The eigenvalues problem for helicity operator for a spin 2 particle in cylindric coordinates, the presence of external magnetic field

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The explicit form of the helicity operator for symmetric 2-nd rank tensor describing the spin 2 particle, is specified in cylindrical coordinates. After separating the variables, the system of 10 differential first order equations is derived. It is split into two independent subsystems of of 4 and 6 equations. The system of 4 equations is straightforwardly solved in terms of confluent hypergeometric functions: the corresponding eigenvalues and eigenfunctions are determined. The subsystem of 6 equations can be reduced to one ordinary differential equation of the 4-th order. The corresponding 4-th order operator is factorized into permutable 2-nd order operators, and the problem reduces to solving two differential equations of the 2-nd order. Their solutions are constructed in terms of Bessel functions. This analysis is extended to the presence of external uniform magnetic field, when the solutions are constructed in term of confluent hypergeometric functions.

1 Introduction

It is known that the eigenvalue states of the helicity operator play a substantial role in studying any spin particle in external electromagnetic (or gravitational) fields with cylindric symmetry. In the present work we specify this problem for a spin 2 particle (massive or massless) in Minkowski space-time, provided by

$$\Sigma^{cart} = J^{23}\frac{\partial}{\partial x} + J^{31}\frac{\partial}{\partial y} + J^{12}\frac{\partial}{\partial z}, \quad \Sigma^{cart} H^{cart} = \sigma H^{cart}, \tag{1.1}$$

where $H^{cart}(x,y,z)$ stands for the 10 components of the symmetric 2-rank tensor referring to the spin 2 particle.

2 Helicity operator in cylindric basis and separating the variables

We shall apply a covariant description of this field, with respect to cylindric coordinates and to the corresponding tetrad

$$dS^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \; x^\alpha = (t, r, \phi, z), \quad e^{\beta}_{(a)}(x) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}. \tag{2.1}$$

The transition from the Cartesian tetrad to the cylindric one is performed with the use of local transformation from the Lorentz group

$$L_b{}^a(x) = e'^{\beta'}_{(b)}(x')\frac{\partial x^{\alpha}}{\partial x^{\beta'}} e_\alpha{}^{(a)}(x) = e'^{\beta'}_{(b)}(x')\frac{\partial x^{a}}{\partial x^{\beta'}}, \tag{2.2}$$

or, in explicit form,

$$L_b{}^a(x) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & 0 \\ 0 & \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}. \tag{2.3}$$

Therefore, the tensor of the second rank H which describes the spin 2 particle transforms according to the rule

$$H^{cart} = [H^{cart}_{(ab)}], \quad H^{cyl} = (L \otimes L) H^{cart} = L H^{cart} \tilde{L}\,,$$

for which it is convenient to apply the 10-dimensional representation.

Correspondingly, the helicity operator transforms in terms of the cylindric tetrad, as follows

$$H^{cyl} = (L \otimes L) H^{cart} \quad \Longrightarrow \quad H^{cyl} = S H^{cart},$$

$$\Sigma^{cyl} = S(\phi)\Big[J^{23}\big(\cos\phi \frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\big) + J^{31}\big(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\big) + J^{12}\frac{\partial}{\partial z}\Big] S(-\phi)\,,$$

whence we get

$$\Sigma^{cyl} = \Big[S(\phi) J^{23} S_2(-\phi)\cos\phi + S_2(\phi) J^{31} S(-\phi)\sin\phi \Big]\frac{\partial}{\partial r}$$

$$+\frac{1}{r}\Big[- S(\phi) J^{23} S(-\phi)\sin\phi + S_2(\phi) J^{31} S_2(-\phi)\cos\phi\Big]\frac{\partial}{\partial\phi}$$

$$+\frac{1}{r}\Big[-\sin\phi S(\phi) J^{23}\frac{\partial S(-\phi)}{\partial\phi} + \cos\phi S(\phi) J^{31}\frac{\partial S(-\phi)}{\partial\phi}\Big] + S(\phi) J^{12} S(-\phi)\frac{\partial}{\partial z}\,.$$

Taking in mind the structure of the field function (which contains the multipliers $e^{im\phi}$ and e^{ikz}), we can change the last relation to:

$$\Sigma^{cyl} = \Big[S(\phi) J^{23} S(-\phi)\cos\phi + S(\phi) J^{31} S(-\phi)\sin\phi \Big]\frac{d}{dr}$$

$$+\frac{1}{r}\Big[- S(\phi) J^{23} S(-\phi)\sin\phi + S(\phi) J^{31} S(-\phi)\cos\phi\Big] im$$

$$+\frac{1}{r}\Big[-\sin\phi S(\phi) J^{23}\frac{\partial S(-\phi)}{\partial\phi} + \cos\phi S(\phi) J^{31}\frac{\partial S(-\phi)}{\partial\phi}\Big] + S(\phi) J^{12} S(-\phi) ik\,.$$

Then the expression for component in the cylindric tetrad basis is:

$$H^{cyl} = \left| \begin{array}{cccc} f_0 & \cos\phi d_1 + \sin\phi d_2 & \cos\phi d_2 - \sin\phi d_1 & d_3 \\ \cos\phi d_1 + \sin\phi d_2 & f_1\cos^2\phi + \sin 2\phi c_3 + \sin^2\phi f_2 & \cos 2\phi c_3 + \cos\phi\sin\phi(f_2 - f_1) & \sin\phi c_1 + \cos\phi c_2 \\ \cos\phi d_2 - \sin\phi d_1 & \cos 2\phi c_3 + \cos\phi\sin\phi(f_2 - f_1) & f_2\cos^2\phi - \sin 2\phi c_3 + \sin^2\phi f_1 & \cos\phi c_1 - \sin\phi c_2 \\ d_3 & \sin\phi c_1 + \cos\phi c_2 & \cos\phi c_1 - \sin\phi c_2 & f_3 \end{array} \right|,$$

or in 10-dimension form $H^{cyl} = S(\phi) H$:

$$H^{cyl} = \text{the column}(f_1, f_2, f_3, c_1, c_2, c_3, d_1, d_2, d_3, f_0),$$

$$S(\phi) = \left| \begin{array}{cccccccccc} \cos^2\phi & \sin^2\phi & . & . & . & \sin 2\phi & . & . & . & . \\ \sin^2\phi & \cos^2\phi & . & . & . & -\sin 2\phi & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . \\ . & . & . & \cos\phi & -\sin\phi & . & . & . & . & . \\ . & . & . & \sin\phi & \cos\phi & . & . & . & . & . \\ -\cos\phi\sin\phi & \cos\phi\sin\phi & . & . & . & \cos 2\phi & . & . & . & . \\ . & . & . & . & . & . & \cos\phi & \sin\phi & . & . \\ . & . & . & . & . & . & -\sin\phi & \cos\phi & . & . \\ . & . & . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{array} \right|.$$

The inverse matrix equals $S^{-1} = S(-\phi)$. After simple calculation, we derive the following expression for the helicity operator in cylindric basis

$$\Sigma^{cyl} = \left| \begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right| \frac{d}{dr} + \frac{1}{r} \left| \begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right| im +$$

$$+ \left| \begin{array}{cccccccccc} 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right| + \left| \begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right| ik.$$

Taking in mind explicit form of generators for symmetric tensor, we obtain the more reduced form:

$$\Sigma^{cyl} = J^{23} \frac{d}{dr} + \frac{im}{r} J^{31} + ikJ^{12} + \frac{1}{r} \left| \begin{array}{cccccccccc} 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right| . \tag{2.4}$$

It is convenient to use the cyclic representation, in with the generator J^{12} becomes diagonal, and hence we perform the similarity transformation $\bar{\Sigma}^{cyl} = C_2 \Sigma^{cyl} C_2^{-1}$; this leads to

$$\bar{\Sigma}^{cyl} = \bar{J}^{23} \frac{d}{dr} + \frac{im}{r} \bar{J}^{31} + ik\bar{J}^{12} + \frac{1}{r} \left| \begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sqrt{2} & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right| .$$

where

$$\bar{J}^{23} = \frac{1}{\sqrt{2}} \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -2i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -2i & \cdot & -2i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -2i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -i & -i & \cdot & -i & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & \cdot & -i & \cdot & \cdot & \cdot & \cdot \\ -i & -i & \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -i & \cdot & -i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

$$\bar{J}^{31} = \frac{1}{\sqrt{2}} \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -2 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

$$\bar{J}^{12} = \begin{vmatrix} -2i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +2i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{vmatrix},$$

The eigenvalue equation $\bar{\Sigma}\bar{H} = \sigma\bar{H}$ gives the system of 10 equations

$$\begin{aligned} 2c_3' &= -\frac{2(m-1)}{r}c_3 + i\sqrt{2}(\sigma - 2k)f_1, \\ 2\left(c_1' + c_3'\right) &= -\frac{2(m+1)}{r}c_1 + \frac{2(m-1)}{r}c_3 + i\sqrt{2}\sigma f_2, \\ 2c_1' &= i\sqrt{2}(2k+\sigma)f_3 + \frac{2(m+1)}{r}c_1, \\ c_2' + f_2' + f_3' &= i\sqrt{2}(k+\sigma)c_1 + \frac{m}{r}\left(c_2 + f_2\right) - \frac{m+2}{r}f_3, \\ c_1' + c_3' &= -\frac{m+1}{r}c_1 + i\sqrt{2}\sigma c_2 + \frac{m-1}{r}c_3, \\ c_2' + f_1' + f_2' &= i\sqrt{2}(\sigma - k)c_3 + \frac{m-2}{r}f_1 - \frac{m}{r}\left(c_2 + f_2\right), \\ d_2' &= i\sqrt{2}(\sigma - k)d_1 - \frac{m}{r}d_2, \\ d_1' + d_3' &= \frac{m-1}{r}d_1 + i\sqrt{2}\sigma d_2 - \frac{m+1}{r}d_3, \\ d_2' &= i\sqrt{2}(k+\sigma)d_3 + \frac{m}{r}d_2, \\ \sqrt{2}\sigma f_0 &= 0. \end{aligned} \tag{2.5}$$

It is convenient to apply special notations for the 8 differential operators

$$\frac{1}{\sqrt{2}}(\frac{d}{dr} \pm \frac{m}{r}) = a^{\pm}_{m}, \quad \frac{1}{\sqrt{2}}(\frac{d}{dr} \pm \frac{m+1}{r}) = a^{\pm}_{m+1}, \quad \frac{1}{\sqrt{2}}(\frac{d}{dr} \pm \frac{m-1}{r}) = a^{\pm}_{m-1},$$
$$\frac{1}{\sqrt{2}}(\frac{d}{dr} + \frac{m+2}{r}) = a^{+}_{m+2}, \qquad \frac{1}{\sqrt{2}}(\frac{d}{dr} - \frac{m-2}{r}) = a^{-}_{m-2}. \tag{2.6}$$

Then the above system may be written in the form of three independent subsystems:

$$\begin{aligned} I \qquad a^{+}_{m} d_2 &= i(\sigma - k) d_1, \\ a^{-}_{m-1} d_1 + a^{+}_{m+1} d_3 &= i\sigma d_2, \\ a^{-}_{m} d_2 &= i(\sigma + k) d_3; \end{aligned} \tag{2.7}$$

$$\begin{aligned} II \qquad a^{+}_{m-1} c_3 &= i(\frac{\sigma}{2} - k) f_1, \\ a^{+}_{m+1} c_1 + a^{-}_{m-1} c_3 &= i\frac{\sigma}{2} f_2, \\ a^{-}_{m+1} c_1 &= i(\frac{\sigma}{2} + k) f_3, \\ a^{-}_{m} c_2 + a^{-}_{m} f_2 + a^{+}_{m+2} f_3 &= i(\sigma + k) c_1, \\ a^{+}_{m+1} c_1 + a^{-}_{m-1} c_3 &= i\sigma c_2, \\ a^{+}_{m} c_2 + a^{-}_{m-2} f_1 + a^{+}_{m} f_2 &= i(\sigma - k) c_3; \end{aligned} \tag{2.8}$$

$$III \qquad \sigma f_0 = 0 \quad \Longrightarrow \quad \sigma \neq 0, f_0 = 0. \tag{2.9}$$

First consider the system I; after eliminating the variables d_1 and d_3, we get an equation for the primary function

$$\left[\frac{1}{i(\sigma - k)} a^{-}_{m-1} a^{+}_{m} + \frac{1}{i(\sigma + k)} a^{+}_{m+1} a^{-}_{m} - i\sigma \right] d_2 = 0. \tag{2.10}$$

Similarly, in the system II one can eliminate the variables f_1, f_2, f_3:

$$f_1 = \frac{2}{i(\sigma - 2k)} a^{+}_{m-1} c_3, \quad f_2 = \frac{2}{i\sigma} a^{+}_{m+1} c_1 + \frac{2}{i\sigma} a^{-}_{m-1} c_3, \quad f_3 = \frac{2}{i(\sigma + 2k)} a^{-}_{m+1} c_1,$$

which results in the system for c_1, c_2, c_3:

$$a^{-}_{m} c_2 + \frac{2}{i\sigma} a^{-}_{m} a^{+}_{m+1} c_1 + \frac{2}{i\sigma} a^{-}_{m} a^{-}_{m-1} c_3 + \frac{2}{i(\sigma + 2k)} a^{+}_{m+2} a^{-}_{m+1} c_1 = i(\sigma + k) c_1,$$

$$a^{+}_{m} c_2 + \frac{2}{i(\sigma - 2k)} a^{-}_{m-2} a^{+}_{m-1} c_3 + \frac{2}{i\sigma} a^{+}_{m} a^{+}_{m+1} c_1 + \frac{2}{i\sigma} a^{+}_{m} a^{-}_{m-1} c_3 = i(\sigma - k) c_3,$$

$$a^{+}_{m+1} c_1 + a^{-}_{m-1} c_3 = i\sigma c_2$$

in which, with the help of third equation we may eliminate the variable c_2, and hence one can derive the system for c_1, c_2:

$$\begin{aligned} \left[3a^{-}_{m} a^{+}_{m+1} + \frac{2\sigma}{(\sigma + 2k)} a^{+}_{m+2} a^{-}_{m+1} + \sigma(\sigma + k) \right] c_1 + 3a^{-}_{m} a^{-}_{m-1} c_3 = 0, \\ \left[3a^{+}_{m} a^{-}_{m-1} c_3 + \frac{2\sigma}{(\sigma - 2k)} a^{-}_{m-2} a^{+}_{m-1} + \sigma(\sigma - k) \right] c_3 + 3a^{+}_{m} a^{+}_{m+1} c_1 = 0. \end{aligned} \tag{2.11}$$

In order to take into account the presence of the external magnetic field, it suffices to make the change $m \Longrightarrow m + eBr^2/2$, which leads to the new 8 operators

$$a^{\pm}_m = \frac{1}{\sqrt{2}}(\frac{d}{dr} \pm \frac{m + eBr^2/2}{r}),$$
$$a^{\pm}_{m+1}, \quad \frac{1}{\sqrt{2}}(\frac{d}{dr} \pm \frac{m + eBr^2/2 + 1}{r}), \; a^{\pm}_{m-1} = \frac{1}{\sqrt{2}}(\frac{d}{dr} \pm \frac{m + eBr^2/2 - 1}{r}), \tag{2.12}$$
$$a^{+}_{m+2} = \frac{1}{\sqrt{2}}(\frac{d}{dr} + \frac{m + eBr^2/2 + 2}{r}), \; a^{-}_{m-2} = \frac{1}{\sqrt{2}}(\frac{d}{dr} - \frac{m + eBr^2/2 - 2}{r}).$$

3 The study of the system for the case I, the free particle

Let us turn to the case I. First consider eq. (2.10) for a free particle

$$\left[\frac{1}{i(\sigma - k)} a^{-}_{m-1} a^{+}_m + \frac{1}{i(\sigma + k)} a^{+}_{m+1} a^{-}_m - i\sigma\right] d_2 = 0. \tag{3.1}$$

Allowing for the identities

$$a^{-}_{m-1} a^{+}_m = \frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{2r}\frac{d}{dr} - \frac{m^2}{2r^2}, \qquad a^{+}_{m-1} a^{-}_m = \frac{1}{2}\frac{d^2}{dr^2} - \frac{1}{2r}\frac{d}{dr} + \frac{(2-m)m}{2r^2}$$

we get

$$\left[\frac{1}{i(\sigma - k)}(\frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{2r}\frac{d}{dr} - \frac{m^2}{2r^2}) + \frac{1}{i(\sigma + k)}(\frac{1}{2}\frac{d^2}{dr^2} - \frac{1}{2r}\frac{d}{dr} + \frac{(2-m)m}{2r^2}) - i\sigma\right] d_2 = 0, \tag{3.2}$$

or, differently,

$$\left[\frac{d^2}{dr^2} + \frac{k}{\sigma r}\frac{d}{dr} - \frac{mk + m(m-1)\sigma}{\sigma r^2} - \left(k^2 - \sigma^2\right)\right] d_2 = 0. \tag{3.3}$$

Close to the point $r = 0$, the equation becomes simpler

$$\left[\frac{d^2}{dr^2} + \frac{k}{\sigma r}\frac{d}{dr} - \frac{mk + m(m-1)\sigma}{\sigma r^2}\right] d_2 = 0, \quad d_2 = r^A;$$

so that

$$A(A-1) + \frac{k}{\sigma}A - \frac{mk + m(m-1)\sigma}{\sigma} = 0 \quad \Longrightarrow \quad A = m, \quad A = -\frac{k}{\sigma} - m + 1. \tag{3.4}$$

Near the point $r \to \infty$, we have

$$\left[\frac{d^2}{dr^2} + \frac{k}{\sigma r}\frac{d}{dr} - \left(k^2 - \sigma^2\right)\right] d_2 = 0, \quad d_2 = e^{\pm i\sqrt{\sigma^2 - k^2}r}. \tag{3.5}$$

Here we have an equation of confluent hypergeometric type. Its solutions should be searched in the form $d_2 = r^A e^{Cr} f$, which yields

$$f'' + \left(2C + \frac{2A}{r} + \frac{k}{\sigma r}\right) f' + \left[\frac{A(A-1)}{\sigma r^2} + \frac{kA}{\sigma}\frac{1}{r^2} - \frac{mk + m(m-1)\sigma}{\sigma}\frac{1}{r^2}\right] f +$$
$$+ [C^2 - (k^2 - \sigma^2)] f + \frac{2AC}{r} f + \frac{kC}{\sigma}\frac{1}{r} f = 0.$$

Imposing the known restrictions

$$A(A-1)+\frac{k}{\sigma}A-\frac{mk+m(m-1)\sigma}{\sigma}=0, \qquad C^2-(k^2-\sigma^2)=0,$$

we reduce the above equation to the form

$$r\frac{d^2f}{dr^2}+\left(2A+\frac{k}{\sigma}+2Cr\right)\frac{df}{dr}++\left(2AC+\frac{k}{\sigma}C\right)f=0.$$

The possible expressions for A, C are known. By the change of variable $z=-2Cr$, we reduce the equation to its hypergeometric form (we take $C=+\sqrt{k^2-\sigma^2}$):

$$z\frac{d^2f}{dz^2}+(2A+\frac{k}{\sigma}-z)\frac{df}{dz}-(A+\frac{k}{2\sigma})f=0,$$
$$d_2=z^Ae^{-z/2}\Phi(c,a,z), \quad a=A+\frac{k}{2\sigma}, \quad c=2A+\frac{k}{\sigma}=2a. \tag{3.6}$$

The solutions will be regular at the point $r=0$, if we take different expressions for A depending on the sign of m:

$$(a) \quad m>0, \quad d_2\sim z^A=z^m\longrightarrow 0;$$
$$(b) \quad m<0, \quad d_2\sim z^A=z^{1-m-\frac{k}{\sigma}}, \quad 1-m>k/\sigma\,. \tag{3.7}$$

These solutions may be presented in terms of Bessel functions. Indeed, starting with the equation (3.3), let us make the substitution $d_2(r)=\varphi(r)\bar{d}_2(r)$:

$$\frac{d}{dr}d_2=\varphi'\bar{d}_2+\varphi\bar{d}_2', \quad \frac{d^2}{dr^2}d_2=\varphi''\bar{d}_2+2\varphi'\bar{d}_2'+\varphi\bar{d}_2''.$$

Then we get:

$$\bar{d}_2''+\left(2\frac{\varphi'}{\varphi}+\frac{k}{\sigma r}\right)\bar{d}_2'+\left(\frac{k}{\sigma r}\frac{\varphi'}{\varphi}+\frac{\varphi''}{\varphi}-\frac{mk+m(m-1)\sigma}{\sigma r^2}-(k^2-\sigma^2)\right)\bar{d}_2=0.$$

Impose the constraint

$$2\frac{\varphi'}{\varphi}+\frac{k}{\sigma r}=\frac{1}{r} \quad\Longrightarrow\quad \frac{\varphi'}{\varphi}=\frac{1}{r}(1-k/\sigma)/2 \quad\Longrightarrow\quad \varphi=r^{(1-k/\sigma)/2};$$

according to the above equation takes the form

$$\bar{d}_2''+\frac{1}{r}\bar{d}_2'+\left(-(k^2-\sigma^2)-\frac{(m-1/2+k/2\sigma)^2}{r^2}\right)\bar{d}_2=0.$$

In the variable $y=i\sqrt{k^2-\sigma^2}\,r$, it has the structure of Bessel equation

$$\left(\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}+1-\frac{\nu^2}{y^2}\right)\bar{d}_2'=0, \quad \nu=m-\frac{1}{2}+\frac{k}{2\sigma},$$
$$\bar{d}_2'(y)=C_1J_{+\nu}(y)+C_2J_{-\nu}(y), \quad d_2(y)=r^{\frac{1-k/\sigma}{2}}\bar{d}_2(z)\,. \tag{3.8}$$

4 The case I, the particle in magnetic field

Let us take into account the presence of the external magnetic field. Then the above equation for d_2 takes the form (for brevity we simplify the notation, $eB \Longrightarrow B$):

$$\Big[\frac{d^2}{dr^2} + \frac{k}{r\sigma}\frac{d}{dr} + \frac{-2\sigma\left(B(2m-1)+2k^2\right)+2Bk+4\sigma^3}{4\sigma} - \frac{B^2}{4}r^2 + \frac{-km-(m-1)m\sigma}{r^2\sigma}\Big]d_2 = 0. \quad (4.1)$$

Near the point $r = 0$, we get

$$\Big[\frac{d^2}{dr^2} + \frac{k}{r\sigma}\frac{d}{dr} - \frac{km+(m-1)m\sigma}{\sigma r^2\sigma}\Big]d_2 = 0,$$

which coincides with that from the case of a free particle, $d_2 = r^A, A = m, A = -\frac{k}{\sigma} - m + 1$. Near the point $r = \infty$, we have

$$\Big(\frac{d^2}{dr^2} + \frac{k}{r\sigma}\frac{d}{dr} - \frac{1}{4}B^2r^2\Big)d_2 = 0, \quad d_2 = e^{Cr^2}, \quad C = \pm\frac{B}{4}.;$$

In eq. (4.1), let us make the change of variable $r^2 = x$. Then the equation takes the form

$$\Big[\frac{d^2}{dx^2} + \frac{(k+\sigma)}{2\sigma x}\frac{d}{dx} - \frac{1}{16}B^2 + \frac{B(k-2m\sigma+\sigma)}{8\sigma x} - \frac{(k^2-\sigma^2)}{4x} + \frac{m(\sigma-k)}{4\sigma x^2} - \frac{m^2}{4x^2}\Big]d_2 = 0. \quad (4.2)$$

This is is an equation of confluent hypergeometric type. Its solutions are searched in the form $d_2 = x^A e^{Cx} f$. For $f(x)$, the equation changes to:

$$f'' + \Big(\frac{2A}{x} + \frac{(k+\sigma)}{2\sigma x} + 2C\Big)f' + \Big\{\frac{4A(A-1)-m^2}{4x^2} + \frac{m(\sigma-k)+2A(k+\sigma)}{4\sigma x^2}$$
$$+\frac{B(k-2m\sigma+\sigma)+4C(k+\sigma)}{8\sigma x} - \frac{(k^2-\sigma^2)-8AC}{4x} + C^2 - \frac{1}{16}B^2\Big\}f = 0.$$

By imposing the constraints

$$4A(A-1)\sigma - \sigma m^2 + m(\sigma-k) + 2A(k+\sigma) = 0, \quad C^2 - \frac{1}{16}B^2 = 0;$$

we get the expected results

$$A = \frac{m}{2}, \quad A = \frac{1-m}{2} - \frac{k}{2\sigma}; \qquad C = -\frac{B}{4}, +\frac{B}{4}. \quad (4.3)$$

We assume below that $C = -\frac{B}{4}$ $(B > 0)$. The main equation becomes simpler

$$xf'' + \Big(2A + \frac{(k+\sigma)}{2\sigma} + 2Cx\Big)f'$$
$$+\Big[\frac{B(k-2m\sigma+\sigma)+4C(k+\sigma)}{8\sigma} - \frac{(k^2-\sigma^2)-8AC}{4}\Big]f = 0.$$

In the variable $2Cx = -z$, it reads as an equation of confluent hypergeometric type

$$zf'' + (c-z)f' - af = 0, \quad z\frac{d^2f''}{dz^2} + \Big(2A + \frac{k+\sigma}{2\sigma} - z\Big)\frac{df}{dz} - \Big(\frac{m}{2} + \frac{k^2-\sigma^2}{2B} + A\Big)f = 0. \quad (4.4)$$

By imposing the usual constraint to get polynomials

$$a = \frac{m}{2} + \frac{k^2-\sigma^2}{2B} + A = -n, \quad n = 0, 1, 2, 3, \ldots;$$

we find expressions for σ:

$$\sigma = \pm\sqrt{k^2 + (A + \frac{m}{2} + n)2B}\,. \tag{4.5}$$

Depending on the sign of m, we have two possibilities

$$\begin{aligned} m > 0, \quad \sigma = \pm\sqrt{k^2 + (m+n)2B}, \\ m < 0, \quad \sigma = \pm\sqrt{k^2 + (n + \frac{1 - k/\sigma}{2})2B}. \end{aligned} \tag{4.6}$$

The second equations determine σ in non-explicit form. Assuming that the solutions are regular at the point $z = 0$, we follow two possibilities depending on the sign of m:

$$(a) \qquad m > 0, \quad d_2 \sim z^{m/2} e^{-z/2} \Phi(-n, c, z), \quad c = m + \frac{1}{2} + \frac{k}{2\sigma}\ ; \tag{4.7}$$

$$(b) \qquad m < 0, \quad d_2 \sim z^{\frac{1-m}{2} - \frac{k}{2\sigma}} e^{-z/2} \Phi(-n, c', z), \quad c = \frac{3}{2} - m, \frac{1-m}{2} > \frac{k}{2\sigma}. \tag{4.8}$$

5 The case II, the free particle

Let us turn to the case II for a free particle, the system (2.11). Taking into account the expressions for the first order operators, we obtain the following equations

$$\Big[\frac{3}{2}\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+1)^2}{r^2}\Big) + \frac{\sigma}{(\sigma+2k)}\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+1)^2}{r^2}\Big) + \sigma(\sigma+k)\Big] c_1$$

$$+ \frac{3}{2}\Big(\frac{d^2}{dr^2} + \frac{(1-2m)}{r}\frac{d}{dr} + \frac{(m^2-1)}{r^2}\Big) c_3 = 0,$$

$$\Big[\frac{3}{2}\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-1)^2}{r^2}\Big) + \frac{\sigma}{(\sigma-2k)}\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-1)^2}{r^2}\Big) + \sigma(\sigma-k)\Big] c_3$$

$$+ \frac{3}{2}\Big(\frac{d^2}{dr^2} + \frac{(1+2m)}{r}\frac{d}{dr} + \frac{(m^2-1)}{r^2}\Big) c_1 = 0.$$

They can be transformed to other form

$$\Big[(3 + \frac{2\sigma}{\sigma+2k})\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+1)^2}{r^2}\Big) + 2\sigma(\sigma+k)\Big] c_1$$

$$+3\Big(\frac{d^2}{dr^2} + \frac{(1-2m)}{r}\frac{d}{dr} + \frac{(m^2-1)}{r^2}\Big) c_3 = 0, \tag{5.1}$$

$$\Big[(3 + \frac{2\sigma}{\sigma-2k})\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-1)^2}{r^2}\Big) + 2\sigma(\sigma-k)\Big] c_3$$

$$+3\Big(\frac{d^2}{dr^2} + \frac{(1+2m)}{r}\frac{d}{dr} + \frac{(m^2-1)}{r^2}\Big) c_1 = 0. \tag{5.2}$$

This system may be considered as linear with respect to the variables c_1'', c_3'':

$$\left|\begin{array}{cc} 3 + \frac{2\sigma}{2k+\sigma} & 3 \\ 3 & 3 - \frac{2\sigma}{2k-\sigma} \end{array}\right| \left|\begin{array}{c} c_1'' \\ c_3'' \end{array}\right| =$$

$$
= - \left| \begin{array}{c} \Big[(3+\frac{2\sigma}{\sigma+2k})(\frac{1}{r}\frac{d}{dr}-\frac{(m+1)^2}{r^2})+2\sigma(k+\sigma)\Big]c_1+3\Big(\frac{(1-2m)}{r}\frac{d}{dr}+\frac{(m^2-1)}{r^2}\Big)c_3 \\ 3\Big(\frac{(1+2m)}{r}\frac{d}{dr}+\frac{(m^2-1)}{r^2}\Big)c_1+\Big[(3+\frac{2\sigma}{\sigma-2k})\Big(\frac{1}{r}\frac{d}{dr}-\frac{(-m+1)^2}{r^2}\Big)-2\sigma(k-\sigma)\Big]c_3 \end{array} \right| ,
$$

evidently this system is symmetric under the change

$$
c_3 \Leftrightarrow c_1, \quad m \Leftrightarrow -m, \quad \sigma \Leftrightarrow -\sigma. \tag{5.3}
$$

Its solution has the form

$$
c_1'' = \frac{1}{8}\,\frac{(9\,m-8)\sigma^2 - 36\,k^2 m}{\sigma^2 r}\,c_1' +
$$
$$
+\Big[\frac{1}{8}\,\frac{8\,\sigma\,k^2+12\,k^3-9\,\sigma^2 k-5\,\sigma^3}{\sigma} - \frac{1}{8}\,\frac{(m+1)(-17\,\sigma^2 m-8\,\sigma^2+36\,k^2 m)}{\sigma^2 r^2}\Big]c_1
$$
$$
-\frac{3}{8}\,\frac{m(-4\,\sigma\,k+12\,k^2-5\,\sigma^2)}{\sigma^2 r}\,c_3' + \Big[\frac{3}{8}\,\frac{(2\,k^2+\sigma^2-3\,\sigma\,k)(\sigma+2\,k)}{\sigma} +
$$
$$
+\frac{3}{8}\,\frac{m(-4\,\sigma\,mk+12\,k^2 m+5\,\sigma^2+4\,\sigma\,k-12\,k^2-5\,\sigma^2 m)}{\sigma^2 r^2}\Big]c_3\,,
$$

$$
c_3'' = \frac{3}{8}\,\frac{m(4\,\sigma\,k+12\,k^2-5\,\sigma^2)}{\sigma^2 r}\,c_1' +
$$
$$
+\Big[-\frac{3}{8}\,\frac{(\sigma+k)(-\sigma^2+4\,k^2)}{\sigma} + \frac{3}{8}\,\frac{m(6\,k+5\,\sigma)(-\sigma+2\,k)(m+1)}{\sigma^2 r^2}\Big]c_1 +
$$
$$
+\frac{1}{8}\,\frac{36\,k^2 m-9\,\sigma^2 m-8\,\sigma^2}{\sigma^2 r}\,c_3' +
$$
$$
+\Big[-\frac{1}{8}\,\frac{-9\,\sigma^2 k+5\,\sigma^3-8\,\sigma\,k^2+12\,k^3}{\sigma} - \frac{1}{8}\,\frac{(-17\,\sigma^2 m+8\,\sigma^2+36\,k^2 m)(m-1)}{\sigma^2 r^2}\Big]c_3\,;
$$

the above symmetry may be noted in this formulas, as well. Shortly, we can write

$$
\begin{aligned}
\frac{d^2}{dr^2}c_1 &= K_1\frac{d}{dr}c_1 + (\frac{L_1}{r^2}+M_1)c_1 + (\frac{F_1}{r}\frac{d}{dr}+\frac{G_1}{r^2}+H_1)c_3,\\
\frac{d^2}{dr^2}c_3 &= K_3\frac{d}{dr}c_3 + (\frac{L_3}{r^2}+M_3)c_3 + (\frac{F_3}{r}\frac{d}{dr}+\frac{G_3}{r^2}+H_3)c_1.
\end{aligned} \tag{5.4}
$$

Let us eliminate from this system the variable c_3. To this end, first we write this function as $c_3 = \varphi\bar{c}_3$, and require that

$$
(\frac{F_1}{r}\frac{d}{dr}+\frac{G_1}{r^2}+H_1)\varphi\bar{c}_3 = \varphi\,\frac{F_1}{r}\frac{d}{dr}\,\bar{c}_3,
$$

whence it follows

$$
\frac{F_1}{r}\frac{\varphi'}{\varphi}\bar{c}_3 + \frac{F_1}{r}\bar{c}_3' + \frac{G_1}{r^2}\bar{c}_3 + H_1\bar{c}_3 = \frac{F_1}{r}\frac{d}{dr}\,\bar{c}_3,
$$

or, differently,

$$
\frac{F_1}{r}\bar{c}_3' + \Big(\frac{F_1}{r}\frac{\varphi'}{\varphi}+\frac{G_1}{r^2}+H_1\Big)\bar{c}_3 = \frac{F_1}{r}\frac{d}{dr}\,\bar{c}_3.
$$

We further derive the equation for determining φ:

$$
\frac{F_1}{r}\frac{\varphi'}{\varphi} = -\frac{G_1}{r^2} - H_1, \quad \Longrightarrow \frac{d}{dr}\ln\varphi = -\frac{G_1}{F_1}\frac{1}{r} - \frac{H_1}{F_1}r,
$$

and we obtain

$$
\ln\varphi = -\frac{G_1}{F_1}\ln r - \frac{H_1}{2F_1}r^2 \quad \Longrightarrow \quad \ln\Big(\varphi r^{G_1/F_1}\Big) = -\frac{H_1}{2F_1}r^2,
$$

whence it follows

$$\varphi r^{G_1/F_1} = e^{-\frac{H_1}{2F_1}r^2} \quad \Longrightarrow \quad \varphi = r^{-G_1/F_1} e^{-\frac{H_1}{2F_1}r^2}. \tag{5.5}$$

Therefore, the initial system reads

$$\begin{aligned} \frac{d^2}{dr^2}c_1 &= K_1 \frac{d}{dr}c_1 + (\frac{L_1}{r^2} + M_1)c_1 + \varphi\, \frac{F_1}{r}\frac{d}{dr}\,\bar{c}_3, \quad c_3 = \varphi \bar{c}_3, \\ \frac{d^2}{dr^2}\varphi\bar{c}_3 &= K_3 \frac{d}{dr}\varphi\bar{c}_3 + (\frac{L_3}{r^2} + M_3)\varphi\bar{c}_3 + (\frac{F_3}{r}\frac{d}{dr} + \frac{G_3}{r^2} + H_3)c_1. \end{aligned} \tag{5.6}$$

From eq.(5.6), we express the derivative $\frac{d}{dr}\,\bar{c}_3$:

$$\frac{d}{dr}\,\bar{c}_3 = \frac{1}{F_1}\frac{r}{\varphi}\left(\frac{d^2}{dr^2}c_1 - K_1\frac{d}{dr}c_1 - (\frac{L_1}{r^2} + M_1)c_1\right);$$

below we will need the second derivative as well,

$$\frac{d^2}{dr^2}\,\bar{c}_3 = \frac{d}{dr}\Big\{\frac{1}{F_1}\frac{r}{\varphi}\left(\frac{d^2}{dr^2}c_1 - K_1\frac{d}{dr}c_1 - (\frac{L_1}{r^2} + M_1)c_1\right)\Big\}.$$

The expression for the function c_3 itself is determined by integrating

$$\bar{c}_3 = \int \frac{1}{F_1}\frac{r}{\varphi}\left(\frac{d^2}{dr^2}c_1 - K_1\frac{d}{dr}c_1 - (\frac{L_1}{r^2} + M_1)c_1\right) dr.$$

Now we turn to the second equation from (5.6)

$$\frac{d^2}{dr^2}\varphi\bar{c}_3 = K_3 \frac{d}{dr}\varphi\bar{c}_3 + (\frac{L_3}{r^2} + M_3)\varphi\bar{c}_3 + (\frac{F_3}{r}\frac{d}{dr} + \frac{G_3}{r^2} + H_3)c_1.$$

Taking into account the two identities

$$\frac{d^2}{dr^2}\varphi\bar{c}_3 = \frac{d}{dr}\Big[\varphi'\bar{c}_3 + \varphi\bar{c}_3'\Big] = \varphi''\bar{c}_3 + 2\varphi'\bar{c}_3' + \varphi\bar{c}_3'',$$

$$K_3\frac{d}{dr}\varphi\bar{c}_3 = K_3\Big[\varphi'\bar{c}_3 + \varphi\bar{c}_3'\Big],$$

we transform the above equation to its alternative form

$$\varphi''\bar{c}_3 + 2\varphi'\bar{c}_3' + \varphi\bar{c}_3'' = K_3\varphi'\bar{c}_3 + K_3\varphi\bar{c}_3' + (\frac{L_3}{r^2} + M_3)\varphi\bar{c}_3 + \frac{F_3}{r}c_1' + \frac{G_3}{r^2}c_1 + H_3 c_1,$$

whence after re-grouping the terms, we obtain

$$\varphi\bar{c}_3'' + (2\varphi' - K_3\varphi)\bar{c}_3' + \left(\varphi'' - K_3\varphi' - (\frac{L_3}{r^2} + M_3)\varphi\right)\bar{c}_3 - \frac{F_3}{r}c_1' - \frac{G_3}{r^2}c_1 - H_3 c_1 = 0,$$

which is equivalent to

$$\Big[\bar{c}_3'' + (2\frac{\varphi'}{\varphi} - K_3)\bar{c}_3'\Big] + \left(\frac{\varphi''}{\varphi} - K_3\frac{\varphi'}{\varphi} - (\frac{L_3}{r^2} + M_3)\right)\bar{c}_3 - \Big[\frac{1}{\varphi}\frac{F_3}{r}c_1' + \frac{1}{\varphi}\frac{G_3}{r^2}c_1 + \frac{1}{\varphi}H_3 c_1\Big] = 0.$$

With the help of the temporary notation

$$\Delta(r) = \left(\frac{\varphi''}{\varphi} - K_3\frac{\varphi'}{\varphi} - (\frac{L_3}{r^2} + M_3)\right),$$

we re-write the last equation differently

$$\frac{1}{\Delta(r)}\Big[\bar{c}_3'' + (2\frac{\varphi'}{\varphi} - K_3)\bar{c}_3'\Big] + \bar{c}_3 - \frac{1}{\Delta(r)}\Big[\frac{1}{\varphi}\frac{F_3}{r}c_1' + \frac{1}{\varphi}\frac{G_3}{r^2}c_1 + \frac{1}{\varphi}H_3 c_1\Big] = 0.$$

After differentiating this equation, we obtain

$$\frac{d}{dr}\Big\{\frac{1}{\Delta(r)}\Big[\bar{c}_3'' + (2\frac{\varphi'}{\varphi} - K_3)\bar{c}_3'\Big]\Big\} + \bar{c}_3' - \frac{d}{dr}\Big\{\frac{1}{\Delta(r)}\Big[\frac{1}{\varphi}\frac{F_3}{r}c_1' + \frac{1}{\varphi}\frac{G_3}{r^2}c_1 + \frac{1}{\varphi}H_3 c_1\Big]\Big\} = 0, \tag{5.7}$$

which is the fourth order equation for the function $c_1(r)$. Recall that

$$\begin{aligned} \bar{c}_3' &= \frac{1}{F_1}\frac{r}{\varphi}\Big(\frac{d^2}{dr^2}c_1 - K_1\frac{d}{dr}c_1 - (\frac{L_1}{r^2} + M_1)c_1\Big), \\ \bar{c}_3'' &= \frac{d}{dr}\Big\{\frac{1}{F_1}\frac{r}{\varphi}\Big(\frac{d^2}{dr^2}c_1 - K_1\frac{d}{dr}c_1 - (\frac{L_1}{r^2} + M_1)c_1\Big)\Big\}. \end{aligned} \tag{5.8}$$

Now the equations of the 4-th order for two (initial) functions $c_1(r)$, $c_3(r)$ in explicit form read

$$\begin{aligned} &\frac{d^4 c_1}{dr^4} + \frac{2}{r}\frac{d^3 c_1}{dr^3} + \Big(\frac{5}{4}\sigma^2 - 2k^2 + \frac{-3 - 4m - 2m^2}{r^2}\Big)\frac{d^2 c_1}{dr^2} \\ &+\Big(\frac{1}{4}\frac{5\sigma^2 - 8k^2}{r} + \frac{3 + 4m + 2m^2}{r^3}\Big)\frac{d c_1}{dr} \\ &+\Big[-\frac{5}{4}\sigma^2 k^2 + \frac{1}{4}\sigma^4 + k^4 + \frac{1}{4}\frac{\left(-5\sigma^2 + 8k^2\right)(m+1)^2}{r^2} \\ &+\frac{(m+3)(m-1)(m+1)^2}{r^4}\Big]c_1 = 0, \end{aligned}$$

$$\begin{aligned} &\frac{d^4 c_3}{dr^4} + \frac{2}{r}\frac{d^3 c_3}{dr^3} + \Big(\frac{5}{4}\sigma^2 - 2k^2 + \frac{-3 + 4m - 2m^2}{r^2}\Big)\frac{d^2 c_3}{dr^2} \\ &+\Big(\frac{1}{4}\frac{5\sigma^2 - 8k^2}{r} + \frac{3 - 4m + 2m^2}{r^3}\Big)\frac{d c_3}{dr} \\ &+\Big[-\frac{5}{4}\sigma^2 k^2 + \frac{1}{4}\sigma^4 + k^4 + \frac{1}{4}\frac{\left(-5\sigma^2 + 8k^2\right)(m-1)^2}{r^2} \\ &+\frac{(m-3)(m+1)(m-1)^2}{r^4}\Big]c_3 = 0. \end{aligned}$$

Both equations may be factorized:
the first becomes

$$\Big\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \Big[-k^2 + \frac{1}{4}\sigma^2 - \frac{(m+1)^2}{r^2}\Big]\Big\}\Big\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \Big[-k^2 + \sigma^2 - \frac{(m+1)^2}{r^2}\Big]\Big\}c_1 = 0,$$

where the two factors are permutable;
the second:

$$\Big\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \Big[-k^2 + \frac{1}{4}\sigma^2 - \frac{(m-1)^2}{r^2}\Big]\Big\}\Big\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \Big[-k^2 + \sigma^2 - \frac{(m-1)^2}{r^2}\Big]\Big\}c_3 = 0,$$

where the two factors are permutable as well.
It suffices to solve the following two 2-nd order equations for c_1:

$$I, \quad \Big[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - k^2 + \sigma^2 - \frac{(m+1)^2}{r^2}\Big]c_1 = 0, \tag{5.9}$$

$$II, \quad \Big[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - k^2 + \frac{1}{4}\sigma^2 - \frac{(m+1)^2}{r^2}\Big] c_1 = 0\,; \tag{5.10}$$

They are solved in terms of Bessel functions

$$I, \quad x = i\sqrt{k^2 - \sigma^2}, \quad \Big[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{(m+1)^2}{x^2}\Big] c_1^I = 0,$$

$$II, \quad y = i\sqrt{k^2 - \sigma^2/4}, \quad \Big[\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} + 1 - \frac{(m+1)^2}{y^2}\Big] c_1^{II} = 0.$$

The solutions for c_3 are constructed in term of Bessel functions (with other indices)

$$I, \quad x = i\sqrt{k^2 - \sigma^2}, \quad \Big[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{(m-1)^2}{x^2}\Big] c_3^I = 0,$$

$$II, \quad y = i\sqrt{k^2 - \sigma^2/4}, \quad \Big[\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} + 1 - \frac{(m-1)^2}{y^2}\Big] c_3^{II} = 0.$$

Recall that these solutions are fixed up to some numerical multipliers.

6 The case II, the presence of magnetic field

In this case, we obtain the following system of equations for the variables c_1, c_3:

$$\Big\{\frac{3}{2}\Big[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m + eBr^2/2 + 1)^2}{r^2} + eB\Big]$$

$$+\frac{\sigma}{(\sigma + 2k)}\Big[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m + eBr^2/2 + 1)^2}{r^2} - eB\Big] + \sigma(\sigma + k)\Big\} c_1$$

$$+\frac{3}{2}\Big\{\frac{d^2}{dr^2} + \frac{1 - 2(m + eBr^2/2)}{r}\frac{d}{dr} + \frac{(m + eBr^2/2)^2 - 1}{r^2} - eB\Big\} c_3 = 0,$$

$$\Big\{\frac{3}{2}\Big[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m + eBr^2/2 - 1)^2}{r^2} - eB\Big]$$

$$+\frac{\sigma}{(\sigma - 2k)}\Big[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m + eBr^2/2 - 1)^2}{r^2} + eB\Big] + \sigma(\sigma - k)\Big\} c_3$$

$$+\frac{3}{2}\Big\{\frac{d^2}{dr^2} + \frac{1 + 2(m + eBr^2/2)}{r}\frac{d}{dr} + \frac{(m + eBr^2/2)^2 - 1}{r^2} + eB\Big\} c_1 = 0.$$

They can be transformed to their alternative form

$$\Big[\Big(3 + \frac{2\sigma}{\sigma + 2k}\Big)\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m + eBr^2/2 + 1)^2}{r^2}\Big) + \Big(3 - \frac{2\sigma}{\sigma + 2k}\Big) eB + 2\sigma(\sigma + k)\Big] c_1$$

$$+3\Big[\frac{d^2}{dr^2} + \frac{1 - 2(m + eBr^2/2)}{r}\frac{d}{dr} + \frac{(m + eBr^2/2)^2 - 1}{r^2} - eB\Big] c_3 = 0,$$

$$\Big[\Big(3 + \frac{2\sigma}{\sigma - 2k}\Big)\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m + eBr^2/2 - 1)^2}{r^2}\Big) + \Big(-3 + \frac{2\sigma}{\sigma - 2k}\Big) eB + 2\sigma(\sigma - k)\Big] c_3$$

$$+3\Big[\frac{d^2}{dr^2} + \frac{1 + 2(m + eBr^2/2)}{r}\frac{d}{dr} + \frac{(m + eBr^2/2)^2 - 1}{r^2} + eB\Big] c_1 = 0,$$

whence there follows the 4-th order equations for c_1 and c_3:

$$\frac{d^4c_1}{dr^4}+\frac{2}{r}\,\frac{d^3c_1}{dr^3}+\left(-\frac{1}{2}\,e^2B^2r^2+\frac{1}{4}\,\frac{5\,\sigma^3-8\,\sigma\,k^2-8\,\sigma\,meB+8\,\sigma\,eB+12\,keB}{\sigma}-\frac{2\,m^2+4\,m+3}{r^2}\right)\frac{d^2c_1}{dr^2}$$
$$+\left(-\frac{3}{2}\,e^2B^2r+\frac{1}{4}\,\frac{5\,\sigma^3-8\,\sigma\,k^2-8\,\sigma\,meB+8\,\sigma\,eB+12\,keB}{r\sigma}+\frac{2\,m^2+4\,m+3}{r^3}\right)\frac{dc_1}{dr}$$
$$+\left[\frac{1}{16}\,e^4B^4r^4-\frac{1}{16}\,\frac{e^2B^2\left(5\,\sigma^3-8\,\sigma\,k^2-8\,\sigma\,meB+8\,\sigma\,eB+12\,keB\right)r^2}{\sigma}\right.$$
$$-\frac{1}{4}\,\frac{(m+1)^2\left(5\,\sigma^3-8\,\sigma\,k^2-8\,\sigma\,meB+8\,\sigma\,eB+12\,keB\right)}{r^2\sigma}+\frac{(m+3)\,(m-1)\,(m+1)^2}{r^4}$$
$$-\frac{1}{4\sigma}\,\left(-\sigma^5+5\,\sigma^3Bem-5\,\sigma^3eB+5\,\sigma^3k^2-21\,\sigma^2keB-6\,\sigma\,m^2e^2B^2+\right.$$
$$\left.\left.+4\,\sigma\,e^2B^2m+16\,e^2B^2\sigma-8\,\sigma\,ek^2Bm+8\,\sigma\,ek^2B-4\,\sigma\,k^4+12\,ke^2B^2m-12\,ke^2B^2+12\,k^3eB\right)\right]c_1=0\,;$$

$$\frac{d^4c_3}{dr^4}+\frac{2}{r}\,\frac{d^3c_3}{dr^3}+\left(-\frac{1}{2}\,e^2B^2r^2+\frac{1}{4}\,\frac{5\,\sigma^3-8\,\sigma\,k^2-8\,\sigma\,meB-8\,\sigma\,eB+12\,keB}{\sigma}-\frac{2\,m^2-4\,m+3}{r^2}\right)\frac{d^2c_3}{dr^2}$$
$$+\left(-\frac{3}{2}\,e^2B^2r+\frac{1}{4}\,\frac{5\,\sigma^3-8\,\sigma\,k^2-8\,\sigma\,meB-8\,\sigma\,eB+12\,keB}{r\sigma}+\frac{2\,m^2-4\,m+3}{r^3}\right)\frac{dc_3}{dr}$$
$$+\left[\frac{1}{16}\,e^4B^4r^4-\frac{1}{16}\,\frac{e^2B^2\left(5\,\sigma^3-8\,\sigma\,k^2-8\,\sigma\,meB-8\,\sigma\,eB+12\,keB\right)r^2}{\sigma}-\right.$$
$$-\frac{1}{4}\,\frac{(m-1)^2\left(5\,\sigma^3-8\,\sigma\,k^2-8\,\sigma\,meB-8\,\sigma\,eB+12\,keB\right)}{r^2\sigma}+\frac{(m-3)\,(m+1)\,(m-1)^2}{r^4}$$
$$-\frac{1}{4\sigma}\,\left(-\sigma^5+5\,\sigma^3eBm+5\,\sigma^3eB+5\,\sigma^3k^2-21\,\sigma^2keB-6\,\sigma\,m^2e^2B^2\right.$$
$$\left.\left.-4\,\sigma\,e^2B^2m+16\,e^2B^2\sigma-8\,\sigma\,ek^2Bm-8\,\sigma\,ek^2B-4\,\sigma\,k^4+12\,ke^2B^2+12\,ke^2B^2m+12\,k^3eB\right)\right]c_3=0\,.$$

Both equations can be factorized.

For the variable c_1 we get

$$\left\{\frac{d^2}{dr^2}+\frac{1}{r}\,\frac{d}{dr}+\left[-\frac{1}{4}\,e^2B^2r^2-\frac{(m+1)^2}{r^2}+\right.\right.$$
$$\left.\left.+\frac{1}{8}\,\frac{\sigma\,\left(5\,\sigma^2-8\,k^2\right)-8\,eB\sigma\,(m-1)+12\,keB\pm 3\,\sqrt{\sigma^6-24\,keB\sigma^3+16\,e^2\,(2\,\sigma^2+k^2)\,B^2}}{\sigma}\right]\right\}\times$$
$$\times\left\{\frac{d^2}{dr^2}+\frac{1}{r}\,\frac{d}{dr}+\left[-\frac{1}{4}\,e^2B^2r^2-\frac{(m+1)^2}{r^2}+\right.\right.$$
$$\left.\left.+\frac{1}{8}\,\frac{\sigma\,\left(5\,\sigma^2-8\,k^2\right)-8\,eB\sigma\,(m-1)+12\,keB\mp 3\,\sqrt{\sigma^6-24\,keB\sigma^3+16\,e^2\,(2\,\sigma^2+k^2)\,B^2}}{\sigma}\right]\right\}c_1=0\,,$$

and for the variable c_3, we have a slightly different equation (note the change $m-1\Longrightarrow m+1$):

$$\left\{\frac{d^2}{dr^2}+\frac{1}{r}\,\frac{d}{dr}+\left[-\frac{1}{4}\,e^2B^2r^2-\frac{(m-1)^2}{r^2}+\right.\right.$$

$$+\frac{1}{8}\,\frac{\sigma\,\left(5\,\sigma^2-8\,k^2\right)-8\,eB\sigma\,\left(m+1\right)+12\,keB\pm3\,\sqrt{\sigma^6-24\,keB\sigma^3+16\,e^2\left(2\,\sigma^2+k^2\right)B^2}}{\sigma}\Bigg]\Bigg\}\times$$

$$\times\left\{\frac{d^2}{dr^2}+\frac{1}{r}\,\frac{d}{dr}+\left[-\frac{1}{4}\,e^2B^2r^2-\frac{(m-1)^2}{r^2}+\right.\right.$$

$$\left.\left.+\frac{1}{8}\,\frac{\sigma\,\left(5\,\sigma^2-8\,k^2\right)-8\,eB\sigma\,\left(m+1\right)+12\,keB\mp3\,\sqrt{\sigma^6-24\,keB\sigma^3+16\,e^2\left(2\,\sigma^2+k^2\right)B^2}}{\sigma}\right]\right\}c_3=0\,.$$

It suffices to examine one equation:

$$\left\{\frac{d^2}{dr^2}+\frac{1}{r}\,\frac{d}{dr}+\left[-\frac{1}{4}\,e^2B^2r^2-\frac{(m+1)^2}{r^2}\right.\right.$$

$$\left.\left.+\frac{1}{8}\,\frac{\sigma\,\left(5\,\sigma^2-8\,k^2\right)-8\,eB\sigma\,\left(m-1\right)+12\,keB\pm3\,\sqrt{\sigma^6-24\,keB\sigma^3+16\,e^2\left(2\,\sigma^2+k^2\right)B^2}}{\sigma}\right]\right\}c_1=0\,.$$

In the new variable

$$x=\frac{1}{2}\,eB\,r^2\,,$$

the last equation reads

$$\frac{d^2c_1}{dx^2}+\frac{1}{x}\,\frac{dc_1}{dx}+\left[-\frac{1}{4}-\frac{1}{4}\,\frac{(m+1)^2}{x^2}\right.$$

$$\left.+\frac{1}{16}\,\frac{\sigma\,\left(5\,\sigma^2-8\,k^2\right)-8\,eB\sigma\,\left(m-1\right)+12\,keB\pm3\,\sqrt{\sigma^6-24\,keB\sigma^3+16\,e^2\left(2\,\sigma^2+k^2\right)B^2}}{\sigma\,eBx}\right]c_1=0\,.$$

With the use of the substitution

$$c_1(x)=e^{Ax}x^CF(x)\,,$$

we get the following equation for $F)x)$:

$$x\frac{d^2F}{dx^2}+(2\,Ax+2\,C+1)\,\frac{dF}{dx}+\left\{\left(A^2-\frac{1}{4}\right)x+\frac{1}{4}\,\frac{4\,C^2-(m+1)^2}{x}\right.$$

$$+\frac{1}{16}\,\frac{1}{\sigma eB}\Big[16\,A\sigma(1+2\,C)\,eB+\sigma\,(5\,\sigma^2-8\,k^2)-8\,eB\sigma\,(m-1)+12\,keB$$

$$\left.\pm3\,\sqrt{\sigma^6-24\,keB\sigma^3+16\,e^2\left(2\,\sigma^2+k^2\right)B^2}\Big]\right\}F=0\,.$$

By imposing the following restrictions on parameters

$$A=-\frac{1}{2}\,,\quad C=\pm|\frac{m+1}{2}|$$

we simplify the above equation to the form

$$x\frac{d^2F}{dx^2}+(2\,C+1-x)\,\frac{dF}{dx}+$$

$$+\frac{1}{16}\,\frac{1}{\sigma eB}\Big[-8\sigma(1+2\,C)\,eB+\sigma\,(5\,\sigma^2-8\,k^2)-8\,eB\sigma\,(m-1)+12\,keB$$

$$\pm3\,\sqrt{\sigma^6-24\,keB\sigma^3+16\,e^2\left(2\,\sigma^2+k^2\right)B^2}\Big]F=0\,.$$

This is identified with the confluent hypergeometric equation, having the parameters

$$\alpha = -\frac{1}{16}\,\frac{1}{\sigma\, eB}\Big[-8\sigma(1+2\,C)\,eB + \sigma\,(5\,\sigma^2 - 8\,k^2) - 8\,eB\sigma\,(m-1) + 12\,keB$$

$$\pm 3\,\sqrt{\sigma^6 - 24\,keB\sigma^3 + 16e^2(2\sigma^2 + k^2)B^2}\Big], \qquad \gamma = 2\,C + 1\,.$$

The known polynomial condition gives the following quantization rule

$$\frac{1}{16}\,\frac{1}{\sigma\, eB}\Big[-8\sigma eB(1+2\,C) - 8\,eB\sigma\,(m-1) + \sigma\,(5\,\sigma^2 - 8\,k^2) + 12\,keB$$

$$\pm 3\,\sqrt{\sigma^6 - 24\,keB\sigma^3 + 16\,e^2\,(2\,\sigma^2 + k^2)\,B^2}\Big] = n, \quad n = 0, 1, 2, ...$$

or, differently, (we use $C = +|\frac{m+1}{2}|$):

$$\sigma(5\sigma^2 - 8k^2) + 12keB \pm 3\sqrt{\sigma^6 - 24keB\sigma^3 + 16e^2(2\sigma^2 + k^2)B^2} = 8\sigma eB(2n + m + |m+1|). \quad (6.1)$$

Let us change the notation

$$eB \Longrightarrow B, \quad 2n + m + |m+1| = N.$$

Then we have

$$\pm 3\sqrt{\sigma^6 - 24kB\sigma^3 + 16(2\sigma^2 + k^2)B^2} = 8\sigma BN - \sigma\,(5\sigma^2 - 8k^2) - 12kB \quad (6.2)$$

After squaring the above equation, we obtain

$$\sigma^6 - 5(BN + k^2)\sigma^4 + 21kB\sigma^3 + (4k^4 + 4B^2N^2 + 8BNk^2 - 18B^2)\sigma^2 - (12B^2Nk + 12k^3)\sigma = 0$$

or

$$\sigma^6 - 5\,\left(BN + k^2\right)\sigma^4 + 21\,kB\sigma^3 + \Big[4\,\left(BN + k^2\right)^2 - 18\,B^2\Big]\,\sigma^2 - 12\,kB\,\left(BN + k^2\right)\sigma = 0\,.$$

or (let $BN + k^2 = \gamma$, the root a $\sigma = 0$ is nonphysical)

$$\sigma^5 - 5\,\gamma\sigma^3 + 21\,kB\sigma^2 + \left[4\,\gamma^2 - 18\,B^2\right]\sigma - 12\,kB\gamma = 0\,. \quad (6.3)$$

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