^2 - m^2 - e^2 + 2ieE} + \sqrt{E^2 - m^2 - e^2 - 2ieE} > 0 \ ;$$

$$(+,-) \qquad \alpha + \beta = \sqrt{E^2 - m^2 - e^2 + 2ieE} - \sqrt{E^2 - m^2 - e^2 - 2ieE} \ \ \textbf{imaginary} \ ;$$

$$(-,+) \quad \alpha + \beta = -\sqrt{E^2 - m^2 - e^2 + 2ieE} + \sqrt{E^2 - m^2 - e^2 - 2ieE} \ \ \textbf{imaginary} \ ;$$

the only two first variants may give multiplier tending to zero, this requires the inequality M + α + β < 0, which is definitely true for the case (-,-).

Now we turn to our equation for g(z):

$$\frac{d^2g}{dz^2}+\left(\frac{P_1}{z}+\frac{P_2}{z-1}+\frac{P_3}{z+1}-\frac{1}{z-z_1}-\frac{1}{z-z_2}\right)\frac{dg}{dz}+$$
$$+\left(\frac{Q_1}{z}+\frac{Q_2}{z-1}+\frac{Q_3}{z+1}+\frac{Q_4}{z-z_1}+\frac{Q_5}{z-z_2}\right)g=0\,.$$

Solutions for $g(z)$ are constructed as power series with 6-term recurrent relations

$$k\geq 4,\qquad (Q_1+Q_2+Q_3+Q_4+Q_5)\;d_{k-4}+$$
$$+\,[(k-3)(k-4)+(P_1+P_2+P_3-2)\;(k-3)+$$
$$+\,(-Q_1-Q_2-Q_3-Q_5)\,z_1+(-Q_1-Q_2-Q_3-Q_4)\,z_2+Q_2-Q_3]\;d_{k-3}+$$
$$+\,[\{(1-P_1-P_2-P_3)\,z_1+(1-P_1-P_2-P_3)\,z_2+P_2-P_3\}\;(k-2)+$$
$$+\,(-z_1-z_2)\;(k-2)(k-3)+Q_3\,z_1\,z_2+Q_2\,z_1\,z_2+Q_3\,z_2-$$
$$-Q_1-Q_4-Q_5+Q_1\,z_1\,z_2+Q_3\,z_1-Q_2\,z_2-Q_2\,z_1]\;d_{k-2}+$$
$$+\,[(2-P_1-P_2z_1+P_3z_2+P_2z_1z_2+P_1z_1z_2+P_3z_1z_2+P_3z_1-P_2z_2)\,(k-1)+$$
$$+\,(z_1z_2-1)\,(k-1)(k-2)+Q_1\,z_2+Q_2\,z_1\,z_2+Q_5\,z_1+Q_1\,z_1-Q_3\,z_1\,z_2+Q_4\,z_2]\;d_{k-1}+$$
$$+\,[(z_1+z_2)\;k\,(k-1)-Q_1\,z_1\,z_2+(-z_1-z_2+P_1\,z_1-P_3\,z_1\,z_2+P_1\,z_2+P_2\,z_1\,z_2)\,k]\;d_k+$$
$$+\,[-z_1\,z_2\,(k+1)\,k-P_1\,z_1\,z_2\,(k+1)]\;d_{k+1}=0\,.\quad(10)$$

Possible convergence radii are

$$R_{\text{conv}} = \left|\frac{1}{R}\right| = +1\,,\ +\infty\,,\ |z_1|\,,\ |z_2|. \tag{11}$$

It is readily checked that the coefficient at d_{k-4} (10) vanishes identically, so in (10) we have 5-term recurrent relations

$$k \geq 4, \quad S_{k-3}d_{k-3} + S_{k-2}d_{k-2} + S_{k-1}d_{k-1} + S_k d_k + S_{k+1}d_{k+1} = 0\ . \tag{12}$$

As quantization rule, we apply the known transcendency condition

$$k \geq 3, \quad S_{k-3} = 0, \qquad (k-3)(k-4) + (P_1 + P_2 + P_3 - 2)\ (k-3) +$$
$$+ (-Q_1 - Q_2 - Q_3 - Q_5)\, z_1 + (-Q_1 - Q_2 - Q_3 - Q_4)\, z_2 + Q_2 - Q_3 = 0\,, \tag{13}$$

which gives

$$k^2 + 2\,k\,(M + \alpha + \beta - 3) - 2\,i\sigma\,\nu\ (z_1 + z_2) + 2\ (M + \beta - 3)\,\alpha + 2\ (M - 3)\,\beta +$$
$$+9 - 6\,M - 2\,m^2 - 4\,\nu\,\sigma^2 - 2\,e^2 + 2\,E^2 = 0\,. \tag{14}$$

We will follow two possibilities. The first one is

$$M = \sqrt{\nu^2 - e^2}\,, \quad \alpha = +\sqrt{(E + ie)^2 - m^2}\,, \quad \beta = +\sqrt{(E - ie)^2 - m^2}\,, \tag{15}$$

and then eq. (14) takes the form

$$\left(\sqrt{(E+ie)^2-m^2}+\sqrt{(E-ie)^2-m^2}+k-3\right)\times$$
$$\times\left(\sqrt{(E+ie)^2-m^2}+\sqrt{(E-ie)^2-m^2}+k-3+2\sqrt{\nu^2-e^2}\right)=0\,.$$

Here there arise two equations which both are of small physical interest (let $n=k-3$):

$$\sqrt{(E+ie)^2-m^2}+\sqrt{(E-ie)^2-m^2}+n=0\,,$$
$$\sqrt{(E+ie)^2-m^2}+\sqrt{(E-ie)^2-m^2}+n+2\sqrt{\nu^2-e^2}=0\,.$$

Now consider the second variant

$$M=\sqrt{\nu^2-e^2}\,,\quad \alpha=-\sqrt{(E+ie)^2-m^2}\,,\quad \beta=-\sqrt{(E-ie)^2-m^2}\,, \tag{16}$$

and then we have the transcendency condition in the form

$$\times\left(\sqrt{(E+ie)^2-m^2}+\sqrt{(E-ie)^2-m^2}-(k-3)\right)\times$$
$$\times\left(\sqrt{(E+ie)^2-m^2}+\sqrt{(E-ie)^2-m^2}-(k-3)-2\sqrt{\nu^2-e^2}\right)=0\,.$$

So we obtain two alternative equations:

$$\sqrt{(E+ie)^2-m^2}+\sqrt{(E-ie)^2-m^2}-n=0\,, \tag{17}$$
$$\sqrt{(E+ie)^2-m^2}+\sqrt{(E-ie)^2-m^2}-n-2\sqrt{\nu^2-e^2}=0\,. \tag{18}$$

Only second one (18) is interesting; it yields:

$$N \sqrt{E^2 - m^2 - e^2 - 2\, iEe} = -iEe + N^2\,.$$

Further, we obtain

$$N^2(E^2 - m^2 - e^2 - 2ieE) = N^4 - 2ieEN^2 - e^2E^2,$$

whence there follows the needed energy spectrum:

$$E = m\sqrt{\frac{1 + (e^2 + N^2)/m^2}{1 + e^2/N^2}}, \qquad N = \frac{n}{2} + \sqrt{\nu^2 - e^2}, \qquad m = \frac{Mc\rho}{\hbar}\,. \tag{19}$$

This spectrum coincides with that found in [1], when studying the same problem for Dirac equation in Riemann space within the semi-classical approach. We have shown that there does not exist the possibility to get the energy spectrum (19) as polynomials.

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