On the matrix equation for a spin 2 particle in pseudo-Riemannian space-time. II. Separating the variables in spherical coordinates

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In the present study we develop the theory of the massive spin 2 field, extended to the generally covariant theory within the Tetrode-Weyl-Fock-Ivanenko tetrad method.

Such an equation is specified in spherical coordinates of the Minkowski space. We separate the variables by diagonalizing the square and the third projection of the total angular momentum, at this the formalism of Wigner D-function is applied instead of spin-weight harmonics.

As a result, we derive the radial system of differential equations of the first order. From these we derive the 2-nd order radial equation for components referring to symmetric tensor of and scalar involved in description of the spin 2 field.

The radial system is divided into two more simple subsystem which describe states with opposite spacial parities. We find in closed form some exact solutions for such subsystems. The restriction in radial equation to the massless spin 2 field is possible.

The extension of the developed procedure of separating the variables to arbitrary space-time model with spherical symmetry does not require new ideas.

Key words: Spin 2 field; matrix approach; spherical coordinates; total angular momentum, Wigner D-functions; spatial reflection, separation of the variables, exact solutions.

1 Introduction

After the investigation by Pauli and Fierz [11], [23], the theory of massive and massless fields with spin 2 has always attracted much attention ([1],[3]-[10],[12]-[16], [19]-[21],[25]-[26]; also see [2], [18]). Most of the studies were performed in the framework of 2-nd order differential equations. It is known that many specific difficulties may be avoided if from the very beginning we start with 1-st order systems. Apparently, the first systematic study of the theory of spin 2 fields within that formalism was performed by F.I. Fedorov [7]. It turns out that this description requires a field function with 30 independent components. This theory was re-discovered by Regee in [25].

2 Basic equation in spherical coordinates

We will consider the matrix equation [28] for spin 2 field in spherical coordinates, $x^\alpha = (t, r, \theta, \phi)$ and tetrad $e^\beta_{(b)}$ in Minkowski space,

$$dS^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \tag{2.1}$$

The basic equation takes the form (see [28])

$$\Big\{ \Gamma^0 \partial_t + \Big(\Gamma^3 \partial_r + \frac{\Gamma^1 J^{31} + \Gamma^2 J^{32}}{r} \Big) + \frac{1}{r} \Sigma_{\theta,\phi} - m \Big\} \Psi = 0\,,$$
$$\Sigma_{\theta,\phi} = \Gamma^1 \partial_\theta + \Gamma^2 \frac{\partial_\phi + J^{12} \cos\theta}{\sin\theta}\,, \qquad \Psi = \left| \begin{array}{c|c} H & 1 \\ H_1 & 4 \\ H_2 & 10 \\ H_3 & 24 \end{array} \right| . \tag{2.2}$$

3 Separation of variables

We will construct solutions which satisfy four groups of equations

$$\begin{aligned} \mathbf{J}^2 H = j(j+1) H, &\quad J_3 H = m H; \\ \mathbf{J}^2_{(1)} H_1 = j(j+1) H_1, &\quad J^{(1)}_3 H_1 = m H_1; \\ \mathbf{J}^2_{(2)} H_2 = j(j+1) H_2, &\quad J^{(2)}_3 H_2 = m H_2; \\ \mathbf{J}^2_{(3)} H_3 = j(j+1) H_3, &\quad J^{(3)}_3 H_2 = m H_3. \end{aligned} \tag{3.1}$$

We will apply the Wigner D-functions technique [28]

$$D^j_{-m,-s_3}(\phi, \theta, 0) = D_{-s_3}; \quad J = 0, 1, 2, 3, ...$$

We take into account the known recurrent relations [27].

For a correct choice of the substitution for Ψ (see the general approach in [28]) we determine the eigenvalues of the third projection of the spin operators:

$$S^{(1)}_3 H_1 = \sigma H_1, \quad S^{(2)}_3 H_2 = \sigma H_2 \quad S^{(3)}_3 H_3 = \sigma H_3, \tag{3.2}$$

where H_1 corresponds to the vector (4 components), H_2 corresponds to the symmetric tensor (10 components), and H_3 refers to 3-rank tensor (24 components).

These eigenvalues are determined by the known diagonal structure of these operators in cyclic basis [29].

We shall use the substitutions

$$
H = h(r)D_0, \quad H_1 = \begin{vmatrix} h_0(r)\ D_0 \\ h_1(r)\ D_{-1} \\ h_2(r)\ D_0 \\ h_3(r)\ D_{+1} \end{vmatrix}, \quad H_2 = \begin{vmatrix} f_1(r)\ D_{-2} \\ f_2(r)\ D_0 \\ f_3(r)\ D_{+2} \\ c_1(r)\ D_{+1} \\ c_2(r)\ D_0 \\ c_3(r)\ D_{-1} \\ d_1(r)\ D_{-1} \\ d_2(r)\ D_0 \\ d_3(r)\ D_{+1} \\ f_0(r)\ D_0 \end{vmatrix},
$$

$$
\varphi_0 = \begin{vmatrix} E_{10}(r)D_{-1} \\ E_{20}(r)D_0 \\ E_{30}(r)D_{+1} \\ B_{10}(r)D_{+1} \\ B_{20}(r)D_0 \\ B_{30}(r)D_{-1} \end{vmatrix}, \quad \varphi_1 = \begin{vmatrix} E_{11}(r)D_{-2} \\ E_{21}(r)D_{-1} \\ E_{31}(r)D_0 \\ B_{11}(r)D_0 \\ B_{21}(r)D_{-1} \\ B_{31}(r)D_{-2} \end{vmatrix},
$$

$$
\varphi_2 = \begin{vmatrix} E_{12}(r)D_{-1} \\ E_{22}(r)D_0 \\ E_{32}(r)D_{+1} \\ B_{12}(r)D_{+1} \\ B_{22}(r)D_0 \\ B_{32}(r)D_{-1} \end{vmatrix}, \quad \varphi_3 = \begin{vmatrix} E_{13}(r)D_0 \\ E_{23}(r)D_{+1} \\ E_{33}(r)D_{+2} \\ B_{13}(r)D_{+2} \\ B_{23}(r)D_{+1} \\ B_{33}(r)D_0 \end{vmatrix}
$$

where the common multiplier $e^{-i\epsilon t}$ is omitted

After rather long calculations, we derive the system of radial equations

$$
H \qquad - i\epsilon h_0 - 2\frac{1}{r}h_2 - h_2' - \frac{a}{r\sqrt{2}}(h_1 + h_3) = mh;
$$

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$$
\frac{1}{6r}\left(-3ir\epsilon h + \sqrt{2}ad_1 + \sqrt{2}bd_3 + 4d_2 + 2ir\epsilon f_0 + 2rd_2'\right) = mh_0,
$$

$$
\frac{1}{6r}\left(\sqrt{2}a\,(c_1 + c_3) + 4c_2 + 2ir\epsilon d_2 + 4f_2 + 3rh' + 2rf_2'\right) = mh_2,
$$

$$
\frac{1}{12r}\left[\sqrt{2}a\,(2c_2 + 3h) + 2\left(\sqrt{2}bf_1 + 6c_3 + 2r\,(i\epsilon d_1 + c_3')\right)\right] = mh_1,
$$

$$
\frac{1}{12r}\left[\sqrt{2}a\,(2c_2 - 3h) + 2\left(\sqrt{2}bf_3 + 6c_1 + 2r\,(i\epsilon d_3 + c_1')\right)\right] = mh_3;
$$

– –

$$-\frac{1}{2r}\left(\sqrt{2}b\left(B_{21}+2h_1\right)-2\left(B_{31}+r\left(B'_{31}-i\epsilon E_{11}\right)\right)\right)=mf_1,$$

$$\frac{1}{2r}\left(\sqrt{2}b\left(B_{23}-2h_3\right)-2(B_{13}+r\left(B'_{13}+i\epsilon E_{33}\right))\right)=mf_3,$$

$$\frac{1}{8r}\Big[4B_{11}+\sqrt{2}a(3B_{12}+B_{21}-B_{23}-3B_{32}+E_{10}+E_{30}-2(h_1+h_3))$$

$$-2(2B_{33}-2E_{20}+4h_2+r(i\epsilon(E_{13}+3E_{22}+E_{31}+2h_0)+$$

$$+B'_{11}-B'_{33}-E'_{20}-6h'_2))\Big]=mf_2,$$

$- -$

$$\frac{1}{4r}\Big(\sqrt{2}bB_{13}+\sqrt{2}a(B_{22}-B_{33}-2h_2)+$$

$$+2(B_{23}-2h_3+r(-i\epsilon(E_{23}+E_{32})-B'_{12}+2h'_3)))\Big)=mc_1,$$

$$\frac{1}{4r}\Big(-\sqrt{2}bB_{31}+\sqrt{2}a(-B_{22}+B_{11}-2h_2)+2(-B_{21}-2h_1+$$

$$+r(-i\epsilon(E_{12}+E_{21})+B'_{32}+2h'_1)\Big)=mc_3,$$

$$-\frac{1}{8r}\Big(\sqrt{2}a(-B_{12}-B_{21}+B_{23}+B_{32}+E_{10}+E_{30}+2(h_1+h_3))+2(2(E_{20}+2h_2)+$$

$$+r(i\epsilon(E_{13}+E_{22}+E_{31}-2h_0)+B'_{11}-B'_{33}+E'_{20}-2h'_2))\Big)mc_2,$$

$- -$

$$\frac{1}{4r}\Big(2B_{30}+\sqrt{2}bE_{11}+2E_{12}+4E_{21}+\sqrt{2}a(-B_{20}+E_{31}-2h_0)+$$

$$+2r(-i\epsilon(E_{10}+2h_1)+B'_{30}+E'_{21})\Big)=md_1,$$

$$\frac{1}{4r}\Big(-2B_{10}+4E_{23}+2E_{32}+\sqrt{2}bE_{33}+\sqrt{2}a(B_{20}+E_{13}-2h_0)+$$

$$+2r(-i\epsilon(E_{30}+2h_3)-B'_{10}+E'_{23})\Big)=md_3,$$

$$\frac{1}{4r}\Big(\sqrt{2}a(B_{10}-B_{30}+E_{12}+E_{32})+2(E_{13}+2E_{22}+E_{31}+$$

$$+r(-i\epsilon(E_{20}+2h_2)+E'_{22}+2h'_0))\Big)=md_2,$$

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$$\frac{1}{8r}\Big(4B_{11}+\sqrt{2}a(B_{12}-B_{21}+B_{23}-B_{32}+3E_{10}+3E_{30}+2(h_1+h_3))+$$
$$+2(-2B_{33}+6E_{20}+4h_2+$$
$$+r(i\epsilon(E_{13}-E_{22}+E_{31}-6h_0)+B'_{11}-B'_{33}+3E'_{20}+2h'_2))\Big)=mf_0;$$

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and yet the 24 equations

$$\frac{1}{6r}\left(6c_3-4ir\epsilon d_1+\sqrt{2}a\left(c_2+3f_0\right)+\sqrt{2}bf_1+2rc'_3\right)=mE_{10},$$
$$\frac{1}{6r}\left(6c_1-4ir\epsilon d_3+\sqrt{2}a\left(c_2+3f_0\right)+\sqrt{2}bf_3+2rc'_1\right)=mE_{30},$$
$$(\frac{ad_2}{\sqrt{2}r}+\frac{d_3}{r}+d'_3)=mB_{10},-(\frac{ad_2}{\sqrt{2}r}+\frac{d_1}{r}+d'_1)=mB_{30},$$
$$\frac{1}{6r}\left(4c_2+\sqrt{2}a\left(c_1+c_3\right)+4f_2+2r\left(-2i\epsilon d_2-3f'_0+f'_2\right)\right)=mmE_{20},$$
$$\frac{a}{\sqrt{2}r}\left(d_3-d_1\right)=mB_{20},$$
$$\left(-i\epsilon c_3+\frac{d_1}{r}+\frac{ad_2}{\sqrt{2}r}\right)=mE_{12},\quad\left(-i\epsilon c_1+\frac{d_3}{r}+\frac{ad_2}{\sqrt{2}r}\right)=mE_{32},$$
$$\frac{1}{6r}\left(4d_2+\sqrt{2}a\left(d_1+d_3\right)+2ir\epsilon\left(f_0-3f_2\right)-4rd'_2\right)=mE_{22},$$
$$\frac{a}{\sqrt{2}r}\left(c_1-c_3\right)=mB_{22},$$
$$-\frac{1}{6r}\left(-6c_1+2ir\epsilon d_3+\sqrt{2}a\left(c_2-3f_2\right)+\sqrt{2}bf_3-4rc'_1\right)=mB_{12},$$
$$\frac{1}{6r}\left(-6c_3+2ir\epsilon d_1+\sqrt{2}a\left(c_2-3f_2\right)+\sqrt{2}bf_1-4rc'_3\right)=mB_{32},$$
$$(\frac{bd_1}{\sqrt{2}r}-i\epsilon f_1)=mE_{11},\quad(\frac{bd_3}{\sqrt{2}r}-i\epsilon f_3)=mE_{33},$$
$$(-i\epsilon c_3-d'_1)=mE_{21},\quad(-i\epsilon c_1-d'_3)=mE_{23},$$
$$\frac{1}{6r}\left(2d_2+\sqrt{2}a\left(2d_1-d_3\right)-2ir\epsilon\left(3c_2+f_0\right)-2rd'_2\right)=mmE_{31},$$
$$\frac{1}{6r}\left(2d_2+\sqrt{2}a\left(2d_3-d_1\right)-2ir\epsilon\left(3c_2+f_0\right)-2rd'_2\right)=mE_{13},$$
$$\frac{1}{3r}\left(2\sqrt{2}ac_2+ir\epsilon d_1-\sqrt{2}bf_1+rc'_3\right)=mB_{21},$$

$$-\frac{1}{3r}\left(2\sqrt{2}ac_2 + ir\epsilon d_3 - \sqrt{2}bf_3 + rc_1'\right) = mB_{23},$$

$$-(\frac{bc_3}{\sqrt{2}r} + \frac{f_1}{r} + f_1') = mB_{31}, \quad (\frac{bc_1}{\sqrt{2}r} + \frac{f_3}{r} + f_3') = mB_{13},$$

$$\frac{1}{6r}\left(2\left(c_2 - ir\epsilon d_2 + f_2 + 3rc_2' - rf_2'\right) - \sqrt{2}a\left(c_1 - 2c_3\right)\right) = mB_{11},$$

$$-\frac{1}{6r}\left(2\left(c_2 - ir\epsilon d_2 + f_2 + 3rc_2' - rf_2'\right) - \sqrt{2}a\left(c_3 - 2c_1\right)\right) = mB_{33}.$$

4 The diagonalisation of the space reflection operator

As known, the diagonalisation of space reflection operator permits to reduce the general radial system to simpler subsystems.

After rather long calculations we derive the following constraints for states with different parities:

$P = -\alpha = (-1)^{j+1}$,

$$h = 0, \quad h_0 = 0, \; h_2 = 0, \; h_3 = -h_1,$$

$$f_0 = 0, f_2 = 0, c_2 = 0, d_2 = 0, \; f_3 = -f_1, c_3 = -c_1, d_3 = -d_1,$$

$$E_{30} = -E_{10}, \; E_{20} = 0, \; B_{30} = +B_{10},$$

$$E_{32} = -E_{12}, \; E_{22} = 0, \; B_{32} = +B_{12},$$

$$E_{13} = -E_{31}, \; E_{23} = -E_{21}, \; E_{33} = -E_{11},$$

$$B_{13} = +B_{31}, \; B_{23} = +B_{21}, \; B_{33} = +B_{11};$$

$P = +\alpha = (-1)^{j}$,

$$h_3 = +h_1, \quad f_3 = +f_1, \quad c_3 = +c_1, \quad d_3 = +d_1,$$

$$B_{30} = -B_{10}, \quad B_{20} = 0, \quad E_{30} = +E_{10},$$

$$B_{32} = -B_{12}, \quad B_{22} = 0, \quad E_{32} = +E_{12},$$

$$B_{13} = -B_{31}, \quad B_{23} = -B_{21}, \quad B_{33} = -B_{11},$$

$$E_{13} = +E_{31}, \quad E_{23} = +E_{21}, \quad E_{33} = +E_{11}.$$

Let us turn back to the complete radial system and take into account the restrictions related to parity. We have proved that the above parity restrictions are consistent with the radial system, we derive two independent subsystems. All the details are omitted.

5 Second order radial equations

In accordance with the Pauli – Fierz approach, let us eliminate the variables related to tensors of the first and third ranks. In this way, we derive two different 2-nd order systems:

$P = -\alpha = (-1)^{j+1}$,

$$-\epsilon^2 f_1 r^2 - f_1'' r^2 + \sqrt{2} b c_1' r - 2 f_1' r + \sqrt{2} b c_1 = m^2 r^2 f_1 ,$$

$$-\frac{b f_1' r}{\sqrt{2}} + \frac{1}{4}\left(3a^2 + b^2 - 4r^2\epsilon^2 - 6\right) c_1 = m^2 r^2 c_1 ,$$

$$i\epsilon c_1' r^2 + 3i\epsilon c_1 r - \frac{ib\epsilon f_1 r}{\sqrt{2}} = 0, \quad d_1 = 0.$$

We may note that this system cannot describe the states with $j = 0$, because all the equations take the form of identity $0 \equiv 0$. By eliminating the variables c_1, c_1', we derive an equation for f_1:

$$f_1'' + \frac{1}{r}\big(2 + \frac{8b^2}{3a^2 + b^2 - 2[2r^2(m^2 + \epsilon^2) + 3]}\big) f_1' +$$

$$+\big(\epsilon^2 + m^2 - \frac{b^2}{r^2}\big) f_1 = 0, \quad (m = iM).$$

Allowing for

$$a = \sqrt{j\,(j+1)}\,, \quad b = \sqrt{(j-1)(j+2)}, \quad x = -\frac{\epsilon^2 - M^2}{4} r^2 ,$$

we get the following equation

$$f_1'' + \left(\frac{5}{2x} - \frac{4}{j^2 + j - 2 + 4\,x}\right) f_1' + \left(-\frac{1}{x} - \frac{1}{4}\frac{j^2 + j - 2}{x^2}\right) f_1 = 0\,.$$

Making the additional change of the independent variable

$$\gamma = \frac{j^2 + j - 2}{4}, \quad z = -\frac{x}{\gamma}, \quad z = +\frac{\epsilon^2 - M^2}{j^2 + j - 2} r^2 \in (0, +\infty),$$

we get

$$\Big[\frac{d^2}{dz^2} + (\frac{5/2}{z} - \frac{1}{z-1})\frac{d}{dz} + (\frac{1}{z} - \frac{\gamma}{z^2})\Big] f_1 = 0.$$

With the use of the substitution $f_1(z) = z^\rho e^{\sigma z} F(Z)$, we yield the equation

$$\frac{d^2 F}{dz^2} + \big(2\,\sigma + \frac{1}{2}\,\frac{4\,\rho + 5}{z} + \frac{1}{1-z}\big)\frac{dF}{dz} +$$

$$+\big(\sigma^2+\frac{1}{2}\,\frac{4\,\rho\,\sigma+5\,\sigma+2\,\rho+2\,\gamma}{z}+\frac{1}{2}\,\frac{2\,\rho^2+3\,\rho-2\,\gamma}{z^2}+\frac{\rho+\sigma}{1-z}\big)F=0\,.$$

B imposing the restrictions

$$\sigma^2=0\,,\quad 2\,\rho^2+3\,\rho-2\,\gamma=0\qquad\Longrightarrow$$

$$\sigma=0\,,\qquad \rho=-\frac{3}{4}\pm\frac{1}{4}\sqrt{16\gamma+9}=-\frac{3}{4}\pm\frac{2j+1}{4}=\frac{j-1}{2},-\frac{j+2}{2}\,,$$

we arrive at the confluent Heun equation

$$\frac{d^2F}{dz^2}+\big(\frac{1}{2}\,\frac{4\,\rho+5}{z}+\frac{1}{1-z}\big)\frac{dF}{dz}+\big(\frac{\rho+\gamma}{z}+\frac{\rho}{1-z}\big)F=0.=0\,.$$

The most interesting is the regular solution $f_1(z)=z^{(j-1)/2}F(Z)$.

There exists a much more simple possibility. Let us express the function f_1 from the third equation, then find c_1' from the second equation. In this way from the first equation we derive an equation for c_1:

$$\frac{d^2c_1}{dr^2}+\frac{4}{r}\,\frac{dc_1}{dr}+\big(\epsilon^2+m^2-\frac{j^2+j-2}{r^2}\big)c_1=0\,,$$

which is the Bessel function.

For states with parity $P=(-1)^j$, we get the following system for the independent variables

$$f_1,\quad f_2,\quad c_1,\quad c_2,\quad d_1,\quad d_2,\quad f_0$$

$$1)\qquad -2\left(a^2+2\right)c_2+d_1\left(-3i\sqrt{2}ar\epsilon-i\sqrt{2}br\epsilon\right)-$$
$$-2abf_1-4\sqrt{2}arc_1'-8\sqrt{2}ac_1-4rc_2'-4ir^2\epsilon d_2'-$$
$$-8ir\epsilon d_2+2r^2\epsilon^2f_0-2r^2f_2''-8rf_2'-4f_2=0,$$

$$2)\quad -\epsilon^2f_1r^2-f_1''r^2-i\sqrt{2}b\epsilon d_1r-\sqrt{2}bc_1'r-2f_1'r-\sqrt{2}bc_1-abc_2=m^2r^2f_1,$$

$$3)\quad i\epsilon d_2'r^2-\frac{1}{2}c_2''r^2-\frac{1}{4}f_0''r^2+\frac{3}{4}f_2''r^2-2i\epsilon d_2r+\sqrt{2}ac_1'r+2c_2'r-\frac{1}{2}f_0'r+\frac{1}{2}f_2'r+$$
$$+\frac{ac_1}{2\sqrt{2}}-\frac{1}{2}\left(r^2\epsilon^2+2\right)c_2-\frac{ac_3}{2\sqrt{2}}+\frac{1}{12}\left(-11i\sqrt{2}ar\epsilon-i\sqrt{2}br\epsilon\right)d_1+$$
$$+\frac{1}{12}\left(3a^2+3r^2\epsilon^2\right)f_0-\frac{1}{2}abf_1+\frac{1}{12}\left(9a^2-9r^2\epsilon^2-12\right)f_2=m^2r^2f_2,$$

$$4)\qquad i\epsilon d_2'r^2-\frac{1}{2}c_2''r^2+\frac{1}{4}f_0''r^2+\frac{1}{4}f_2''r^2+\frac{1}{2}f_0'r-\frac{1}{2}f_2'r-$$

$$-\frac{3ac_1}{2\sqrt{2}}-\frac{1}{2}\left(r^2\epsilon^2+2\right)c_2-\frac{ac_3}{2\sqrt{2}}+\frac{1}{12}\left(i\sqrt{2}br\epsilon-i\sqrt{2}ar\epsilon\right)d_1+$$

$$+\frac{1}{12}\left(-3a^2-3r^2\epsilon^2\right)f_0-\frac{1}{2}abf_1+\frac{1}{12}\left(3a^2-3r^2\epsilon^2-12\right)f_2=m^2r^2c_2,$$

$$5)\quad i\epsilon d_1'r^2-i\epsilon d_1r-\frac{ia\epsilon d_2r}{\sqrt{2}}+\frac{ac_2'r}{\sqrt{2}}+\frac{bf_1'r}{\sqrt{2}}-\frac{af_2'r}{\sqrt{2}}+$$

$$+\frac{1}{4}\left(-a^2+b^2-4r^2\epsilon^2-6\right)c_1-\sqrt{2}ac_2=m^2r^2c_1,$$

$$6)\qquad -i\epsilon c_1'r^2-d_1''r^2-3i\epsilon c_1r-\frac{ia\epsilon c_2r}{\sqrt{2}}-\frac{ia\epsilon f_0r}{\sqrt{2}}-\frac{ib\epsilon f_1r}{\sqrt{2}}-2d_1'r-$$

$$-\frac{ad_2'r}{\sqrt{2}}+\frac{1}{12}(6-(a-b)(a+3b))d_1=m^2r^2d_1,$$

$$7)\qquad d_2a^2-i\sqrt{2}r\epsilon c_1a-2ir\epsilon c_2+\frac{(7a-b)d_1}{3\sqrt{2}}-2ir\epsilon f_2+$$

$$+\frac{(5a+b)rd_1'}{3\sqrt{2}}+ir^2\epsilon f_0'-ir^2\epsilon f_2'=m^2r^2d_2,$$

$$8)\qquad -i\epsilon d_2'r^2+\frac{1}{2}c_2''r^2-\frac{3}{4}f_0''r^2+\frac{1}{4}f_2''r^2-2i\epsilon d_2r+\sqrt{2}ac_1'r+2c_2'r-\frac{3}{2}f_0'r+$$

$$+\frac{3}{2}f_2'r+\frac{7ac_1}{2\sqrt{2}}+\frac{1}{2}\left(r^2\epsilon^2+2\right)c_2+\frac{ac_3}{2\sqrt{2}}+\frac{1}{4}\left(-3i\sqrt{2}ar\epsilon-i\sqrt{2}br\epsilon\right)d_1+$$

$$+\frac{1}{4}\left(3a^2+3r^2\epsilon^2\right)f_0+\frac{1}{2}abf_1+\frac{1}{4}\left(a^2-r^2\epsilon^2+4\right)f_2=m^2r^2f_0.$$

The system is rather complicated. We can construct its solution for the minimal $j=0$.

6 State with minimal $j = 0$

Let us consider the simplest case $j=0$. To this end, in the complete first order system of radial equations, we take into account the following restrictions on radial functions

$$h_1=0,h_3=0,f_1=0,f_3=0,c_1=0,c_3=0,d_1=0,d_3=0,$$

$$
\begin{array}{llll}
E_{10}=0 & E_{11}=0 & E_{12}=0 & E_{23}=0 \\
E_{30}=0 & E_{21}=0 & E_{32}=0 & E_{33}=0 \\
B_{10}=0 & B_{21}=0 & B_{12}=0 & B_{13}=0 \\
B_{30}=0 & B_{31}=0 & B_{32}=0 & B_{23}=0
\end{array}
\tag{6.1}
$$

In this way we get 13 equations for 13 variables,

$$
h\,,\; h_0\,,\; h_2\,,\; d_2\,,\; f_0\,,\; f_2\,,\; c_2\,,\; B_{11}\,,\; B_{33}\,,\; E_{20}\,,\; E_{13}\,,\; E_{22}\,,\; E_{31}:
$$

$$
-i\epsilon\, h_0 - \frac{2\,h_2}{r} - h_2' = mh\,,
$$

$$
\frac{1}{6\,r}\;(-3\,ir\epsilon\,h + 4\,d_2 + 2\,ir\epsilon\,f_0 + 2\,r\,d_2') = mh_0\,,
$$

$$
\frac{1}{6\,r}\;(4\,c_2 + 2\,ir\epsilon\,d_2 + 4\,f_2 + 3\,r\,h' + 2\,r\,f_2') = mh_2\,,
$$

$$
\frac{1}{8r}\Big\{4B_{11} - 4B_{33} + 4E_{20} - 8h_2 - 2r[i\epsilon(E_{13} + 3E_{22} + E_{31} + 2h_0) +
$$

$$
+B_{11}' - B_{33}' - E_{20}' - 6h_2']\Big\} = mf_2\,,
$$

$$
-\frac{1}{8\,r}\Big\{4\,E_{20} + 8\,h_2 + 2\,r[i\epsilon\,(E_{13} + E_{22} + E_{31} - 2\,h_0) +
$$

$$
+B_{11}' - B_{33}' + E_{20}' - 2\,h_2']\Big\} = mc_2\,,
$$

$$
\frac{1}{4\,r}\Big\{2\,E_{13} + 4\,E_{22} + 2\,E_{31} + 2\,r\,[-i\epsilon\;(E_{20} + 2\,h_2) + E_{22}' + 2\,h_0']\Big\} = md_2\,,
$$

$$
\frac{1}{8r}\Big\{4B_{11} - 4B_{33} + 12E_{20} + 8h_2 + 2r[i\epsilon(E_{13} - E_{22} + E_{31} - 6h_0) +
$$

$$
+B_{11}' - B_{33}' + 3E_{20}' + 2h_2']\Big\} = mf_0\,,
$$

$$
\frac{1}{6\,r}\;[4\,c_2 + 4\,f_2 + 2\,r\,(-2\,i\epsilon\,d_2 - 3\,f_0' + f_2')] = mE_{20}\,,
$$

$$
\frac{1}{6\,r}\;[4\,d_2 + 2\,ir\epsilon\,(f_0 - 3\,f_2) - 4\,r\,d_2'] = mE_{22}\,,
$$

$$
\frac{1}{6\,r}\;[2\,d_2 - 2\,ir\epsilon\;(3\,c_2 + f_0) - 2\,r\,d_2'] = mE_{31}\,,
$$

$$
\frac{1}{6\,r}\;[2\,d_2 - 2\,ir\epsilon\;(3\,c_2 + f_0) - 2\,r\,d_2'] = mE_{13}\,,
$$

$$
\frac{1}{6\,r}\;(2\,c_2 - 2\,ir\epsilon\,d_2 + 2\,f_2 + 6\,r\,c_2' - 2\,r\,f_2') = mB_{11}\,,
$$

$$-\frac{1}{6\,r}\ (2\,c_2 - 2\,ir\epsilon\, d_2 + 2\,f_2 + 6\,r\,c_2' - 2\,r\,f_2') = mB_{33}\,.$$

With the help of the six last equations, we eliminate the variables with two indices:

$$-i\epsilon\, h_0 - \frac{2\,h_2}{r} - h_2' = mh\,,$$

$$\frac{1}{6\,r}\ (-3\,ir\epsilon\, h + 4\,d_2 + 2\,ir\epsilon\, f_0 + 2\,r\,d_2') = mh_0\,,$$

$$\frac{1}{6\,r}\ (4\,c_2 + 2\,ir\epsilon\, d_2 + 4\,f_2 + 3\,r\,h' + 2\,r\,f_2') = mh_2\,,$$

$$\frac{1}{3}\,\frac{8-6\,\epsilon^2 r^2}{r^2}\,c_2 - \frac{16}{3}\,\frac{i\epsilon}{r}\,d_2 + \frac{1}{3}\,\epsilon^2 f_0 + \frac{1}{3}\,\frac{-9\,\epsilon^2 r^2+8}{r^2}\,f_2 - 2\,i\epsilon m\,h_0 - \frac{4\,m}{r}\,h_2 +$$
$$+\frac{4}{r}\,c_2' + \frac{8}{3}\,i\epsilon\,d_2' - \frac{2}{r}\,f_0' - \frac{2}{3}\,\frac{1}{r}\,f_2' + 6\,m\,h_2' - 2\,c_2'' - f_0'' + f_2'' = 4\,m^2 f_2\,,$$

$$-6\,\epsilon^2 c_2 - \epsilon^2 f_0 - 3\,\epsilon^2 f_2 + 6\,i\epsilon\, m h_0 - \frac{12\,m}{r}\,h_2 -$$
$$-\frac{4}{r}\,c_2' + 8\,i\epsilon\,d_2' + \frac{6}{r}\,f_0' - \frac{6}{r}\,f_2' + 6\,m\,h_2' - 6\,c_2'' + 3\,f_0'' + f_2'' = 12\,m^2 c_2\,,$$

$$\frac{-4\,i\epsilon}{r}\,c_2 + \frac{(2-\epsilon^2 r^2)}{r^2}\,d_2 - \frac{4\,i\epsilon}{r}\,f_2 - 3\,i\epsilon\,m\,h_2 - \frac{2}{r}\,d_2' +$$
$$+2\,i\epsilon\,f_0' - 2\,i\epsilon\,f_2' + 3\,m\,h_0' - d_2'' = 3\,m^2 d_2\,,$$

$$\frac{6\,\epsilon^2 r^2+8}{r^2}\,c_2 - \frac{16\,i\epsilon}{r}\,d_2 + 3\,\epsilon^2 f_0 + \frac{-3\,\epsilon^2 r^2+8}{r^2}\,f_2 - 18\,i\epsilon\,m\,h_0 + \frac{12\,m}{r}\,h_2 +$$
$$+\frac{20}{r}\,c_2' - 8\,i\epsilon\,d_2' - \frac{18}{r}\,f_0' + \frac{10}{r}\,f_2' + 6\,m\,h_2' + 6\,c_2'' - 9\,f_0'' + f_2'' = 12\,m^2 f_0\,.$$

On can readily verify that the function h, h_0, h_2 vanish identically, the the system becomes simpler

$$4\,d_2 + 2\,ir\epsilon\,f_0 + 2\,r\,d_2' = 0\,, \qquad (1')$$

$$4\,c_2 + 2\,ir\epsilon\,d_2 + 4\,f_2 + 2\,r\,f_2' = 0\,, \qquad (2')$$

$$\frac{1}{3}\,\frac{8-6\,\epsilon^2 r^2}{r^2}\,c_2 - \frac{16}{3}\,\frac{i\epsilon}{r}\,d_2 + \frac{1}{3}\,\epsilon^2 f_0 + \frac{1}{3}\,\frac{-9\,\epsilon^2 r^2+8}{r^2}\,f_2 +$$
$$+\frac{4}{r}\,c_2' + \frac{8}{3}\,i\epsilon\,d_2' - \frac{2}{r}\,f_0' - \frac{2}{3}\,\frac{1}{r}\,f_2' - 2\,c_2'' - f_0'' + f_2'' = 4\,m^2 f_2\,, \qquad (3')$$

$$-6\,\epsilon^2 c_2 - \epsilon^2 f_0 - 3\,\epsilon^2 f_2 - \frac{4}{r}\,c_2' + 8\,i\epsilon\,d_2' + \frac{6}{r}\,f_0' -$$
$$-\frac{6}{r}\,f_2' - 6\,c_2'' + 3\,f_0'' + f_2'' = 12\,m^2 c_2\,, \qquad (4')$$

$$\frac{-4\,i\epsilon}{r}\,c_2+\frac{(2-\epsilon^2r^2)}{r^2}\,d_2-\frac{4\,i\epsilon}{r}\,f_2-\frac{2}{r}\,d_2'+2\,i\epsilon\,f_0'-2\,i\epsilon\,f_2'-d_2''=3\,m^2d_2\,, \qquad (5')$$

$$\frac{6\,\epsilon^2r^2+8}{r^2}\,c_2-\frac{16\,i\epsilon}{r}\,d_2+3\,\epsilon^2 f_0+\frac{-3\,\epsilon^2r^2+8}{r^2}\,f_2+$$
$$+\frac{20}{r}\,c_2'-8\,i\epsilon\,d_2'-\frac{18}{r}\,f_0'+\frac{10}{r}\,f_2'+6\,c_2''-9\,f_0''+f_2''=12\,m^2f_0\,. \qquad (6')$$

Let us express the variable f_0 from eq. $(1')$, from eq. $(2')$ – the variable c_2, and substitute results in $(5')$, this yields

$$d_2''+\frac{2}{r}\,d_2'+\left(\epsilon^2+m^2-\frac{2}{r^2}\right)d_2=0 \quad (m=iM);$$

remaining functions are determines through d_2 by the formulas

$$c_2=\frac{-im^2r\,d_2'-2\,i\epsilon^2d_2-im^2d_2}{2\,r\epsilon\,m^2+2\,r\epsilon^3},$$

$$f_0=\frac{ir\,d_2'+2\,id_2}{r\epsilon},\quad f_2=\frac{-i\epsilon^2r\,d_2'-im^2d_2}{-r\epsilon\,m^2-r\epsilon^3}\,.$$

As a primary function, one can take c_2. Then we obtain

$$c_2''+[\frac{4}{r}-\frac{2rm^4}{m^4r^2+4\epsilon^2-2m^2}]c_2'+[\epsilon^2+m^2-2\frac{m^2\,(-2\epsilon^2+m^2)}{m^4r^2+4\epsilon^2-2m^2}]c_2=0\,,$$

$$d_2=-\frac{2\,i\epsilon\,r^2m^2}{m^4r^2+4\,\epsilon^2-2\,m^2}\,c_2'+\frac{-2\,i\epsilon\,rm^2+4\,i\epsilon^3r}{m^4r^2+4\,\epsilon^2-2\,m^2}\,c_2\,,$$

$$f_0=\frac{-4\,\epsilon^2r+2\,m^2r}{m^4r^2+4\,\epsilon^2-2\,m^2}\,c_2'+\frac{-12\,\epsilon^2-2\,m^2\epsilon^2r^2+6\,m^2-2\,m^4r^2\,(r)}{m^4r^2+4\,\epsilon^2-2\,m^2}\,c_2\,,$$

$$f_2=+\frac{-4\,\epsilon^2r+2\,m^2r}{m^4r^2+4\,\epsilon^2-2\,m^2}\,c_2'+\frac{2\,m^2-2\,m^2\epsilon^2r^2-4\,\epsilon^2}{m^4r^2+4\,\epsilon^2-2\,m^2}\,c_2\,.$$

As a primary function one can take f_0, then we obtain

$$f_0''+\frac{2}{r}\,f_0'+\left(\epsilon^2+m^2\right)f_0=0,\ ,$$

$$c_2=\frac{1}{2}\,\frac{2\,\epsilon^2-m^2}{\left(\epsilon^2+m^2\right)^2r}\,f_0'-\frac{1}{2}\,\frac{m^2}{\epsilon^2+m^2}\,f_0\,,\quad d_2=\frac{i\epsilon}{\epsilon^2+m^2}\,f_0'\,,$$

$$f_2=\frac{2\,\epsilon^2-m^2}{\left(\epsilon^2+m^2\right)^2r}\,f_0'+\frac{\epsilon^2}{\epsilon^2+m^2}\,f_0\,.$$

As a primary function one can take f_2. Then we obtain

$$f_2''+\left[\frac{4}{r}-\frac{2\,r\epsilon^4}{m^2-2\,\epsilon^2+\epsilon^4r^2}\right]f_2'+\left[\epsilon^2+m^2+2\,\frac{\epsilon^2\,(-2\,\epsilon^2+m^2)}{m^2-2\,\epsilon^2+\epsilon^4r^2}\right]f_2=0\,,$$

$$c_2 = \frac{2\,\epsilon^2 r - m^2 r}{2\,m^2 - 4\,\epsilon^2 + 2\,\epsilon^4 r^2}\, f_2' + \frac{-2\,m^2 + 4\,\epsilon^2 - m^2\epsilon^2 r^2}{2\,m^2 - 4\,\epsilon^2 + 2\,\epsilon^4 r^2}\, f_2\,,$$

$$d_2 = \frac{i\epsilon^3 r^2}{m^2 - 2\,\epsilon^2 + \epsilon^4 r^2}\, f_2' + \frac{-i\epsilon\, r m^2 + 2\,i\epsilon^3 r}{m^2 - 2\,\epsilon^2 + \epsilon^4 r^2}\, f_2\,,$$

$$f_0 = \frac{-2\,\epsilon^2 r + m^2 r}{m^2 - 2\,\epsilon^2 + \epsilon^4 r^2}\, f_2' + \frac{\epsilon^4 r^2 + m^2\epsilon^2 r^2 - 6\,\epsilon^2 + 3\,m^2}{m^2 - 2\,\epsilon^2 + \epsilon^4 r^2}\, f_2\,.$$

Evidently, these are different representations for the same solution, which describes the spin 2 particle with $j = 0$.

7 Conclusions

The matrix tetrad based equation for spin 2 particle has been specified in spherical coordinates of Minkowski space. After separating the variables with the use of the total angular momentum and space reflection operators we have derive two independent systems of radial equations. Some simple solutions are found in explicit form.

It should be stressed that extension to the arbitrary space-time with spherical symmetry

$$dS^2 = e^{\nu}(dt)^2 - e^{\mu}(dr)^2 - r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2], \qquad (7.1)$$

does not require new ideas. This is due to the structure of the main equation for such models,

$$\Big[\Gamma^0\big(e^{-\nu/2}\partial_t + \frac{1}{2}\frac{\partial\nu}{\partial r}e^{-\mu/2}J^{03}\Big) + \Gamma^3\Big(e^{-\mu/2}\partial_r + \frac{1}{2}\frac{\partial\mu}{\partial t}e^{-\nu/2}J^{03}\Big)$$
$$+\frac{1}{r}e^{-\mu/2}\Big(\Gamma^1 J^{12} + \Gamma^2 J^{23}\Big) + \frac{1}{r}\Sigma_{\theta,\phi} - m\Big]\Phi(x) = 0,$$

where we assume the tetrad in the form

$$e^{\beta}_{(0)} = (e^{-\nu/2}, 0, 0, 0), \quad e^{\beta}_{(3)} = (0, e^{-\mu/2}, 0, 0),$$

$$e^{\beta}_{(1)} = (0, 0, \frac{1}{r}, 0), \quad e^{\beta}_{(2)} = (0, 0, 0, \frac{1}{r\sin\theta}).$$

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