Lengths of closed geodesics in conformally flat tori

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Abstract. We show that on a conformally flat n-torus T^n, there exist n closed geodesics such that they are linearly independent in the fundamental group $\mathbb{Z}^n$ of T^n and the product of their lengths is less than or equal to a dimensional constant times the volume of T^n. This generalizes a result of Loewner to high dimensions and multiple geodesics.

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1 Introduction

In Riemannian geometry, systole refers to the shortest length of non-contractible closed curves. Let M be a Riemannian manifold, the systole of M is defined by

$$sys(M) := \min_{0\neq[\gamma]\in\pi_1(M)} length(\gamma). \tag{1.1}$$

Systolic inequalities relate systoles to the area or volume of the manifold. In 1949, Loewner proved the systolic inequality for 2-torus. He showed that for any metric g on the torus T^2, we have

$$sys(T^2, g)^2 \leq \frac{2}{\sqrt{3}} area(T^2, g). \tag{1.2}$$

Following the same approach, Pu proved the systolic inequality for $\mathbb{RP}^2$ in [7]:

$$sys(\mathbb{RP}^2, g)^2 \leq \frac{\pi}{2} area(\mathbb{RP}^2, g). \tag{1.3}$$

Those constants are in fact optimal. In higher dimensions, major breakthroughs were due to Gromov who established the following systolic inequality for tori in any dimensions:

Theorem 1.1. *(Gromov,[2]) Let (T^n, g) be a Riemannian metric on the n-dimensional torus. Then there exists a dimensional constant $C(n)$ such that*

$$sys(T^n, g)^n \leq C(n) Vol(T^n, g),$$

where $Vol(T^n, g)$ is the volume of (T^n, g).

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In Gromov's original paper, the constant is taken to be $C(n) = [6(n+1)n^n\sqrt{(n+1)!}]^n$. Gromov observed that the systole and volume can be related via the filling radius, an important geometric concept invented by him. By showing that the systole is controlled by the filling radius and the filling radius is controlled by the volume, Gromov proved the systolic inequality on higher dimensional tori. Later, by estimating the volume of suitable balls in the torus, Guth was able to reprove Gromov's systolic inequality and improve the constant to $(8n)^n$ [4]. In the same paper, Guth mentioned the following conjecture:

Conjecture 1. *For any Riemannian torus* (T^n, g)*, we have*

$$sys(T^n, g)^n \leq C^n n^{n/2} Vol(T^n, g), \tag{1.4}$$

where $C > 0$ *is an absolute constant.*

He pointed out that the above conjecture is a corollary of a conjecture of Gromov on the volume of balls in a torus. Guth also mentioned that a randomly chosen flat n-torus with volume 1 has systole of order $\approx n^{1/2}$. Thus the constant $C^n n^{n/2}$ in the above conjecture has nearly optimal order.

It is worth mentioning that Gromov also established systolic inequalities for a more general class of manifolds, the so called "essential manifolds" which include all the aspherical manifolds. There are many other variants of the notion of systoles, such as relative systoles, higher dimensional systoles, shortest closed geodesics in a simple connected space, etc. For a detailed survey of systolic inequalities, one can read e.g. [1].

One possible way to generalize the classical systolic inequality is to consider multiple closed geodesics instead of just the shortest one. In this direction, Hebda studied the relationship between the primitive length spectrum and the area of the 2-tori [5][6]. The primitive length spectrum of a Riemannian manifold M refers to the set of lengths of shortest loops in primitive free homotopy classes of unoriented closed curves in M, counted with multiplicity. We may list the primitive length spectrum in an increasing sequence:

$$0 < l_1(g) \leq l_2(g) \leq l_3(g) \leq \cdots . \tag{1.5}$$

The main result of Hebda can be stated as follows:

Theorem 1.2. *For each positive integer* n *there exists a function* $\mu_n(x_1, ..., x_n)$ *such that for every Riemannian metric* g *on the two-dimensional torus* T^2 *with area* $a(g)$ *and first eigenvalue of Laplacian* $\lambda_1(g)$*, one has*

$$\mu_n(l_1^2(g), ..., l_n^2(g)) \leq a^2(g) \tag{1.6}$$

and

$$\lambda_1(g) \leq \frac{4\pi^2}{\mu_n(l_1(g), ..., l_n(g))}. \tag{1.7}$$

Equality holds in either inequality if and only if one of the following holds:

- *$n = 1$, and (T^2, g) is a flat equilateral torus with $l_1(g) = l_2(g) = l_3(g)$;*
- *$n = 2$, and (T^2, g) is a flat isosceles torus with $l_2(g) = l_3(g)$;*
- *$n \geq 3$, and (T^2, g) is a flat torus.*

Hebda also obtained the (rather complicated) expression of the functions μ_n by studying the primitive length spectrum of the flat tori. Inspired by Hebda's work, we find a generalization of Loewner's inequalities to higher dimensional tori with conformally flat metrics.

The main theorem in this paper is as follows.

Theorem 1.3. *Let (T^n, g) be a conformally flat Riemannian metric on the n-dimensional torus. Then there exist a dimensional constant $C(n) > 0$ and n distinct closed geodesics γ_i of length c_i $(1 \leq i \leq n)$ which represent n linearly independent elements in $\pi_1(T^n) \cong \mathbb{Z}^n$, such that the following holds:*

$$\prod_{i=1}^{n} c_i \leq C(n) Vol(g).$$

Here $Vol(g)$ is the volume of (T^n, g). Moreover, $C(n)$ can be taken to be $2^n / Vol(B^n)$, where $Vol(B^n)$ is the volume of the n-dimensional unit ball.

As a corollary, we confirm Conjecture 1 for conformally flat tori.

Corollary 1.4. *For any conformally flat torus (T^n, g), we have*

$$sys(T^n, g)^n \leq \left(\frac{2n}{e\pi}\right)^{n/2} Vol(T^n, g). \tag{1.8}$$

The rest of the paper is organized as follows. In the Preliminary, we will state some facts from the lattice theory and Lie group theory. In Section 3 we will prove the main theorem and the corollary. We first prove it for flat tori and then finish the proof by reducing conformally flat metrics to flat metrics.

2 Preliminaries

Definition 2.1. Let $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ be n linearly independent vectors in $\mathbb{R}^n$. We call

$$\Lambda := \left\{ \sum_{i=1}^{n} k_i \vec{a}_i : k_i \in \mathbb{Z} \right\} \tag{2.1}$$

a *lattice of rank n*, and call $\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}$ a *basis* of Λ.

When $n \geq 2$, a lattice Λ of rank n has infinitely many bases. Suppose that both $\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}$ and $\{\vec{b}_1, \vec{b}_2, ..., \vec{b}_n\}$ are bases of Λ, and let A and B denote their matrices, respectively. Then there is an $n \times n$ integer unimodular matrix U such that

$$B = UA. \tag{2.2}$$

Therefore, although the number of the bases of a given lattice is infinite, the absolute value of the determinants of their corresponding matrices is a constant. It is called the *determinant of the lattice*, and is denoted by $det(\Lambda)$.

In fact we have

$$det(\Lambda) = vol\left(\left\{\sum_{i=1}^{n} \lambda_i \vec{a}_i :\ 0 \leq \lambda_i \leq 1\right\}\right). \tag{2.3}$$

We also recall the notion of "successive minima" in the lattice theory. Let Λ denote a lattice in $\mathbb{R}^n$ and B denote the unit n-ball centered at the origin. Let λ_1 be the smallest positive number such that $\lambda_1 B \cap \Lambda$ contains one pair of nonzero vectors. Similarly, let λ_i be the smallest positive number such that $\lambda_i B \cap \Lambda$ contains i pairs of linearly independent vectors. Then we call λ_i the *i-th successive minimum* with respect to the lattice Λ.

Now we are ready to state Minkowski's second theorem:

Theorem 2.1. *Let λ_i be the i-th successive minimum with respect to the lattice Λ. Then we have*

$$\frac{2^n}{n! Vol(B)} det(\Lambda) \leq \prod_{i=1}^{n} \lambda_i \leq \frac{2^n}{Vol(B)} det(\Lambda). \tag{2.4}$$

There are many proofs of this theorem. One concise proof can be found in [3], pp. 58–61. We remark that Minkowski's second theorem holds for any Minkowski norm in $\mathbb{R}^n$, but we only need the version for Euclidean norm. However, the constant $\frac{2^n}{Vol(B)}$ is not sharp, and it is quite challenging to determine the sharp bound.

Finally we recall the notion of Haar measure on topological groups. Any locally compact Hausdorff topological group has a unique (up to scalars) nonzero left invariant measure which is finite on compact sets. If the group is Abelian or compact, then this measure is also right invariant and is known as the *Haar measure*. In particular, the torus T^n, viewed as an Abelian Lie group, admits the Haar measure, which can be taken as the volume measure with respect to a flat metric on T^n. We also note that a metric on T^n is flat if and only if it is homogeneous, that is, invariant under left translations.

3 Proof of the main theorem

Recall the main theorem we want to prove:

Theorem 3.1. *Let (T^n, g) be a conformally flat Riemannian metric on the n-dimensional torus. Then there exist a dimensional constant $C(n) > 0$ and n distinct closed geodesics γ_i of length c_i $(1 \leq i \leq n)$ which represent n linearly independent elements in $\pi_1(T^n) \cong \mathbb{Z}^n$, such that the following holds:*

$$\prod_{i=1}^{n} c_i \leq C(n) Vol(g).$$

Here $Vol(g)$ is the volume of (T^n, g). Moreover, $C(n)$ can be taken to be $2^n/Vol(B^n)$, where $Vol(B^n)$ is the volume of the n-dimensional unit ball.

We divide the proof into two parts. We first prove it for flat tori. Any flat torus is isometric to $\mathbb{R}^n/\Lambda$ for a certain lattice Λ in $\mathbb{R}^n$. Any primitive closed geodesic of $\mathbb{R}^n/\Lambda$ passing through the origin is represented by a nonzero vector in Λ. Let $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ denote the n vectors in Λ realizing the successive minima. By definition they are linearly independent over $\mathbb{Z}$, and thus they represent n distinct closed geodesics which are $\mathbb{Z}$-linearly independent in $\pi_1(T^n)$. By Minkowski's second theorem, we have

$$\prod_{i=1}^{n} ||\vec{v}_i|| \leq \frac{2^n}{Vol(B)} det(\Lambda). \tag{3.1}$$

Also notice $det(\Lambda) = Vol(\mathbb{R}^n/\Lambda)$. Thus we have established the desired inequality for flat tori.

In the next step we study the general case. Let $g = f(p)^2 g_0$ be a conformally flat metric on T^n, where p is a point on T^n, $f(p)$ is the conformal factor and g_0 is a flat metric on T^n. Let's consider the "averaged metric" $h = \bar{f}(p)^2 g_0$, where

$$\bar{f}(p) = \int_{T^n} f(\sigma p) d\mu(\sigma), \tag{3.2}$$

σ ranges over translations on T^n and μ is the Haar measure on T^n (viewed as the group of translations). Here we normalize μ so that $\int_{T^n} d\mu(\sigma) = 1$. The metric h is flat since $\bar{f}$ is constant (by the definition of Haar measure). Thus we can find n closed geodesics γ_i^0 $(1 \leq i \leq n)$ with respect to h satisfying the desired inequality. We choose the g-minimizers γ_i in the homotopy classes of γ_i^0. We claim that these γ_i's satisfy

$$\prod_{i=1}^{n} L(\gamma_i) \leq C(n) Vol(g),$$

where $L(\gamma_i)$ is the g-length of γ_i, which would finish the proof.

We compare the volume and lengths of curves for the metrics g and h. We first show the following lemma:

Lemma 3.2. *The volume of h is less than or equal to the volume of g.*

Proof. Let $dvol_0$ denote the volume element for g_0. Then

$$Vol(g) = \int_{T^n} f(p)^n dvol_0(p), \quad Vol(h) = \int_{T^n} \Big(\int_{T^n} f(\sigma p) d\mu(\sigma) \Big)^n dvol_0(p). \tag{3.3}$$

By Holder's inequality:

$$\int_{T^n} f(\sigma p) d\mu(\sigma) \leq \Big(\int_{T^n} f(\sigma p)^n d\mu(\sigma) \Big)^{1/n} \Big(\int_{T^n} 1 d\mu(\sigma) \Big)^{(n-1)/n} = \Big(\int_{T^n} f(\sigma p)^n d\mu(\sigma) \Big)^{1/n}. \tag{3.4}$$

Thus we have

$$\begin{aligned} Vol(h) &\le \int_{T^n}\int_{T^n} f(\sigma p)^n d\mu(\sigma) dvol_0(p) \\ &= \int_{T^n}\int_{T^n} f(\sigma p)^n dvol_0(p)\ d\mu(\sigma) \\ &= \int_{T^n}\int_{T^n} f(p)^n dvol_0(p)\ d\mu(\sigma) \\ &= \int_{T^n} Vol(g) d\mu(\sigma) \\ &= Vol(g). \end{aligned} \tag{3.5}$$

Here we used Fubini's theorem to interchange the order of integration, and $\sigma^*(dvol_0) = dvol_0$ since translations are isometries for flat metrics. □

Next, let $h(\gamma)$ and $g(\gamma)$ denote the length of the curve γ in the metric h and g, respectively. We claim $h(\gamma_i^0) \ge g(\gamma_i)$, where γ_i and γ_i^0 are defined at the beginning of this section.

Indeed, by the construction of h we have

$$h(\gamma_i^0) = \int_{T^n} g(\sigma\gamma_i^0) d\mu(\sigma). \tag{3.6}$$

Here $\sigma\gamma_i^0$ is the image of γ_i^0 under the group action of σ. Note that each $\sigma\gamma_i^0$ is homotopic to γ_i^0 since σ lies in a connected group T^n. Thus by the definition of γ_i, $g(\sigma\gamma_i^0) \ge g(\gamma_i)$. Thus $h(\gamma_i^0) \ge g(\gamma_i)$.

Now we have come to the end of the proof. For the flat metric h, we have proven $\prod_{i=1}^n h(\gamma_i^0) \le C(n)Vol(h)$ with $C(n)$ given by Minkowski's second theorem in the previous section. In this section, we proved $Vol(g) \ge Vol(h)$ and $g(\gamma_i) \le h(\gamma_i^0)$. Combining all these, we conclude the proof of the main theorem.

Proof of Corollary 1.4. Since $sys(T^n, g)^n \le \prod_{i=1}^n c_i$, we only need to calculate the growth rate of $C(n) = 2^n/Vol(B^n)$. It is well known that

$$Vol(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}, \tag{3.7}$$

where $\Gamma(\cdot)$ is the Gamma function. By Stirling's formula: $\Gamma(\frac{n}{2}+1) \approx (\frac{n}{2e})^{\frac{n}{2}}$. Substituting it into the expression of $C(n)$, we obtain $C(n) \approx (\frac{2n}{e\pi})^{\frac{n}{2}}$, thus proving the corollary.

Remark 3.1. The construction of h is more like "averaging $g^{1/2}$" instead of averaging the metric g itself. We hope to find a way to average more general classes of metrics so that the above argument goes through. We also hope to establish similar inequalities on other types of manifolds. It is also interesting to investigate the optimal constant for each conformal class of flat metrics.

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