$\star-$Ricci tensor on normal metric contact pair manifolds

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Abstract. In this paper we study NMCP manifolds, which are $(2p+2q+2)-$dimensional differential manifolds with an normal metric contact pair structure. The aim of the study is to examine the Riemannian geometry of normal metric contact pair manifolds under certain conditions related to the $\star-$Ricci tensor. We prove that a $\star-$Ricci-semi-symmetric NMCP manifold is $\star-$Ricci-flat, and that a $\star-$generalized quasi-Einstein NMCP manifold cannot be $\star-$Ricci-semi-symmetric. Finally, we consider the concircular curvature tensor on NMCP manifolds and prove that a concircular flat NMCP manifold is locally isometric to $(2p+2q+2)$-dimensional hyperbolic space.

M.S.C. 2010: 53C05, 53C10, 53C25.
Key words: normal metric contact pair manifold; $\star-$Ricci tensor; generalized quasi-Einstein manifold; concircular curvature tensor.

1 Introduction

The Riemannian geometry of contact manifolds has been extensively studied with a tensorial point of view since the early 1960s. Many geometric properties that occur in complex structures were examined on contact structures. In addition, important results were obtained regarding the geometric properties of the contact structures themselves. The found results are applied to various fields, especially to theoretical physics. Although there are some similarities between contact structures and complex structures, there are very important differences as well. By using the geometric properties of complex structures, contact structures are studied. It is also possible to transfer the results obtained from contact structures to complex geometry. There are various subclasses of complex and contact structures that differ according to their geometrical properties. For example, Kähler manifolds are one of the most important types of complex manifolds, while Sasakian manifolds, which are regarded as their analogs, are the most important class of contact manifolds. These two classes are considered canonical structures in these two domains. However, it is clear that not

Balkan Journal of Geometry and Its Applications, Vol.27, No.1, 2022, pp. 178-189.

every complex structure has to be Kähler, and not every contact structure has to be Sasakian. In this case, there examples of non-Sasakian contact structures, and of non-Kählerian complex structures.

It is known that a $2p+1$-dimensional sphere has a contact structure. On the other hand, Calabi-Eckman showed that the product of two spheres has a complex structure [9]. Such manifolds are called Calabi-Eckman manifolds. Blair, Ludden and Yano [8] studied complex manifolds whose complex structures are similar to the complex structure on Calabi-Eckman manifolds. In [8] the authors defined a new structure on Hermitian manifolds which is called bicontact structures. They proved that *"A Hermitian bicontact manifold is locally the product of two normal contact manifolds M^{2p+1} and M^{2q+1}."* Hermitian bicontact manifolds were studied by Abe [1]. Abe obtained many useful results for complex manifolds by using the notion of Hermitian bicontact manifolds.

Bicontact structures, which have different features from classical contact structures, did not attract attention for a while. The work by Blair, Ludden and Yano was done on a Hermitian manifold. The generalization of bicontact structures to any Riemannian manifold has been done by Bande and Hadjar, holdingthe name of contact pairs[2]. They constructed an almost contact structure on a contact pair manifold and defined the associated metric [3]. In 2013, the normality of an almost contact metric pair structure was studied [4]. Later, certain details of *normal contact metric pair* (NMCP) manifolds were studied by Bande, Hadjar and Blair in [7, 5, 6]. In 2020, one of the authors defined in [14] the notion of generalized quasi-Einstein normal metric contact pair manifold, and obtained some results on curvature relations. Also, the same author worked on certain flatness conditions [15] and some semi-symmetry conditions [16].

In the Riemannian geometry of manifolds, one of the most basic reference points is the concept of curvature. While the Riemann curvature of a manifold gives the measure of the non-flatness of the manifold, some different geometric properties of the manifold can be studied with the tensor tools related to Riemann curvature. One of them is Ricci curvature, which is the trace of the Riemann curvature. The interpretation of curvatures can be expanded by adding the structure on the manifold. The Ricci curvature defined by using the complex structure on a complex manifold is called $\star$-Ricci curvature. Tachibana [13] defined the $\star$-Ricci tensor $Ric^{\star}$ on almost Hermitian manifold. A similar definition can be given for the contact structure. Hamada [10] gave the definition of the $\star$-Ricci tensor for the contact case. In contact pair structures, the notion of $\star$-Ricci tensor has been studied in [7]. In this study, we aim to study normal contact metric pairs by using the geometric properties of the $\star$-Ricci tensor. We consider $\star$-Ricci-semi-symmetric normal metric contact pair manifold s. We give the definition of $\star$-Ricci generalized quasi-Einstein manifolds and we obtain several results. Finally, we consider the concircular curvature tensor on NMCP manifolds, and we prove that a concircular flat NMCP manifold is locally isometric to $(2p+2q+2)$-dimensional hyperbolic space.

2 Preliminaries

In this section we give a brief survey on normal metric contact pair manifolds (for details, see [2, 3, 4]).

Definition 2.1. A differentiable manifold $M^{2p+2q+2}$ is called a contact pair manifold if we have

- $\alpha_1 \wedge (d\alpha_1)^p \wedge \alpha_2 \wedge (d\alpha_2)^q \neq 0$,
- $(d\alpha_1)^{p+1} = 0$ and $(d\alpha_2)^{q+1} = 0$,

for two 1-forms α_1, α_2 [2]. We recall (α_1, α_2) as (p,q)-type contact pairs.

Two canonical examples of contact pair manifolds are given in the following [2].

Example 2.2. Let take $x_1, ..., x_{2p+1}, y_1, ..., y_{2q+1}$ are coordinate functions on $\mathbb{R}^{2p+2q+2}$. Then, two 1-forms

$$\alpha_1 = dx_{2p+1} + \sum_{i=1}^{p} x_{2i-1} dx_{2i}, \quad \alpha_2 = dy_{2q+1} + \sum_{j=1}^{q} y_{2i-1} dy_{2i}$$

defines a (p,q)-type contact pairs. $(\mathbb{R}^{2p+2q+2}, \alpha_1, \alpha_2)$ is an example of contact pair manifolds.

Example 2.3. Let us take two contact manifolds: (M_1^{2p+1}, α_1) and (M_2^{2q+1}, α_2), and let M be the product of M_1^{2p+1} and M_2^{2q+1}. Then, (α_1, α_2) is a (p,q)-type contact pair, and $(M = M_1^{2p+1} \times M_2^{2q+1}, \alpha_1, \alpha_2)$ is called *product contact pairs.*

As we know, the kernel of a contact form defines a distribution which is called *contact distribution.* For contact pairs, since we have two $1-$forms α_1 and α_2, we have two integrable subbundles of TM as $\mathcal{D}_1 = \ker \alpha_1$, $\mathcal{D}_2 = \ker \alpha_2$. We can naturally associate to it the distribution of vectors on which α_1 and $d\alpha_1$ vanish, and the one of vectors on which α_2 and $d\alpha_2$ vanish. Then for (α_1, α_2) are Pfaffian forms of constant classes $2p+1$ and $2q+1$, respectively, whose characteristic foliations are transverse and complementary, such that α_1 and α_2 restrict to contact forms on the leaves of the characteristic foliations of α_1 and α_2, respectively. We determine these foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. These distributions are involutive. Also they are of codimension $2p+1$ and $2q+1$, respectively, and their leaves are contact manifolds [2]. For them we shall use the name of contact pairs. These two characteristic foliations of M, are denoted by

$$\mathcal{F}_1 = \mathcal{D}_1 \cap ker d\alpha_1 \text{ and } \mathcal{F}_2 = \mathcal{D}_2 \cap ker d\alpha_2.$$

The Reeb vector fields Z_1 and Z_2 of contact pair (α_1, α_2) are determined by the following equations:

$$\alpha_1(Z_1) = \alpha_2(Z_2) = 1, \; \alpha_1(Z_2) = \alpha_2(Z_1) = 0$$
$$i_{Z_1} d\alpha_1 = i_{Z_1} d\alpha_2 = i_{Z_2} d\alpha_2 = 0,$$

where i_X is the contraction with the vector field X.

Let define two subbundle of TM by

$$T\mathcal{G}_i = ker d\alpha_i \cap ker \alpha_1 \cap ker \alpha_2, \; i = 1, 2.$$

Then we can write

$$T\mathcal{F}_i = T\mathcal{G}_i \oplus \mathbb{R} Z_i,$$

and so

$$TM = T\mathcal{G}_1 \oplus T\mathcal{G}_2 \oplus \mathbb{R}Z_1 \oplus \mathbb{R}Z_2.$$

Thus, the horizontal and vertical subbundles are defined by $\mathcal{H} = T\mathcal{G}_1 \oplus T\mathcal{G}_2$ and $\mathcal{V} = \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$, respectively. Finally, we have $TM = \mathcal{H} \oplus \mathcal{V}$ [3].

Any $X \in \Gamma(TM)$ could be written as $X = X^{\mathcal{H}} + X^{\mathcal{V}}$, where $X^{\mathcal{H}} \in \mathcal{H},\ X^{\mathcal{V}} \in \mathcal{V}$. Equivalently, we can write $X = X^1 + X^2$ for $X^1 \in T\mathcal{F}_1$ and $X^2 \in T\mathcal{F}_2$. Also, we can state $X^1 = X^{1^h} + \alpha_2(X^1)Z_2$ and $X^2 = X^{2^h} + \alpha_1(X^2)Z_1$, where X^{1^h} and X^{2^h} are horizontal parts of X^1, X^2, respectively. From all these decomposition of X finally we get

$$X = X^{1^h} + X^{2^h} + \alpha_1(X^2)Z_1 + \alpha_2(X^1)Z_2$$
$$\alpha_1(X^{1^h}) = \alpha_1(X^{2^h}) = 0, \quad \alpha_2(X^{1^h}) = \alpha_2(X^{2^h}) = 0.$$

Let define $(1,1)-$tensor field ϕ such as

$$\text{(2.1)}\quad \phi^2 = -I + \alpha_1 \otimes Z_1 + \alpha_2 \otimes Z_2, \quad \phi Z_1 = \phi Z_2 = 0,\ \alpha_1 \circ \phi = \alpha_2 \circ \phi = 0.$$

If $\phi T\mathcal{F}_i = T\mathcal{F}_i$, then ϕ is said to be decomposable, i.e., $\phi = \phi_1 + \phi_2$. From the decomposability of ϕ, we infer that (α_1, Z_1, ϕ_1) (resp. (α_2, Z_2, ϕ_2)) induces an almost contact structure on the leaves of $\mathcal{F}_2$ (resp.$\mathcal{F}_1$) [3]. Throughout this study, it is assumed that ϕ is decomposable. Finally, we recall $(\phi_1, \phi_2, g, Z_1, Z_2, \alpha_1, \alpha_2, \phi_1, \phi_2)$ the contact pair structure.

A Riemannian metric g on $(M, \phi, Z_1, Z_2, \alpha_1, \alpha_2)$ is called compatible if $g(\phi X_1, \phi X_2) = g(X_1, X_2) - \alpha_1(X_1)\alpha_1(X_2) - \alpha_2(X_1)\alpha_2(X_2)$ for all $X_1, X_2 \in TM$, and associated, if $g(X_1, \phi X_2) = (d\alpha_1 + d\alpha_2)(X_1, X_2)$ and $g(X_1, Z_i) = \alpha_i(X_1)$, for $i = 1, 2$. The 4-tuple $(\alpha_1, \alpha_2, \phi, g)$ is called metric contact pair structure on M.

Normality of almost contact metric structures is an important notion in contact geometry. As we know, a normal contact metric manifold is called a Sasakian manifold. A Sasakian manifold can be seen as an odd-dimensional Kähler manifold. Similarly, we have many subclasses of complex contact manifolds which are normal. A complex Sasakian manifold is also a normal complex contact metric manifold [17]. The normality of a metric contact pair manifold was studied in [4]. We have two almost complex structures:

$$\text{(2.2)}\qquad \mathcal{J} = \phi - \alpha_2 \otimes Z_1 + \alpha_1 \otimes Z_2, \quad \mathcal{T} = \phi + \alpha_2 \otimes Z_1 - \alpha_1 \otimes Z_2;$$

$\mathcal{J}$ and $\mathcal{T}$ are called the almost complex structure associated to $(\alpha_1, \alpha_2, \phi)$. If $\mathcal{J}$ and $\mathcal{T}$ are integrable, then M is normal. It is obvious that

$$\mathcal{J}Z_1 = Z_2,\ \mathcal{J}Z_2 = -Z_1 \text{ and } \mathcal{T}Z_1 = -Z_2,\ \mathcal{T}Z_2 = Z_1$$

On the other hand, the integrability of $\mathcal{J}$ and $\mathcal{T}$ is determined by the following condition

$$[\phi, \phi](X_1, X_2) + 2d\alpha_1(X_1, X_2)Z_1 + 2d\alpha_2(X_1, X_2)Z_2 = 0,$$

for all $X_1, X_2 \in \Gamma(TM)$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ [4]. For the sake of brevity, we shall use the abbreviation NMCP instead of the term *normal metric contact pair*.

In [7], it was proved that

Lemma 2.1. *On a normal metric contact pair,*

$$\begin{aligned} g(R(X_1X_2)Z, X_3) &= d\alpha_1(\phi X_3, X_1)\alpha_1(X_2) + d\alpha_2(\phi X_3, X_1)\alpha_2(X_2) \\ &\quad -d\alpha_1(\phi X_3, X_2)\alpha_1(X_1) - d\alpha_2(\phi X_3, X_2)\alpha_2(X_1), \end{aligned}$$

where R is the Riemannian curvature of M.

Some of the curvature relations on a NMCP manifold are given by

$$\begin{aligned} R(X_1, Z)X_2 &= -g(\phi X_1, \phi X_2)Z, \\ R(X_1, X_2, Z, X_3) &= d\alpha_1(\phi X_3, X_1)\alpha_1(X_2) + d\alpha_2(\phi X_3, X_1)\alpha_2(X_2) \\ &\quad -d\alpha_1(\phi X_3, X_2)\alpha_1(X_1) - d\alpha_2(\phi X_3, X_2)\alpha_2(X_1), \\ R(X_1, Z)Z &= -\phi^2 X_1. \end{aligned}$$

for $X_1, X_2, X_3 \in \Gamma(TM)$, and $Z = Z_1 + Z_2$ for the Reeb vector fields Z_1, Z_2 [7]. Let us take an orthonormal basis of M,

$$\{E_1, E_2, ..., E_p, \phi E_1, \phi E_2, ..., \phi E_p, E_{p+1}, E_{p+2}, ..., E_{p+q}, \phi E_{p+1}, \phi E_{p+2}, ..., \phi E_{p+q}, Z_1, Z_2\}.$$

Then, for all $X_1 \in \Gamma(TM)$, we get the Ricci curvature of M as ([6]):

$$Ric(X_1, Z) = \sum_{i=1}^{2p+2q} d\alpha_1(\phi E_i, E_i)\alpha_1(X_1) + d\alpha_2(\phi E_i, E_i)\alpha_2(X_1).$$

Also, the Ricci curvature of M has the following properties [7];

$$Ric(X_1, Z) = 0, \quad \text{for } X_1 \in \Gamma(\mathcal{H}), \tag{2.3}$$

$$Ric(Z, Z) = 2p + 2q. \tag{2.4}$$

$$Ric(Z_1, Z_1) = 2p, \ Ric(Z_2, Z_2) = 2q, \ Ric(Z_1, Z_2) = 0. \tag{2.5}$$

Definition 2.4. A NMCP manifold is called a generalized quasi-Einstein (GQE) manifold if the Ricci curvature of M has the following form;

$$Ric(X_1, X_2) = \lambda g(X_1, X_2) + \beta\alpha_1(X_1)\alpha_1(X_2) + \gamma\alpha_2(X_1)\alpha_2(X_2)$$

where λ, β, γ are scalar field on M and $X_1, X_2 \in \Gamma(TM)$ [14].

Thus, from (2.4) and (2.5), we have

$$Ric(X_1, X_2) = \lambda g(X_1, X_2) + (2p - \lambda)\alpha_1(X_1)\alpha_1(X_2) + (2q - \lambda)\alpha_2(X_1)\alpha_2(X_2)$$

for all $X_1, X_2 \in \Gamma(TM)$.

3 $\star-$Ricci tensor on normal metric contact pair manifolds

On an almost Hermitian manifold with almost complex structure J, the $\star$-Ricci tensor is defined by

$$Ric^\star(X, Y) = \sum_i R(X_1, E_i, JE_i, JX_2)$$

where E_i is an arbitrary orthonormal basis [13]. In general $Ric^{\star}$ is not symmetric, but it satisfies the relation

$$Ric^{\star}(X_1, X_2) = Ric^{\star}(JX_2, JX_1).$$

The trace of $Ric^{\star}$, denoted by $\tau^{\star}$ is called the $\star$-scalar curvature. A Hermitian manifold is $\star$-Einstein if we have $g(Q^{\star}X_1, X_2) = \lambda g(X_1, X_2)$, where λ is a constant.

A real hypersurface of a complex space form with the Kähler metric of constant holomorphic sectional curvature $4c$ carries an almost contact structure [10]. Hamada [10] studied such hypersurfaces and gave the definition of $\star$-Ricci tensor for contact case. Ivey and Ryan in [12] extended the work of Hamada and studied the equivalence of $\star$-Einstein condition with other geometric conditions such as the pseudo-Einstein and the pseudo-Ryan conditions. By using the concept of $\star$-Ricci tensor, Venkatesha and his group (see, [18] and [11]) recently studied some of the curvature properties on Sasakian manifolds and generalized (κ, μ)-space-forms.

As we now, a NMCP manifold cannot support a Hermitian metric. So we have no possible complex structure. But the normality tensors $\mathcal{J}$ and $\mathcal{T}$ defines almost complex structures on TM. In [7], the authors gave the definition of the $\star$-Ricci tensor by using the almost complex structure $\mathcal{J}$. On the other hand, since we have another complex structure $\mathcal{T}$ one can also fulfill this definition by using $\mathcal{T}$. Finally we have the following two formulas for the $\star$-Ricci tensor of a NMCP manifold;

$$Ric^{\star}_{\mathcal{J}}(X_1, X_2) = \sum_i R(X_1, E_i, \mathcal{J}E_i, \mathcal{J}X_2) \tag{3.1}$$

and

$$Ric^{\star}_{\mathcal{T}}(X_1, X_2) = \sum_i R(X_1, E_i, \mathcal{T}E_i, \mathcal{T}X_2). \tag{3.2}$$

On the other hand, for the contact case, it can be given as

$$Ric^{\star}_{\phi}(X_1, X_2) = \sum_i R(X_1, E_i, \phi E_i, \phi X_2). \tag{3.3}$$

In contact pair structures, the notion of $\star$-Ricci tensor has been studied in [7]. This paper is on Bochner and conformal flatness of normal metric contact pair manifold s. The Bochner tensor field is a formal analogue of the Weyl conformal curvature tensor. The authors use the definition from (3.1) in the mentioned paper. We also follow their results in this study. We do not use the subscript $\mathcal{J}$ from (3.1) in the following sections. On the other hand, same results can be obtained for the definition from (3.2). But also by using the third definition, many different results can be obtained from contact properties. This can be a subject for further research.

Although the $\star$-Ricci tensor of a Hermitian manifold is not in general, it is symmetric in the contact pair case. On the other hand, it is $\mathcal{J}-$invariant. The relation between the $\star$-Ricci and the Ricci tensor of a normal metric contact pair manifold is given in [7] by the following;

$$\begin{aligned} Ric^{\star}(X_1, X_2) \;=\; & Ric(X_1, X_2) - (2p-1)g(\phi_1 X_1, \phi_1 X_2) \\ & -(2q-1)g(\phi_2 X_1, \phi_2 X_2) \\ & -2p\alpha_1(X_1)\alpha_1(X_2) - 2q\alpha_2(X_1)\alpha_2(X_2) \end{aligned} \tag{3.4}$$

The $\star-$scalar curvature of a NMMCP manifold is the trace of $\star$-Ricci tensor. By direct computation we infer

$$\tau^\star = \tau - 4(p^2 + q^2). \tag{3.5}$$

The following lemma presents some useful properties of the $\star$-Ricci tensor on NMCP manifolds.

Lemma 3.1 ([7])**.** *Ric is $\mathcal{J}-$invaraint on horizontal vectors: $Ric(\mathcal{J}X_1, \mathcal{J}X_2) = Ric(X_1, X_2)$ and for $Ric_{ij} = Ric(Z_i, Z_j)$ we have*

$$Ric_{11} = 2p,\ Ric_{22} = 2q,\ Ric_{12} = 0, \quad Ric^\star_{11} = Ric^\star_{21} = Ric^\star_{12} = 0.$$

In a NMCP manifold we know that the Ricci tensor cannot be zero. However, the result from above indicates that such an obstacle may not exist for the $\star$-Ricci tensor.

4 $\star-$Ricci-semi-symmetry on normal metric contact pair manifolds

In this section we examine $\star-$Ricci-semi-symmetry conditions for NMCP manifolds. A normal metric contact pair manifold is $\star-$Ricci-semi-symmetric if we have $R\cdot Ric^\star = 0$.

Let us take the vector fields X_1, X_2, X_3, X_4 on a normal metric contact pair manifold M. Then, we have

$$(R(X_1, X_2)\cdot Ric^\star)(X_3, X_4) = -Ric^\star(R(X_1, X_2)X_3, X_4) - Ric^\star(X_3, R(X_1, X_2)X_4).$$

From (3.4), we get

$$\begin{aligned}(R(X_1, X_2)\cdot Ric^\star)(X_3, X_4) &= (R(X_1, X_2)\cdot Ric)(X_3, X_4)\\ &+(2p-1)[g(\phi_1 R(X_1, X_2)X_3, \phi_1 X_4)\\ &+g(\phi_1 X_3, \phi_1 R(X_1, X_2)X_4)]\\ &+(2q-1)[g(\phi_2 R(X_1, X_2)X_3, \phi_2 X_4)\\ &+g(\phi_2 X_3, \phi_2 R(X_1, X_2)X_4)]\\ &+2p[\alpha_1(R(X_1, X_2)X_3)\alpha_1(X_4)\\ &+\alpha_1(X_3)\alpha_1(R(X_1, X_2)X_4)]\\ &+2q[\alpha_2(R(X_1, X_2)X_3)\alpha_2(X_4)\\ &+\alpha_2(X_3)\alpha_2(R(X_1, X_2)X_4)]\end{aligned}$$

Then, from Lemma 2.1, we can state the following result.

Lemma 4.1. *Let take X_1, X_2, X_3, X_4 vector fields on a normal metric contact pair manifold M. Then we have*

$$\begin{aligned}(R(X_1, X_2)\cdot Ric^\star)(X_3, X_4) - (R(X_1, X_2)\cdot Ric)(X_3, X_4)) &= +\alpha_1(X_4)\alpha_1(R(X_1, X_2)X_3)\\ &+\alpha_1(X_3)\alpha_1(R(X_1, X_2)X_4)\\ &+\alpha_2(X_4)\alpha_2(R(X_1, X_2)X_3)\\ &+\alpha_2(X_3)\alpha_2(R(X_1, X_2)X_4)\end{aligned}$$

This result shows that the Ricci-semi-symmetry and the $\star-$Ricci-semi-symmetry conditions are not equivalent. It easy to see that for horizontal subbundles they coincide. We state this by the following result.

Corollary 4.2. *On the horizontal bundle of a NMCP manifold, the Ricci-semi-symmetry and the $\star-$Ricci-semi-symmetry coincide.*

Suppose that a normal metric contact pair manifold M is $\star-$Ricci-semi-symmetric, i.e., $(R(X_1, X_2) \cdot Ric^\star)(X_3, X_4) = 0$. Then we have

$$Ric^\star(R(X_1, X_2)X_3, X_4) + Ric^\star(X_3, R(X_1, X_2)X_4) = 0.$$

By taking $X_1 = X_3 = Z$ and by using (2.1), we obtain

$$d\alpha_1(\phi Q^\star X_4, X_2) + d\alpha_2(\phi Q^\star X_4, X_2) = 0$$

and this equals

$$(d\alpha_1 + d\alpha_2)(\phi Q^\star X_4, X_2) = 0.$$

From the definition of associated metric, we get

$$g(\phi^2 Q^\star X_4, X_2) = 0,$$

and finally by using (2.1), we infer

$$-Ric^\star(X_4, X_2) + Ric^\star(X_4, Z_1)\alpha_1(X_2) + Ric^\star(X_4, Z_2)\alpha_2(X_2) = 0.$$

Now, we are ready to present the following result.

Theorem 4.3. *A $\star-$Ricci-semi-symmetric normal metric contact pair manifold is $\star-$Ricci-flat.*

Definition 4.1. A normal metric contact pair manifold is called $\star-$GQE normal metric contact pair manifold if we have

$$Ric^\star(X_1, X_2) = \lambda^\star g(X_1, X_2) + \beta^\star\alpha_1(X_1)\alpha_1(X_2) + \mu^\star\alpha_2(X_1)\alpha_2(X_2).$$

From Lemma 3.1 for $\star-$GQE normal metric contact pair manifold we have $\beta^\star = \gamma^\star = -\lambda^\star$. Thus the $\star-$Ricci curvature of a $\star-$GQE normal metric contact pair manifold is given by

$$Ric^\star(X_1, X_2) = \lambda^\star(g(X_1, X_2) - \alpha_1(X_1)\alpha_1(X_2) - \alpha_2(X_1)\alpha_2(X_2)).$$

If M is GQE normal metric contact pair manifold, then it is not $\star-$GQE normal metric contact pair manifold. It is obvious that:

A $\star-$GQE normal metric contact pair manifold can not be $\star-$Ricci flat.

It is well known that an Einstein manifold is Ricci-semi-symmetric. But this is not true in general for a GQE manifold. On the other hand, from Theorem 4.3 we obtain the following result:

Theorem 4.4. *A $\star-$GQE normal metric contact pair manifold cannot be $\star-$Ricci-semi-symmetric.*

5 Concircular curvature tensor on normal metric contact pair manifolds

A geodesic circle is defined as a curve whose first curvature is constant and the second curvature is identically zero, and it is not invariant under conformal transformations. For this reason, Yano [19] defined concircular transformations and the concircular curvature tensor, which is invariant under concircular transformations. A manifold is called concircularly flat if this tensor vanishes. The concircular curvature tensor on an $(2p+2q+2)-$dimensional normal metric contact pair manifold is defined as follows:

$$\begin{aligned}\mathcal{W}(X_1,X_2)X_3 &= R(X_1,X_2)X_3 - \frac{\tau}{(2p+2p+2)(2p+2q+1)}[Ric(X_2,X_3)X_1 \\ &\quad -Ric(X_1,X_3)X_2 + g(X_2,X_3)QX_1 - g(X_1,X_3)QX_2],\end{aligned}$$

for $X_1, X_2, X_3 \in \Gamma(TM)$, where Q is the Ricci operator given by $Ric(X_1,X_2) = g(QX_1,X_2)$, and τ is the scalar curvature of M. Yano proved that a concircular flat Riemann manifold is space of constant curvature. Also, an Einstein space is invariant under concircular transformations. In [15], it was proved that a concircularly flat NMCP manifold is Einstein, and in [16] were examined some semi-symmetry conditions related to the concircular curvature tensor on NMCP manifolds. In this section, we examine concircularly flat NMCP manifolds with the help of the $\star-$Ricci tensor.

Suppose that a normal metric contact pair manifold M is concircularly flat; then we have

$$\begin{aligned}R(X_1,X_2)X_3 &= \frac{\tau}{(2p+2p+2)(2p+2q+1)}[Ric(X_2,X_3)X_1 \\ &\quad -Ric(X_1,X_3)X_2 + g(X_2,X_3)QX_1 - g(X_1,X_3)QX_2].\end{aligned} \tag{5.1}$$

Let us choose $X_2 = E_i, X_3 = \mathcal{J}E_i$ and take the inner product with $\mathcal{J}X_4$; then, taking the sum from $i=1$ to $i=2p+2q+2$, we get

$$\begin{aligned}\sum_i R(X_1,E_i,\mathcal{J}E_i,\mathcal{J}X_4) &= \frac{\tau}{(2p+2p+2)(2p+2q+1)}\sum_i[Ric(E_i,\mathcal{J}E_i)g(X_1,\mathcal{J}X_4) \\ &\quad -Ric(X_1,\mathcal{J}E_i)g(E_i,\mathcal{J}X_4) + g(E_i,\mathcal{J}E_i)g(QX_1,\mathcal{J}X_4) \\ &\quad -g(X_1,\mathcal{J}E_i)g(QE_i,\mathcal{J}X_4)].\end{aligned} \tag{5.2}$$

By using (2.2), it is not hard to see the following;

$$g(\mathcal{J}X_1,X_2) = -g(X_1,\mathcal{J}X_2) \text{ and } g(\mathcal{J}X_1,\mathcal{J}X_2) = g(X_1,X_2).$$

Also we have,

$$\sum_i g(\mathcal{J}QX_1,E_i)g(\mathcal{J}X_4,E_i) = g(\mathcal{J}QX_1,X_4) = Ric(X_1,X_4),$$

$$\sum_i g(X_1,\mathcal{J}E_i)g(QE_i,\mathcal{J}X_4) = g(\mathcal{J}X_1,Q\mathcal{J}X_4) = Ric(\mathcal{J}X_1,\mathcal{J}X_4),$$

$$\sum_i g(E_i,\mathcal{J}E_i) = 0.$$

Considering (5.2), we get

$$\begin{aligned} Ric^{\star}(X_1, X_4) &= \frac{\tau}{(2p+2p+2)(2p+2q+1)} \sum_i Ric(E_i, \mathcal{J}E_i) g(X_1, \mathcal{J}X_4) \\ &\quad - Ric(X_1, X_4) + Ric(\mathcal{J}X_1, \mathcal{J}X_4). \end{aligned}$$

Let us take X_1, X_4 as horizontal vector fields. From Lemma 3.1, we get

$$Ric^{\star}(X_1, X_4) = \frac{\tau}{(2p+2p+2)(2p+2q+1)} \sum_i Ric(E_i, \mathcal{J}E_i) g(X_1, \mathcal{J}X_4).$$

By choosing $X_1 = X_4 = E_j$, where the vectors E_j form an orthonormal basis of $\mathcal{H}$, and taking the sum over j, we obtain

$$\sum_{j=1}^{2p+2q} Ric^{\star}(E_j, E_j) = \frac{\tau}{(2p+2p+2)(2p+2q+1)} \sum_j \left\{ \sum_i Ric(E_i, \mathcal{J}E_i) g(E_j, \mathcal{J}E_j) \right\} = 0.$$

Also, we know that $Ric^{\star}(Z_1, Z_1) = Ric^{\star}(Z_2, Z_2) = 0$, hence we get $\tau^{\star} = 0$.

Finally, we are entitled to present the following result.

Theorem 5.1. *The $\star-$scalar curvature of a concircularly flat NMCP manifold vanishes.*

From the above theorem, we have $\tau = 4(p^2+q^2)$. Therefore, from (5.1) we obtain

$$\begin{aligned} Ric(X_1, X_4) &= \frac{4(p^2+q^2)}{(2p+2p+2)(2p+2q+1)} [g(X_1, X_4) - Ric(X_1, X_4) \\ &\quad + (2p+2q+2) Ric(X_1, X_4) - Rix(X_1, X_4)], \end{aligned}$$

and finally

$$Ric(X_1, X_4) = \frac{4A(p^2+q^2)}{1 - A(2p+2q)} g(X_1, X_4),$$

where $A = \frac{4(p^2+q^2)}{(2p+2p+2)(2p+2q+1)}$. Using this in (5.1), we get

$$R(X_1, X_2, X_3, X_4) = c[g(X_1, X_4)g(X_2, X_3) - g(X_1, X_3)g(X_2, X_4)]$$

where $c = \frac{8A^2(p^2+q^2)}{1-A(2p+2q)}$. Since $\tau = 4(p^2+q^2)$, we have

$$c = \frac{64(p^2+q^2)^3}{(p+q+1)(2p+2q+1) - 4(p+q)(p^2+q^2)}.$$

So, we can state that M is a space of constant sectional curvature c. Let us take $u(p,q) = 64(p^2+q^2)^3$ and $\omega(p,q) = [(p+q+1)(2p+2q+1) - 4(p+q)(p^2+q^2)]^{-1}$. Then we have $c = u(p,q)\omega(p,q)$. As clearly seen, $u(p,q)$ is positive for every pairs (p,q). It is not hard to see that $\omega(p,q)$ is negative. So, c becomes negatively constant. By considering the classification theorem from Riemanian Geometry, we can state the following result:

Theorem 5.2. *A concircular flat NMCP manifold is locally isometric to* $(2p+2q+2)$*-dimensional hyperbolic space, i.e.,* $H^{2p+2q+2}(c), c = u(p,q)\omega(p,q)$.

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