Slant curves in 3-dimensional C_{12}-manifolds

G. Beldjilali

Abstract. Slant curves and in particular Legendre curves play a very important and special role in geometry and topology of almost contact manifolds. There are certain results known for these curves in 3-dimensional normal almost contact metric manifolds. In the presented paper, we study the slant curves in the case of 3-dimensional non-normal almost contact metric manifolds, especially, C_{12}-manifolds. Examples are also constructed.

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1 Introduction

In [5], D. Chinea and C. Gonzalez have defined 12 classes of almost contact metric manifolds. In dimension 3, these manifolds are reduced to four classes: $|C|$ class of cosymplectic manifolds, C_5 class of β-Kenmotsu manifolds, C_6 class of α-Sasakian manifolds, C_9-manifolds and C_{12}-manifolds.
Only the last two classes can never be normal. For this reason, all work concerning curves on almost contact metric manifolds focuses on the first three classes.

In the present study, we will focus on slant curves and in particular Legendre curves in C_{12}-manifolds which can be integrable but never normal. Recently this class was studied in [3] where the authors studied the properties of 3-dimensional C_{12}- manifolds with concrete examples and construct some relations between class C_{12} and other classes as C_5 and C_6 or $|C|$.

The present paper is organized as follows:
After the introduction, required preliminaries are given in Section 2.
Section 3 contains new results on 3-dimensional C_{12}-manifolds with a class of concrete illustrative examples. The goal of last section, is to give an investigation of the slant curves and in particular Legendre curves in 3-dimensional C_{12}- manifolds.

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2 Preliminaries

2.1 Almost contact manifolds

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$-tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\begin{cases} \eta(\xi) = 1, \\ \varphi^2(X) = -X + \eta(X)\xi, \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \end{cases} \tag{2.1}$$

for any vector fields X,Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

The fundamental 2-form ϕ is defined by $\phi(X,Y) = g(X, \varphi Y)$. It is known that the almost contact structure (φ, ξ, η) is said to be normal if and only if

$$N^{(1)}(X,Y) = N_\varphi(X,Y) + 2d\eta(X,Y)\xi = 0, \tag{2.2}$$

for any X, Y on M, where N_φ denotes the Nijenhuis torsion of φ, given by

$$N_\varphi(X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{2.3}$$

Given an almost contact structure, one can associate in a natural manner an almost CR-structure $(\mathcal{D}, \varphi|_{\mathcal{D}})$, where $\mathcal{D} := Ker(\eta) = Im(\varphi)$ is the distribution of rank $2n$ transversal to the characteristic vector field ξ. If this almost CR-structure is integrable (i.e., $N_\varphi = 0$) the manifold M^{2n+1} is said to be CR-integrable. It is known that normal almost contact manifolds are CR-manifolds.

For more background on almost contact metric manifolds, we recommend the references [2, 4, 11].

2.2 Slant curves

Let (M, g) be a 3-dimensional Riemannian manifold with Levi-Civita connection ∇. γ is said to be a Frenet curve if there exists an orthonormal frame $\{E_1 = \dot\gamma, E_2, E_3\}$ along γ such that

$$\nabla_{E_1} E_1 = \kappa E_2, \qquad \nabla_{E_1} E_2 = -\kappa E_1 + \tau E_3 \qquad \nabla_{E_1} E_3 = -\tau E_2. \tag{2.4}$$

The curvature κ is defined by the formula

$$\kappa = |\nabla_{\dot\gamma}\dot\gamma|. \tag{2.5}$$

The second unit vector field E_2 is thus obtained by

$$\nabla_{\dot\gamma}\dot\gamma = \kappa E_2. \tag{2.6}$$

Next, the torsion τ and the third unit vector field E_3 are defined by the formulas

$$\tau = |\nabla_{\dot\gamma} E_2 + \kappa E_1| \qquad \textit{and} \qquad \nabla_{\dot\gamma} E_2 + \kappa E_1 = \tau E_3. \tag{2.7}$$

The concept of slant curve γ in almost contact metric geometry was introduced in [6] with the constant angle θ between the tangent $\dot\gamma$ and the structure vector field ξ. The particular case of $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ is very important since we recover the Legendre curves of [1].

Definition 2.1. $\gamma : I \to M$ be a Frenet curve on M. The structural angle of γ is the function $\theta : I \to [0, 2\pi[$ given by

$$\cos\theta = g(\dot{\gamma}, \xi) = \eta(\dot{\gamma}). \tag{2.8}$$

where $\dot{\gamma} = \frac{d\gamma}{ds}$ with s is the arc length parameter. The curve γ is a slant curve if θ is a constant function. Particularly if $\eta(\dot{\gamma}) = 0$ the curve γ is called Legendre curve.

For more background on slant curves and Legendre curves, we recommend the references [1, 6, 7, 8, 9, 10].

3 Three dimensional C_{12}-manifolds

In this section, we are mainly interested in three dimensional C_{12}-manifolds. Below we recall certain results concerning this case basing on [3], then we give a more general study and confirm the results with a class of concrete examples.

The 3-dimensional C_{12}-manifolds can be characterized by:

$$(\nabla_X \varphi)Y = \eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big), \tag{3.1}$$

for any X and Y vector fields on M.

where $\omega = -\big(\nabla_\xi \xi\big)^\flat = -\nabla_\xi \eta$ and ψ is the vector field given by

$$\omega(X) = g(X, \psi) = -g(X, \nabla_\xi \xi),$$

for all X vector field on M.

Moreover, from (3.1) it follow,

$$\begin{cases} \nabla_X \xi = -\eta(X)\psi, \\ \mathrm{d}\eta = \omega \wedge \eta, \\ \mathrm{d}\omega = 0, \end{cases} \tag{3.2}$$

Notice that $\nabla_\xi \xi = -\psi$ which implies that ψ is orthogonal to ξ.

The 3-dimensional C_{12}-manifolds is also characterized by

$$\mathrm{d}\eta = \omega \wedge \eta \qquad \mathrm{d}\phi = 0 \qquad \textit{and} \qquad N_\varphi = 0. \tag{3.3}$$

In [3], the authors studied the 3-dimensional unit C_{12}-manifold i.e. the case where ψ is a unit vector field. We will deal here with the general case, i.e. ψ is not necessarily unitary. For that, taking $V = \mathrm{e}^{-\rho}\psi$ where $\mathrm{e}^{\rho} = |\psi|$, we get immediately that $\{\xi, V, \varphi V\}$ is an orthonormal frame. We refer to this basis as Fundamental basis.

Using this frame, one can get the following:

Proposition 3.1. *For any C_{12}-manifold, for all vector field X on M we have*

1) $\nabla_X \xi = -\mathrm{e}^{\rho}\eta(X)V$

2) $\nabla_\xi V = \mathrm{e}^{\rho}\xi$

3) $\nabla_V V = \varphi V(\rho)\varphi V$

4) $\nabla_\xi \varphi V = 0$

5) $\nabla_V \varphi V = -\varphi V(\rho) V$.

Proof. For the first, using (3.1) for $Y = \xi$ we get

$$\begin{aligned}(\nabla_X \varphi)\xi &= \eta(X)\varphi\psi \\ &= \mathrm{e}^{\rho}\eta(X)\varphi V,\end{aligned}$$

knowing that $(\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi \nabla_X Y$ and applying φ we obtain

$$\begin{aligned}\nabla_X \xi &= \mathrm{e}^{\rho}\eta(X)\varphi^2 V \\ &= -\mathrm{e}^{\rho}\eta(X) V.\end{aligned}$$

For the second, we have

$$\begin{aligned}2\mathrm{d}\omega(\xi, X) = 0 \Leftrightarrow g(\nabla_\xi \psi, X) &= g(\nabla_X \psi, \xi) \\ &= -g(\psi, \nabla_X \xi) \\ &= \mathrm{e}^{2\rho}\eta(X),\end{aligned}$$

which gives $\nabla_\xi \psi = \mathrm{e}^{2\rho}\xi$ and then

$$\begin{aligned}\nabla_\xi V &= \nabla_\xi(\mathrm{e}^{-\rho}\psi) \\ &= -\xi(\rho)V + \mathrm{e}^{\rho}\xi.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\xi(\rho) &= \frac{1}{2}\mathrm{e}^{-2\rho}\xi(\mathrm{e}^{2\rho}) \\ &= \frac{1}{2}\mathrm{e}^{-2\rho}\xi\big(g(\psi, \psi)\big) \\ &= \mathrm{e}^{-2\rho}g(\nabla_\xi \psi, \psi) = 0,\end{aligned}$$

then,

$$\nabla_\xi V = \mathrm{e}^{\rho}\xi.$$

For $\nabla_V V$, we have

$$\begin{aligned}2\mathrm{d}\omega(\psi, X) = 0 \Leftrightarrow g(\nabla_\psi \psi, X) &= g(\nabla_X \psi, \psi) \\ &= \frac{1}{2} X g(\psi, \psi) \\ &= \mathrm{e}^{2\rho}g(\mathrm{grad}\rho, X),\end{aligned}$$

i.e. $\nabla_\psi \psi = \mathrm{e}^{2\rho}\mathrm{grad}\rho$ which gives $\nabla_V V = \mathrm{grad}\rho - V(\rho)V$.
Also, we have

$$\begin{aligned}\mathrm{grad}\rho &= \xi(\rho)\xi + V(\rho)V + \varphi V(\rho)\varphi V \\ &= V(\rho)V + \varphi V(\rho)\varphi V,\end{aligned}$$

then,

$$\nabla_V V = \varphi V(\rho)\varphi V.$$

For the rest, just use the formula $\nabla_X \varphi Y = (\nabla_X \varphi)Y + \varphi \nabla_X Y$ noting that $(\nabla_V \varphi)X = (\nabla_{\varphi V}\varphi)X = 0$. □

It remains to count $\nabla_{\varphi V}V$ and $\nabla_{\varphi V}\varphi V$. For that, we have the following result

Lemma 3.2. *For any 3-dimensional C_{12}-manifold, we have*

1) $\nabla_{\varphi V}V = (-\mathrm{e}^\rho + \mathrm{div}V)\varphi V$,

2) $\nabla_{\varphi V}\varphi V = (\mathrm{e}^\rho - \mathrm{div}V)V$.

Proof. Since $\{\xi, V, \varphi V\}$ is an orthonormal frame, we infer

$$\nabla_{\varphi V}V = a\,\xi + b\,V + c\,\varphi V,$$

Using Proposition 3.1, we have

$$a = g(\nabla_{\varphi V}V, \xi) = -g(V, \nabla_{\varphi V}\xi) = 0$$

and $b = g(\nabla_{\varphi V}V, V) = 0$. To get c, we note that

$$\begin{aligned} \mathrm{div}V &= g(\nabla_\xi V, \xi) + g(\nabla_{\varphi V}\psi, \varphi V) \\ &= \mathrm{e}^\rho + g(\nabla_{\varphi\psi}\psi, \varphi\psi) \Leftrightarrow g(\nabla_{\varphi V}V, \varphi V) = -\mathrm{e}^\rho + \mathrm{div}V, \end{aligned}$$

and then,

$$\nabla_{\varphi V}V = (-\mathrm{e}^\rho + \mathrm{div}V)\varphi V.$$

Applying φ with (3.1), we obtain

$$\nabla_{\varphi V}\varphi V = (\mathrm{e}^\rho - \mathrm{div}V)V.$$

□

According to the Proposition 3.1 and Lemma 3.2, the 3-dimensional C_{12}-manifold is completely controllable. That is:

Corollary 3.3. *For any C_{12}-manifold, we have*

$$\begin{array}{lll} \nabla_\xi\xi = -\mathrm{e}^\rho V, & \nabla_\xi V = \mathrm{e}^\rho\xi, & \nabla_\xi\varphi V = 0, \\ \nabla_V\xi = 0, & \nabla_V V = \varphi V(\rho)\varphi V, & \nabla_V\varphi V = -\varphi V(\rho)V, \\ \nabla_{\varphi V}\xi = 0, & \nabla_{\varphi V}V = (-\mathrm{e}^\rho + \mathrm{div}V)\varphi V, & \nabla_{\varphi V}\varphi V = (\mathrm{e}^\rho - \mathrm{div}V)V. \end{array}$$

To clarify these notions, we give the following class of examples:

Example 3.1. We denote the Cartesian coordinates in a 3-dimensional Euclidean space $M = \mathbb{R}^3$ by (x, y, z) and define a symmetric tensor field g by

$$g = \mathrm{e}^{2f}\begin{pmatrix} \alpha^2+\beta^2 & 0 & -\beta \\ 0 & \alpha^2 & 0 \\ -\beta & 0 & 1 \end{pmatrix},$$

where $f = f(y) \neq const$, $\beta = \beta(x)$ and $\alpha = \alpha(x, y) \neq 0$ every where are functions on $\mathbb{R}^3$ with $f' = \frac{\partial f}{\partial y}$. Further, we define an almost contact metric (φ, ξ, η) on $\mathbb{R}^3$ by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -\beta & 0 \end{pmatrix}, \quad \xi = \mathrm{e}^{-f}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \eta = \mathrm{e}^f(-\beta, 0, 1).$$

The fundamental 1-form η and the 2-form ϕ have the forms,

$$\eta = e^f(dz - \beta dx) \qquad and \qquad \phi = -2\alpha^2 e^{2f} dx \wedge dy,$$

and hence

$$d\eta = f' e^f \Big(\beta dx \wedge dy + dy \wedge dz\Big) = f' dy \wedge \eta,$$
$$d\phi = 0.$$

By direct computation, the non trivial components of $N^{(1)\ i}_{kj}$ are given by

$$N^{(1)\ 3}_{12} = \beta f', \quad N^{(1)\ 3}_{23} = f' \neq 0.$$

But, $\forall i, j, k \in \{1, 2, 3\}$

$$(N_\varphi)^i_{kj} = 0,$$

implying that (φ, ξ, η) becomes integable non normal. We have $\omega = f'dy$ i.e. $d\omega = 0$, and knowing that ω is the g-dual of ψ, i.e., $\omega(X) = g(X, \psi)$, we have immediately that

$$\psi = \frac{f'}{\alpha^2} e^{-2f} \frac{\partial}{\partial y}. \tag{3.4}$$

Thus, $(\varphi, \xi, \psi, \eta, \omega, g)$ is a 3-parameters family of C_{12} structure on $\mathbb{R}^3$.
Notice that

$$|\psi|^2 = \omega(\psi) = g(\psi, \psi) = \frac{f'^2}{\alpha^2} e^{-2f}$$

implies $V = \frac{e^{-f}}{\alpha} \frac{\partial}{\partial y}$ is a unit vector field, then

$$\Big\{\xi = e^{-f} \frac{\partial}{\partial z}, \quad V = \frac{e^{-f}}{\alpha} \frac{\partial}{\partial y}, \quad \varphi V = \frac{e^{-f}}{\alpha} \big(\frac{\partial}{\partial x} + \beta \frac{\partial}{\partial z}\big)\Big\}$$

form an orthonormal basis. To verify the result in (3.1), the components of the Levi-Civita connection corresponding to g are given by:

$$\begin{array}{lll} \nabla_\xi \xi = -\frac{f' e^{-f}}{\alpha} V, & \nabla_\xi V = \frac{f' e^{-f}}{\alpha} \xi, & \nabla_\xi \varphi V = 0, \\ \nabla_V \xi = 0, & \nabla_V V = -\frac{e^{-f}}{\alpha^2} \alpha_1 \varphi V, & \nabla_V \varphi V = -\varphi \nabla_V V, \\ \nabla_{\varphi V} \xi = 0, & \nabla_{\varphi V} V = \frac{e^{-f}}{\alpha^2} (f'\alpha + \alpha_2) \varphi V, & \nabla_{\varphi V} \varphi V = \varphi \nabla_{\varphi V} V, \end{array}$$

where $\alpha_i = \frac{\partial \alpha}{\partial x_i}$. Then, one can easily check that for all $i, j \in \{1, 2, 3\}$

$$\begin{array}{rcl} (\nabla_{e_i} \varphi) e_j & = & \nabla_{e_i} \varphi e_j - \varphi \nabla_{e_i} e_j \\ & = & \eta(e_i) \big(\omega(\varphi e_j) \xi + \eta(e_j) \varphi \psi\big). \end{array}$$

Through the rest of this paper $(M, \varphi, \xi, \psi, \eta, \omega, g)$ always denotes a 3-dimensional C_{12}-manifold and $\{\xi, V, \varphi V\}$ is fundamental frame; γ is a Frenet curve for which we denote the Frenet frame as usual $\{T = \dot\gamma, N, B\}$ and the Frenet equations are:

$$\nabla_{\dot\gamma} \dot\gamma = \kappa N, \qquad \nabla_{\dot\gamma} N = -\kappa \dot\gamma + \tau B \qquad \nabla_{\dot\gamma} B = -\tau N. \tag{3.5}$$

where κ denotes the curvature and τ the torsion.

4 Slant curves in 3-dimensional C_{12}-manifolds

In this section, we investigate slant curves in three dimensional C_{12}-manifolds. Firstly, we find the curvature and torsion of a Legendre curve with respect to the Levi-Civita connection.

Theorem 4.1. *If a Legendre curve $\gamma : I \to M$ is not a geodesic, then it is a plane curve (i.e., $\tau = 0$) and its curvature is given by*

$$\kappa = \begin{cases} |\mathrm{e}^{\rho} - \mathrm{div}V| & if \quad a = 0 \quad and \quad b \neq 0 \\ |\varphi V(\rho)| & if \quad b = 0 \quad and \quad a \neq 0 \\ |\frac{\dot{a}}{b} - a\,\varphi V(\rho) + b(\mathrm{e}^{\rho} - \mathrm{div}V)| & if \quad (a, b) \neq (0, 0), \end{cases}$$

where $\dot{\gamma} = aV + b\varphi V$ with a and b are functions on M.

Proof. Let $\gamma : I \to M$ be a Legendre curve non- geodesic on M. Then we have

$$\begin{cases} \eta(\dot{\gamma}) = 0 \\ g(\dot{\gamma}, \dot{\gamma}) = 1 \end{cases} \Leftrightarrow \begin{cases} \dot{\gamma} = aV + b\,\varphi V \\ a^2 + b^2 = 1, \end{cases}$$

where $a = \mathrm{e}^{-\rho}\omega(\dot{\gamma})$ and $b = -\mathrm{e}^{-\rho}\omega(\varphi\dot{\gamma})$. Then

$$\begin{aligned} \nabla_{\dot{\gamma}}\dot{\gamma} &= \nabla_{\dot{\gamma}}(aV + b\,\varphi V) \\ &= \dot{a}V + a\nabla_{\dot{\gamma}}V + \dot{b}\varphi V + b\nabla_{\dot{\gamma}}\varphi V \\ &= \dot{a}V + a\left(a\nabla_V V + b\nabla_{\varphi V}V\right) + \dot{b}\varphi V + b\left(a\nabla_V\varphi V + b\nabla_{\varphi V}\varphi V\right), \end{aligned}$$

with the help of Corollary 3.3, we get

$$\begin{aligned} \nabla_{\dot{\gamma}}\dot{\gamma} &= \left(\dot{a} - ab\,\varphi V(\rho) + b^2(\mathrm{e}^{\rho} - \mathrm{div}V)\right)V \\ &+ \left(\dot{b} + a^2\varphi V(\rho) - ab(\mathrm{e}^{\rho} - \mathrm{div}V)\right)\varphi V. \end{aligned} \tag{4.1}$$

Here we discuss three cases:
First case:
If $a = 0$ we get $|b| = 1$ because $g(\dot{\gamma}, \dot{\gamma}) = 1$ and formula (4.1) becomes

$$\nabla_{\dot{\gamma}}\dot{\gamma} = (\mathrm{e}^{\rho} - \mathrm{div}V)\,V,$$

then,

$$\kappa = |\nabla_{\dot{\gamma}}\dot{\gamma}| = |\mathrm{e}^{\rho} - \mathrm{div}V|.$$

Second case:
If $b = 0$ we get $|a| = 1$ and formula (4.1) becomes

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \varphi V(\rho)\varphi V,$$

then,

$$\kappa = |\varphi V(\rho)|.$$

Third case:
If $(a, b) \neq (0, 0)$ we know that $a^2 + b^2 = 1$ i.e. $\dot{b}b = -\dot{a}a$ then, the equation (4.1) becomes

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \frac{\lambda}{ab}(bV - a\varphi V),$$

where

$$\lambda = \left(\dot{a}a - a^2 b\, \varphi V(\rho) + ab^2(\mathrm{e}^{\rho} - \mathrm{div}V)\right).$$

Since γ is not geodesic then $\lambda \neq 0$. Therefore

$$\kappa = |\nabla_{\dot{\gamma}}\dot{\gamma}| = |\frac{\lambda}{ab}|.$$

Having (2.6), we get

$$\begin{aligned} E_2 &= \frac{1}{\kappa}\nabla_{\dot{\gamma}}\dot{\gamma} \\ &= \epsilon(bV - a\varphi V), \end{aligned}$$

where $\epsilon = \frac{\lambda}{ab}|\frac{ab}{\lambda}| \in \{-1, 1\}$. Let's compute $\nabla_{\dot{\gamma}}E_2$

$$\begin{aligned} \nabla_{\dot{\gamma}}E_2 &= \epsilon\nabla_{\dot{\gamma}}(bV - a\varphi V) \\ &= \epsilon(\dot{b}V - \dot{a}\varphi V) + \epsilon b\left(a\nabla_V V + b\nabla_{\varphi V}V\right) - \epsilon a\left(a\nabla_V \varphi V + b\nabla_{\varphi V}\varphi V\right) \\ &= \epsilon\left(\dot{b} + a^2\, \varphi V(\rho) - ab(\mathrm{e}^{\rho} - \mathrm{div}V)\right)V \\ &+ \epsilon\left(-\dot{a} + ab\varphi V(\rho) - b^2(\mathrm{e}^{\rho} - \mathrm{div}V)\right)\varphi V \\ &= -\epsilon\frac{\lambda}{ab}\dot{\gamma} \\ &= -\kappa\dot{\gamma}, \end{aligned}$$

therefore

$$\tau E_3 = \nabla_{\dot{\gamma}}E_2 + \kappa\dot{\gamma} = 0.$$

implies $\tau = 0$. □

Remark 4.1. A nice case appear when $V = \psi$ i.e. $\rho = 0$. In this case we get

$$\kappa = \begin{cases} |1 - \mathrm{div}\psi| & if \quad a = 0 \quad and \quad b \neq 0 \\ 0 & if \quad b = 0 \quad and \quad a \neq 0 \\ |\frac{\dot{a}}{b} + b(1 - \mathrm{div}\psi)| & if \quad (a, b) \neq (0, 0). \end{cases}$$

Corollary 4.2. *Taking into account the fundamental basis $\{\xi, V, \varphi V\}$ along γ with Corollary 3.3, one notices two particular cases:*
1) If $E_1 = \dot{\gamma} = V$, $E_2 = \varphi V$ and $E_3 = \xi$ then

$$\eta(\dot{\gamma}) = 0, \qquad \kappa = 0 \qquad and \qquad \tau = 0.$$

2) If $E_1 = \dot{\gamma} = \varphi V$, $E_2 = V$ and $E_3 = \xi$ then

$$\eta(\dot{\gamma}) = 0, \qquad \kappa = |\mathrm{e}^{\rho} - \mathrm{div}V| \qquad and \qquad \tau = 0.$$

Notice that for $E_1 = \dot{\gamma} = \xi$, $E_2 = V$ and $E_3 = \varphi V$ we obtain

$$\eta(\dot{\gamma}) = 1, \qquad \kappa = \mathrm{e}^{\rho} \qquad and \qquad \tau = 0.$$

This last case confirms two things. Firstly, $\dot{\gamma}$ can be pointwise collinear with ξ and γ is not a geodesic. Secondly, γ is a slant curve with $\theta = 0$. So, we will examine these two conclusions in the following.

We know that a Frenet curve $\gamma : I \to M$ in a normal almost contact metric manifold is said to be slant if its tangent vector field makes constant contact angle θ with ξ, i.e., $\eta(\dot{\gamma}) = \cos\theta$ is constant along γ. Of course, this does not mean that $\dot{\gamma}$ is collinear with ξ, otherwise we have

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \cos^2\theta \nabla_\xi \xi = 0,$$

and then γ becomes geodesic. This last reason is not available in C_{12}-manifolds ($\nabla_\xi \xi = -\psi \neq 0$), and then we can ask about the nature of the curve γ in three dimensional C_{12}-manifolds with $\dot{\gamma} = \cos\theta\xi$. To make the first step in this direction at least for a 3-dimensional C_{12}-manifold, let us assume $\dot{\gamma} = \cos\theta\, \xi$. Using (3.3), we get

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \cos^2\theta\, \nabla_\xi \xi = -\mathrm{e}^\rho \cos^2\theta\, V,$$

then,

$$\kappa = |\nabla_{\dot{\gamma}}\dot{\gamma}| = \mathrm{e}^\rho \cos^2\theta.$$

On the other hand,

$$E_2 = \frac{1}{\kappa}\nabla_{\dot{\gamma}}\dot{\gamma} = -V,$$

so,

$$\nabla_{\dot{\gamma}} E_2 = -\cos\theta \nabla_\xi V = -\mathrm{e}^\rho \cos\theta\, \xi,$$

and then,

$$\tau E_3 = \nabla_{\dot{\gamma}} E_2 + \kappa\dot{\gamma} = -\mathrm{e}^\rho \cos\theta\, \sin^2\theta \xi,$$

which gives

$$\tau = \mathrm{e}^\rho \sin^2\theta |\cos\theta|.$$

Therefore, we have the following proposition

Proposition 4.3. *Let $\gamma : I \to M$ be a non-geodesic Frenet curve such that $\dot{\gamma}$ is pointwise collinear with ξ, that is $\dot{\gamma} = c\xi$ with $c =$ constant. Then, its curvature and torsion are given by*

$$\kappa = \mathrm{e}^\rho \cos^2\theta \qquad \textit{and} \qquad \tau = \mathrm{e}^\rho \sin^2\theta |\cos\theta|. \tag{4.2}$$

Here, we report a nice remark. By computing the ratio between the two curvatures κ and τ, we find

$$\frac{\tau}{\kappa} = \frac{1}{|\cos\theta|} - |\cos\theta|.$$

If $\tau = \kappa$ we obtain $|\cos\theta| = -\phi^*$ where $\phi = 1 - \phi^* = \frac{1+\sqrt{5}}{2}$ is the Golden ratio.

Proposition 4.4. *The Frenet curve $\gamma : I \to M$ is a slant curve if and only if, along γ the following relation holds:*

$$\eta(N) = \frac{\cos\theta}{\kappa}\omega(\dot{\gamma}),$$

and a necessary condition for γ to be slant is:

$$|\cos\theta| < \frac{\kappa}{|\omega(\dot{\gamma})|}.$$

Proof. Let γ be a slant curve on M, that is

$$\eta(\dot\gamma) = g(\dot\gamma, \xi) = \cos\theta \qquad and \qquad g(\dot\gamma, \dot\gamma) = 1. \tag{4.3}$$

Let us take the covariant derivative in the relation (4.3) along γ:

$$\begin{aligned} 0 &= g(\nabla_{\dot\gamma}\dot\gamma, \xi) + g(\nabla_{\dot\gamma}\xi, \dot\gamma) \\ &= g(\kappa N, \xi) - \eta(\dot\gamma)\omega(\dot\gamma), \end{aligned}$$

then,

$$\eta(N) = \frac{\cos\theta}{\kappa}\omega(\dot\gamma).$$

The expression of ξ in the Frenet frame is

$$\begin{aligned} \xi &= \eta(T)T + \eta(N)N + \eta(B)B \\ &= \cos\theta\, T + \frac{\cos\theta}{\kappa}\omega(\dot\gamma)N + \eta(B)B, \end{aligned}$$

and since ξ is an unitary vector field, we get that

$$1 = \cos^2\theta + \frac{\cos^2\theta}{\kappa^2}\omega(\dot\gamma)^2 + \eta(B)^2,$$

and then $\eta(B)^2 \leq 0$ implies

$$\cos^2\theta \leq \frac{\kappa^2}{\kappa^2 + \omega(\dot\gamma)^2} < \frac{\kappa^2}{\omega(\dot\gamma)^2},$$

because κ strictly positive, which yields the condition. □

In the following, we suppose that $\gamma : I \to M$ is non-geodesic i.e. $\kappa > 0$ and let $\eta(\dot\gamma) = \sigma$, where $\sigma \in \mathcal{C}^\infty(M)$ such that $|\sigma| < 1$. Then, we consider an orthonormal frame field in TM along γ

$$E_1 = \dot\gamma, \qquad E_2 = \frac{\varphi\dot\gamma}{\sqrt{1-\sigma^2}}, \qquad E_3 = \frac{\xi - \sigma\dot\gamma}{\sqrt{1-\sigma^2}}.$$

Immediately, one can get

$$\varphi E_1 = \sqrt{1-\sigma^2}E_2, \quad \varphi E_2 = -\sqrt{1-\sigma^2}E_1 + \sigma E_3, \quad \varphi E_3 = -\sigma E_2. \tag{4.4}$$

Since $X = \sum_{i=1}^3 g(X, E_i)E_i$, the decompositions of ξ and ψ with respect to this frame are:

$$\xi = \sigma E_1 + \sqrt{1-\sigma^2}E_3, \tag{4.5}$$

$$\psi = \omega(\dot\gamma)E_1 + \frac{\omega(\varphi\dot\gamma)}{\sqrt{1-\sigma^2}}E_2 - \frac{\sigma\omega(\dot\gamma)}{\sqrt{1-\sigma^2}}E_3, \tag{4.6}$$

$$\varphi\psi = -\omega(\varphi\dot\gamma)E_1 + \frac{\omega(\dot\gamma)}{\sqrt{1-\sigma^2}}E_2 + \frac{\sigma\omega(\varphi\dot\gamma)}{\sqrt{1-\sigma^2}}E_3. \tag{4.7}$$

Let's us compute the equations of motion for this orthonormal field of frames.

$$\begin{aligned} \nabla_{E_1} E_1 &= g(\nabla_{\dot\gamma}\dot\gamma, E_2)E_2 + g(\nabla_{\dot\gamma}\dot\gamma, E_3)E_3 \\ &= \frac{1}{\sqrt{1-\sigma^2}}\big(\alpha E_2 + g(\nabla_{\dot\gamma}\dot\gamma, \xi)E_3\big), \end{aligned} \tag{4.8}$$

where $\alpha = g(\nabla_{\dot\gamma}\dot\gamma, \varphi\dot\gamma)$. Differentiating $\eta(\dot\gamma) = \sigma$ along γ we get

$$g(\nabla_{\dot\gamma}\dot\gamma, \xi) = \dot\sigma - g(\nabla_{\dot\gamma}\xi, \dot\gamma),$$

with the help of Proposition 3.1, we get

$$g(\nabla_{\dot\gamma}\dot\gamma, \xi) = \dot\sigma - \sigma\omega(\dot\gamma),$$

replacing in (4.8) we obtain

$$\nabla_{E_1} E_1 = \frac{1}{\sqrt{1-\sigma^2}}\Big(\alpha E_2 + \big(\dot\sigma - \sigma\omega(\dot\gamma)\big)E_3\Big). \tag{4.9}$$

For $\nabla_{E_1} E_2$, we have

$$\begin{aligned} \nabla_{E_1} E_2 &= g(\nabla_{E_1} E_2, E_1)E_1 + g(\nabla_{E_1} E_2, E_3)E_3 \\ &= -g(E_2, \nabla_{E_1} E_1)E_1 + \frac{1}{\sqrt{1-\sigma^2}} g(\nabla_{E_1} E_2, \xi - \sigma\dot\gamma)E_3 \\ &= -g(E_2, \nabla_{E_1} E_1)E_1 \\ &+ \frac{1}{\sqrt{1-\sigma^2}}\big(-g(E_2, \nabla_{\dot\gamma}\xi) + \sigma g(E_2, \nabla_{E_1} E_1)\big)E_3, \end{aligned}$$

using (4.9) and (3.3), we get

$$\nabla_{E_1} E_2 = \frac{-\alpha}{\sqrt{1-\sigma^2}} E_1 + \frac{\sigma}{1-\sigma^2}\big(\alpha - \omega(\dot{\varphi\gamma})\big)E_3. \tag{4.10}$$

With the same reasoning for $\nabla_{E_1} E_3$, we obtain

$$\nabla_{E_1} E_3 = -\frac{1}{\sqrt{1-\sigma^2}}\big(\dot\sigma - \sigma\omega(\dot\gamma)\big)E_1 - \frac{\sigma}{1-\sigma^2}\big(\alpha - \omega\big(\varphi\dot\gamma\big)\big)E_2, \tag{4.11}$$

comparing the formulas (4.9), (4.10) and (4.11) with (2.4) we get the following main result

Theorem 4.5. *Let $\gamma : I \to M$ be a non-geodesic Frenet curve such that $\eta(\dot\gamma) = \sigma$ where $\sigma \in \mathcal{C}^\infty(M)$ with $|\sigma| < 1$. Then, γ realizes the following hypotheses*

$$\begin{cases} \kappa = \frac{|\alpha|}{\sqrt{1-\sigma^2}} \\ \tau = \frac{\sigma}{1-\sigma^2}\big(\alpha - \omega(\varphi\dot\gamma)\big) \\ \dot\sigma - \sigma\omega(\dot\gamma) = 0 \end{cases}$$

where $\alpha = g(\nabla_{\dot\gamma}\dot\gamma, \varphi\dot\gamma)$.

As a consequence of the above Theorem, we immediately obtain the following result:

Proposition 4.6. *Let γ be a curve in M, which satisfies the assumptions of Theorem 4.5. If γ is a plane curve, then it is a Legendre curve or a slant curve with $\alpha = \omega(\varphi\dot{\gamma})$.*

Proposition 4.7. *Let $\gamma : I \to M$ be a non-geodesic Frenet curve such that $\eta(\dot{\gamma}) = \sigma$, where $\sigma \in \mathcal{C}^\infty(M)$ with $|\sigma| < 1$. If $V = E_2$, then γ is a slant curve.*

Proof. The decomposition of ψ with respect to $\{E_1, E_2, E_3\}$ is

$$\psi = \omega(\dot{\gamma})E_1 + \frac{1}{\sqrt{1-\sigma^2}}\omega(\varphi\dot{\gamma})E_2 - \frac{\sigma}{\sqrt{1-\sigma^2}}\omega(\dot{\gamma})E_3,$$

and then

$$|\psi| = \mathrm{e}^{2\rho} = \frac{1}{1-\sigma^2}\big(\omega(\dot{\gamma})^2 + \omega(\varphi\dot{\gamma})^2\big),$$

implies

$$\omega(\dot{\gamma})^2 = (1-\sigma^2)\mathrm{e}^{2\rho} - \omega(\varphi\dot{\gamma})^2.$$

From Theorem 4.5, we have

$$\dot{\sigma} - \sigma\omega(\dot{\gamma}) = 0,$$

which gives

$$\dot{\sigma}^2 = \sigma^2\big((1-\sigma^2)\mathrm{e}^{2\rho} - \omega(\varphi\dot{\gamma})^2\big). \tag{4.12}$$

On the other hand, if $V = E_2$ that is $\varphi\dot{\gamma} = \mathrm{e}^{-\rho}\sqrt{1-\sigma^2}$ i.e.,

$$\omega(\varphi\dot{\gamma}) = \mathrm{e}^{\rho}\sqrt{1-\sigma^2},$$

by replacing in (4.12) we obtain $\dot{\sigma} = 0$, which completes the proof. □

The following corollary confirms the previous results.

Corollary 4.8. *In a 3-dimensional C_{12}-manifolds, we have*

1) If $\sigma = 0$, then γ is a Legendre curve with

$$\kappa = |\alpha| \qquad \textit{and} \qquad \tau = 0.$$

2) If $\sigma = \cos\theta$, then γ is a slant curve with

$$\kappa = |\frac{\alpha}{\sin\theta}| \qquad \textit{and} \qquad \tau = \frac{\cos\theta}{\sin^2\theta}\big(\alpha - \omega(\varphi\dot{\gamma})\big).$$

3) If $E_1 = \dot{\gamma} = \xi$, $E_2 = V$ and $E_3 = \varphi V$, then γ is a slant curve with

$$\alpha = \mathrm{e}^{\rho}\sqrt{1-\sigma^2}, \qquad \kappa = \mathrm{e}^{\rho} \qquad \textit{and} \qquad \tau = 0.$$

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Author's address:

Gherici Beldjilali
Laboratory of Quantum Physics and Mathematical Modeling (LPQ3M),
University of Mascara, Algeria.
E-mail: gherici.beldjilali@univ-mascara.dz