A note on the compatibility of $G_2$-structures with symplectic structures

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Abstract. In this paper we study the relationship between $G_2$-structures and 8-dimensional symplectic structures. We introduce the notion of compatibility of these structures. It is shown that a 7-manifold with $G_2$ structure can be embedded into an 8-dimensional symplectic manifold and with additional conditions, this symplectic structure can be chosen compatible with $G_2$-structure.

Key words: $G_2$-structure; symplectic structure.

1 Introduction

In the classification of Riemannian holonomy groups due to Berger, there are two exceptional cases: $G_2$ and $Spin(7)$. In this paper we concern with manifolds of exceptional holonomy group $G_2$. The compact, simple and simply connected Lie group $G_2$ can be defined as the group of linear transformations of $\mathbb{R}^7$ that preserves the Euclidean metric and a vector cross product. A $G_2$-structure (or an almost $G_2$-structure) on a 7-dimensional manifold $Q$ is a nondegenerate three form $\Omega$ on it. A $G_2$-structure induces a unique Riemannian metric $g$ on $Q$. If furthermore $\text{Hol}(g) \subseteq G_2$, then $Q$ is called a $G_2$-manifold.

The geometry of $G_2$-manifolds has been studied extensively in several papers ([8],[4],[5],[11]). Akbulut and Salur in [1] studied the relationship between Calabi-Yau geometry and $G_2$ geometry. By definition a Calabi-Yau manifold is a Kähler manifold $X$ with $c_1(X) = 0$(of course there are some inequivalent definitions ). Thus a Calabi-Yau manifold is a special symplectic manifold. On the other hand the relation between symplectic geometry and contact geometry is obvious. So it is natural to expect a connection between $G_2$ geometry from one hand and symplectic geometry and contact geometry from another hand. In [2] the relationship between $G_2$ geometry and contact geometry has been studied. The relationship between $G_2$ geometry and symplectic geometry emerged in [9] for the first time. In [9], by using methods of spin geometry, Fernandez and Gray showed that $T^*Q \times \mathbb{R}$ admits a closed $G_2$-structure,
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when $Q$ is an oriented 3-dimensional manifold and in [7], Cho and Salur computed this $G_2$-structure as $\Omega = \text{Re}\Theta + \omega \wedge dt$, where $\Theta$ is a certain complex valued 3-form and $\omega$ is the standard symplectic form on $T^*Q$.

In this paper we investigate the connection between symplectic structures and $G_2$-structures. The paper is organized as follows:

In section 2 we present some preliminaries. In section 3, the compatibility of symplectic structures with $G_2$-structures and its relation with compatibility of contact structures and $G_2$-structures, will be study. In particular the following theorems will be proved.

**Theorem.** Let $(Q, \alpha)$ be a 7-dimensional contact manifold and $\Omega$ be a $G_2$-structure on $Q$ compatible with $\alpha$. Then $\Omega$ is compatible with symplectic form $\omega = d(e^\theta \alpha)$ on $M = Q \times \mathbb{R}$, where $\theta$ denotes the coordinate on $\mathbb{R}$.

**Theorem.** Let $(Q, \Omega)$ be a hypersurface of symplectic manifold $(M, \omega)$ and $\omega$ is compatible with $\Omega$. If furthermore $Q$ is of contact type then $\Omega$ is compatible with contact structure of $Q$.

In section 4 the existence of symplectic structures on $Q \times \mathbb{R}$ and $Q \times S^1$ is discussed, when $Q$ is a 7-manifold with $G_2$-structure. The main results of this section are as follows:

**Theorem.** Let $Q$ be a 7-dimensional manifold with a $G_2$-structure $\Omega$. Then $M = Q \times \mathbb{R}$ admits an almost symplectic structure compatible with $\Omega$. The same statement is true for $M = Q \times S^1$.

**Theorem.** Let $Q$ be a connected 7-dimensional manifold with a $G_2$-structure. Then $M = Q \times \mathbb{R}$ is a symplectic manifold. The same statement is true for $M = Q \times S^1$, when $Q$ is furthermore noncompact.

**Theorem.** In previous Theorem, if $R$ is a vector field on $Q$ such that $\iota_R \varphi$ is exact, then $Q \times \mathbb{R}$ and $Q \times S^1$ admits a symplectic structure compatible with $\varphi$.

## 2 Preliminaries

### 2.1 $G_2$-structures

In this section $V$ is a finite dimensional real vector space and $(\cdot, \cdot)$ is an inner product on $V$.

**Definition 2.1.** A skew symmetric bilinear map

$$V \times V \to V : (u, v) \mapsto u \times v$$

is called a cross product if it satisfies

$$\langle u \times v, u \rangle = \langle u \times v, v \rangle = 0,$$

$$|u \times v|^2 = |u|^2 |v|^2 - \langle u, v \rangle^2$$

for all $u, v \in V$.

It is well known that if $V$ admits a non vanishing cross product, then dimension of $V$ is 3 or 7.
Lemma 2.1. If $\times$ be a cross product on $V$, then the map $\Omega : V \times V \times V \to \mathbb{R}$, defined by
\[
\Omega(u, v, w) = \langle u \times v, w \rangle,
\]
is an alternating 3-form the so called the associative calibration of $V$.

Definition 2.2. Let $V$ be a finite dimensional real vector space. A 3-form $\Omega \in \Lambda^3 V^*$ is called nondegenerate if, $\iota_v \Omega = 0$ implies that $v = 0$. An inner product on $V$ is called compatible with $\Omega$ if the map (2.1) defined by (2.2) is a cross product.

Theorem 2.2. Let $V$ be a 7-dimensional real vector space and $\Omega \in \Lambda^3 V^*$. Then:
(i) $\Omega$ is nondegenerate if and only if it admits a compatible inner product.
(ii) The inner product in (i), if it exists, is uniquely determined by $\Omega$.
(iii) If $\Omega_1, \Omega_2 \in \Lambda^3 V^*$ are nondegenerate, then there is an automorphism $g : V \to V$ such that $g^* \Omega_1 = \Omega_2$.
(iv) If $\Omega$ is compatible with the inner product $\langle \cdot, \cdot \rangle$, then there is an orientation on $V$ such that the associated volume form $d\text{vol} \in \Lambda^7 V^*$ satisfies
\[
\iota_u \Omega \wedge \iota_v \Omega \wedge \Omega = 6 \langle u, v \rangle d\text{vol}
\]
for all $u, v \in V$.

Example 2.3. Identify $\mathbb{R}^7$ with $\text{Im} O$ of imaginary part of octonions. then for $u, v \in \mathbb{R}^7$
\[
u \times v = \text{im} uv
\]
defines a cross product with respect to the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^7$. The associated calibration $\Omega_0$ reads
\[
\Omega_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{275} - e^{347} - e^{356}
\]
where $e^{ijk} = dx_i \wedge dx_j \wedge dx_k$.

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space endowed with a cross product $\times$ and $\Omega$ be it’s associated calibration. The sub group of $G\ell(V)$ that preserve $\Omega$ is denoted by
\[
G(V, \Omega) = \{g \in G\ell(V) : g^* \Omega = \Omega \}.
\]
The group $G(\mathbb{R}^7, \Omega_0)$ will be denoted simply by $G_2$. According Theorem 2.4(iii), for an arbitrary nondegenerate 3-form $\Omega$ on a 7-dimensional vector space $V$, The group $G(V, \Omega)$ is isomorphic to $G_2$.

Definition 2.4. A nondegenerate 3-form $\Omega$ on a smooth 7-dimensional manifold $Q$ is called a $G_2$-structure(or an almost $G_2$-structure).

Remark 2.5. By Theorem 2.4(i, iv) a $G_2$ structure $\Omega$ on $Q$ induces a unique Riemannian metric and a unique orientation on $Q$. Thus each tangent space $T_p Q$ of $Q$ admits a cross product defined by (2.2).

For more information about $G_2$-structures we refer to [13] and [10].
2.2 Almost symplectic structures and Gromov’s Theorem

Let $M$ be a $2n$-dimensional smooth manifold. A nondegenerate two form $\omega$ on $M$ is called an almost symplectic structure. If furthermore $\omega$ is closed, then $\omega$ is called a symplectic structure on $M$. It is well known that an almost symplectic manifold $(M, \omega)$ admits almost complex structures $J$ tamed by $\omega$, i.e., $\omega(v, Jv) > 0$ for all nonzero $v$ in $TM$. The space of such almost complex structures is contractible. The following theorem, due to Gromov, states that an almost symplectic structure is homotopic to a symplectic structure. For a proof of this theorem we refer to Theorem 7.34 of [12].

**Theorem 2.3. (Gromov’s Theorem)** Let $M$ be an open $2n$ dimensional manifold. Let $\tau$ be an almost symplectic structure on $M$ and $a \in H^2(M, \mathbb{R})$. There exists a family of almost symplectic forms $\tau_t$ on $M$ such that $\tau_0 = \tau$ and $\tau_1$ is a symplectic form that represents the class $a$.

2.3 Almost contact structures

Let $M$ be an $(2n+1)$ dimensional smooth manifold. An almost contact structure on $M$ is a triple $(J, R, \alpha)$ consists of a field $J$ of endomorphisms of the tangent bundle, a vector field $R$ and a 1-form $\alpha$ satisfying

1) $\alpha(R) = 1$,
2) $J^2(X) = -X + \alpha(X)R$,

for all $X$ in $TM$.

Let $(J, R, \alpha)$ be an almost contact structure on $M$. A Riemannian metric $g$ on $M$ is called a compatible metric if

$$g(Ju, Jv) = g(u, v) - \alpha(u)\alpha(v),$$

for all $u, v$ in $TM$. An **almost contact metric structure** on $M$ is a quadruple $(J, R, \alpha, g)$, where $(J, R, \alpha)$ is an almost contact symplectic structure and $g$ is a compatible metric.

It is well known that every manifold with an almost contact structure admits a compatible metric. For more details we refer to [3].

3 Compatibility of $G_2$-structures and symplectic structures

In [2], two kind of compatibility of contact structures and $G_2$ structures on a manifold, when both of them exist, has been defined. Here we need one of them, the so called $A$-compatibility, which we simply call it compatible.

**Definition 3.1.** Let $\Omega$ be a $G_2$-structure on 7-dimensional manifold $Q$. A contact structure $\xi$ on $Q$ is said to be compatible with $\Omega$ if there exist a vector field $R$ on $Q$, a contact form $\alpha$ for $\xi$ and a nonzero function $f : Q \to \mathbb{R}$ such that $d\alpha = \iota_R \Omega$ and $fR$ is the Reeb vector field of a contact form for $\xi$.

In this section we consider a hypersurface of a symplectic 8-dimensional manifold, which admits a $G_2$-structure. We want to know how these two structures are related.
Definition 3.2. Let $(M,\omega)$ be an eight dimensional (almost) symplectic manifold and $Q$ be a hypersurface of $M$ with $G_2$-structure $\Omega$. The (almost) symplectic form $\omega$ is called compatible with $\Omega$ if there is a vector field $R : Q \to TQ$ satisfying

$$j^*(\omega) = \iota_R \Omega,$$

where $j : Q \to M$ is the inclusion map.

The following example explains the motivation of this definition.

Example 3.3. Let $(x_1,\ldots,x_8)$ denotes the coordinates on $\mathbb{R}^8$ and consider the symplectic form $\omega$ on $\mathbb{R}^8$ as follows:

$$\omega = dx_1 \wedge dx_8 + dx_2 \wedge dx_3 + dx_4 \wedge dx_5 + dx_6 \wedge dx_7.$$

Consider $\mathbb{R}^7$ as a hypersurface in $\mathbb{R}^8$ with coordinates $(x_1,\ldots,x_7)$. Let $\Omega_0$ be the standard $G_2$-structure on $\mathbb{R}^7$. If $R = \frac{\partial}{\partial x_1}$, we have

$$\iota_R \Omega_0 = dx_2 \wedge dx_3 + dx_4 \wedge dx_5 + dx_6 \wedge dx_7 = j^*(\omega),$$

where $j : \mathbb{R}^7 \to \mathbb{R}^8$ is defined by $j(x_1,\ldots,x_7) = (x_1,\ldots,x_7,0)$.

Theorem 3.1. Let $(Q,\alpha)$ be a 7-dimensional contact manifold and $\Omega$ be a $G_2$-structure on $Q$ compatible with $\alpha$. Then $\Omega$ is compatible with symplectic form $\omega = d(e^\theta \alpha)$ on $M = Q \times \mathbb{R}$, where $\theta$ denotes the coordinate on $\mathbb{R}$.

Proof. By assumption, there is a vector field $R$ on $Q$ such that $\iota_R \Omega = d\alpha$. but $d\alpha = j^*(\omega)$. 

Example 3.4. Let $Q$ be a 3-dimensional oriented Riemannian manifold. Consider the coordinates $(x_1,x_2,x_3,y_1,y_2,y_3)$ on the cotangent bundle $T^*Q$. Assume $\omega = -d\lambda_{\text{can}}$ be the standard symplectic form on $T^*Q$, where $\lambda_{\text{can}} = \sum y_i dx_i$ is the canonical 1-form on $T^*Q$. If $t$ denotes the coordinate on $\mathbb{R}$, define the 3-form $\Omega$ on $T^*Q$ by

$$\Omega = Re(\Theta) + dt \wedge \omega,$$

where $\Theta = (dx_1 + idy_1) \wedge (dx_2 + idy_2 \wedge (dx_3 + idy_3))$ is a complex valued $(3,0)$ form on $T^*Q$. In [7] it is shown that $\Omega$ is a $G_2$-structure on $T^*Q \times \mathbb{R}$. On the other hand it is easy to see that $\alpha = dt + \lambda_{\text{can}}$ defines a contact structure on $T^*Q \times \mathbb{R}$ with the Reeb field $\frac{\partial}{\partial t}$. This contact structure is compatible with $\Omega$. Thus $\Omega$ is compatible with symplectic structure $\omega = d(e^\theta \alpha)$ on $M = T^*Q \times \mathbb{R}^2$.

Definition 3.5. A compact and orientable hypersurface $Q$ of a symplectic manifold $(M,\omega)$ is called of contact type if there exists a 1-form $\alpha$ on $Q$ satisfying

1) $d\alpha = j^*(\omega),$

2) $\alpha(\xi) \neq 0$ for $0 \neq \xi \in \mathcal{L}_Q$,

where $j : Q \to M$ is the inclusion map and $\mathcal{L}_Q$ is the canonical line bundle of $Q$.

Theorem 3.2. Let $(Q,\Omega)$ be a hypersurface of symplectic manifold $(M,\omega)$ and $\omega$ is compatible with $\Omega$. If furthermore $Q$ is of contact type then $\Omega$ is compatible with contact structure of $Q$.
Proof. Since $Q$ is of contact type then there exists a 1-form $\alpha$ on $Q$ such that $d\alpha = j^*(\omega)$ and since $\omega$ is compatible with $\Omega$, there is a vector field $R$ on $Q$ such that $\iota_R \Omega = j^*(\omega) = d\alpha$.

Moreover $\iota_R d\alpha = 0$ and since the restriction of $d\alpha$ to $Ker \alpha$ is symplectic, then $\alpha(R) \neq 0$ and so $fR$ is the Reeb field of $\alpha$, where $f = \frac{1}{\alpha(R)}$.

\[ \square \]

Theorem 3.3. Let $(M, \omega)$ be an 8-dimensional symplectic manifold and $Q \subset M$ be a closed (i.e. compact and without boundary) hypersurface of $M$ with a closed $G_2$-structure $\Omega$. If $H^1(Q) = 0$, then $\omega$ is not compatible with $\Omega$.

Proof. Since $j^*(\omega)$ is closed and $H^1(Q) = 0$, then $j^*(\omega) = d\alpha$ for some 1-form $\alpha$ on $Q$. If $\omega$ is compatible with $\Omega$, then there is a vector field $R$ on $Q$ such that $\iota_R \omega = d\alpha$.

Thus $g(R, R)_{\Omega} = \iota_R(\omega) \wedge (\iota_R \omega) \wedge \Omega$ is exact and hence $\int_Q g(R, R)_{\Omega} = 0$, which is a contradiction.

\[ \square \]

4 $G_2$-structures and existence of symplectic structures

In this section we show that if $Q$ admits a $G_2$-structure, then $Q \times \mathbb{R}$ and $Q \times S^1$ admit a symplectic structure, and hence $Q$ can be embedded in a symplectic manifold.

Lemma 4.1. Let $(2n+1)$-dimensional manifold $Q$ admits an almost contact structure. Then $Q \times \mathbb{R}$ and $Q \times S^1$ admit an almost complex structure.

Proof. Let $(J, R, \alpha)$ be an almost contact structure on $Q$ and $g$ be a Riemannian compatible metric. Let $D$ be the sub bundle of $TQ$ generated by $R$ and $H$ be the orthogonal complement of $D$ with respect to $g$. Thus $TQ = H \oplus D \oplus TR$. So, for $X \in T(Q \times \mathbb{R})$, $X$ splits as $X = X_H + bR + a\frac{\partial}{\partial \theta}$, where $X_H \in H$ and $\theta$ denotes the coordinate on $\mathbb{R}$. Define the automorphism $J' : T(Q \times \mathbb{R}) \to T(Q \times \mathbb{R})$ by

\[ J'(X_H + bR + a\frac{\partial}{\partial \theta}) = J(X_H) + aR - b\frac{\partial}{\partial \theta}. \]

It is easy to see that $J'$ is an almost complex structure on $Q \times \mathbb{R}$.

\[ \square \]

Theorem 4.2. Let $Q$ be a 7-dimensional manifold with a $G_2$-structure $\Omega$. Then $M = Q \times \mathbb{R}$ admits an almost symplectic structure compatible with $\Omega$. The same statement is true for $M = Q \times S^1$.

Proof. Let $g_\Omega$ and $\times \Omega$ denotes, respectively, the Reimannian metric and cross product associative to $\Omega$ on $Q$. Choose a nonzero vector field $R$ on $Q$ with $g_\Omega(R, R) = 1$ and define the 1-form $\alpha$ and endomorphism $J_R : TQ \to TQ$ by

\[ \alpha_R(u) = g_\Omega(R, u), \]

\[ J_R(u) = R \times \Omega u. \]
The quadruple \((J_R, R, \alpha_R, g_\Omega)\) defines an almost contact metric structure on \(Q\). Let \(J\) be the almost complex structure induced by \(J_R\) on \(Q \times \mathbb{R}\). Let \(\theta\) denotes the coordinate on \(\mathbb{R}\) and define the Riemannian metric \(g\) and the two form \(\omega\) on \(M = Q \times \mathbb{R}\) by

\[
g = g_\Omega + d\theta^2,
\]

\[
\omega(u, v) = g(Ju, v).
\]

\(\omega\) is an almost symplectic structure on \(M\) and for \(u, v\) in \(TQ\) we have

\[
\omega(u, v) = g(Ju, v) = g(R \times u, v) = \Omega(R, u, v).
\]

Thus \(\omega\) and \(\Omega\) are compatible. \(\square\)

**Corollary 4.3.** Every connected 7-dimensional manifold with \(G_2\)-structure can be embedded in an 8-dimensional symplectic manifold.

**Proof.** Let \(Q\) be a 7-dimensional manifold with \(G_2\)-structure. By Theorem 4.2, \(Q \times \mathbb{R}\) and \(Q \times S^1\) admit an almost symplectic structure. Now Gromov’s Theorem follows the assertion. \(\square\)

As in Corollary 4.3 mentioned if \(Q\) admits a \(G_2\)-structure, then \(Q \times \mathbb{R}\) and \(Q \times S^1\) (if \(Q\) is not compact) admit a symplectic structure. It seems to be an open question wether or not every \(G_2\)-structure is compatible with a symplectic structure. We could not find counterexample but also did not see how to prove it.

**Definition 4.1.** (see[6]) Let \(\varphi\) be a closed \(G_2\)-structure on \(Q\). The vector field \(R\) on \(Q\) is called a \(G_2\)-vector field if the flow of \(R\) preserves the \(G_2\)-structure. Also \(R\) is called Rochesterian if \(\iota_R \varphi\) is an exact form.

**Corollary 4.4.** Let \((Q, \varphi)\) is a hypersurface of \((M, \omega)\) and \(\omega\) is compatible with \(\varphi\). If \(\varphi\) is closed and \(\iota_R \varphi = j^* (\omega)\), then \(R\) is a \(G_2\)-vector field.

**Corollary 4.5.** In Theorem 4.2, if \(R\) is a vector field on \(Q\) such that \(\iota_R \varphi\) is exact, then \(Q \times \mathbb{R}\) and \(Q \times S^1\) admits a symplectic structure compatible with \(\varphi\).

In [6] it is shown that there is no Rochesterian vector field on a closed 7-dimensional manifold with a closed \(G_2\)-structure. So in the Corollary 4.4, if \(\omega\) is exact, then \(Q\) is assumed to be noncompact or compact without boundary.

**Corollary 4.6.** In Theorem 4.2, if \(\varphi\) is closed and \(R\) is a \(G_2\)-vector field, then there exists a symplectic form \(\omega\) on \(Q \times \mathbb{R}\) such that \([\omega] = [\pi^*(\iota_R \varphi)]\), where \(\pi : Q \times \mathbb{R} \rightarrow Q\) is the projection map. The same result is true for \(Q \times S^1\).

**References**


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