On kernels of second-order elliptic operators defined by Stein-Weiss operators acting on covariant tensors

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Abstract. The article is devoted to the study of the global geometry of symmetric and skew-symmetric higher order tensors on complete Riemannian manifolds using second-order elliptic operators, which are constructed on the basis of Stein-Weiss operators.

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1 Introduction

We consider a real vector bundle $E \to M$ on a differentiable C^∞-manifold M of dimension $n \geq 2$ with a linear connection $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ and a Lie group G, acting in the fibers of the bundles $T^*M \otimes E$ and E. Let $\mathrm{Diff}(E, T^*M \otimes E)$ denote a C^∞-module of first order linear differential operators $D : C^\infty E \to C^\infty(T^*M \otimes E)$ on the space $C^\infty(E)$ of smooth sections of E.

E. Stein and G. Weiss introduced in [20] the *generalized gradient* (in short, *G-gradient*), as the differential operator $D \in \mathrm{Diff}(E, T^*M \otimes E)$, which is the projection of the covariant derivative $\nabla\, s$ on the pointwise G-irreducible subbundle of the bundle $T^*M \otimes E$ for any section $s \in C^\infty(E)$. For example, Maxwell and Dirac equations, are based on these Stein-Weiss gradients (e.g., [20]). Later on, G-gradients were called *Stein-Weiss operators* (see [6]). We will also use this terminology.

Let g be a Riemannian metric on M, then on any real vector bundle $E \to M$ there exists a Riemannian metric, which we also denote by g. In this case, any Stein-Weiss differential operator D admits a formal adjoint operator D^* defined using g (see [3, p. 34]). Based on this fact, we are interested in a special class of second order differential operators D^*D, from which many geometric statements can be derived. In [6, 17], they studied ellipticity of second order differential operators D^*D. Our starting point is the following statement: If D is a differential operator of order k with injective symbol, then D^*D is elliptic. We also consider an elliptic differential operator $\Delta_E = \bar\Delta + t\,\Re$ (of the Weitzenböck decomposition form) for a suitable constant t, see [9], acting on $C^\infty(E)$, where $\bar\Delta = \nabla^*\nabla$ is the *rough* or *Bochner Laplacian*,

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∇^* denotes the formal adjoint of ∇ with respect to g (e.g., [3, p. 53] and [16, p. vii]), and $\Re$ is a smooth symmetric endomorphism of E depending linearly in a known way on the curvature R^{∇} of the connection ∇ on E. An example of a bundle to which the above reasoning applies is the space of differential p-forms, where the role of Δ_E is played by the Hodge-De Rham Laplacian Δ_H. A smooth section $s \in C^{\infty}(E)$ is called Δ_E-*harmonic* if $\Delta_E\, s = 0$ (see [16, p. 104]). Below, we consider the relationship between the operators Δ_E and D^*D and give examples of such harmonic sections.

The article has the following structure. In Section 2, we review the properties of Stein-Weiss operators D defined on differential p-forms ($1 \leq p \leq n-1$) and corresponding second order elliptical operators D^*D, and also the geometry of tensors lying in kernels of such operators. In Sections 3 and 4, we extend the results of [21, 22, 25] for symmetric p-tensors ($p \geq 2$). In Sections 5 and 6, we study the global geometry of traceless symmetric conformal Killing tensors and Codazzi tensors using second-order elliptic operators based on Stein-Weiss operators and the approach of a short article [24], where the question was investigated for tensors of order $p = 2$.

2 Stein-Weiss operators on differential forms

Let a linear group $\mathrm{GL}(n, \mathbb{R})$ act in the fibers of tensor bundles over M. Let $C^{\infty}\Lambda^p M$ denote the space of C^{∞}-sections of the bundle of p-forms on M for $1 \leq p \leq n-1$, and $d : C^{\infty}\Lambda^p M \to C^{\infty}\Lambda^{p+1} M$ the exterior derivative operator (see [3, p. 21]). There is a pointwise $\mathrm{GL}(n, \mathbb{R})$-irreducible decomposition $T^*M \otimes \Lambda^p M = \Lambda^{p+1} M \oplus \ker \Lambda^{p+1}$ for the pointwise algebraic alternation operator $\Lambda^p : T^*M \otimes \Lambda^p M \to \Lambda^{p+1} M$. As a consequence, we have the following pointwise $\mathrm{GL}(n, \mathbb{R})$-irreducible decomposition:

$$\nabla\omega = L_1\,\omega + L_2\,\omega \tag{2.1}$$

for any $\omega \in C^{\infty}\Lambda^p M$, where $L_1 = (p+1)^{-1} d$ and $L_2 = \nabla - (p+1)^{-1} d$ (see [21]). Due to [20, 11], these L_1 and L_2 are $\mathrm{GL}(n, \mathbb{R})$-gradients, or, Stein-Weiss operators, defined on $C^{\infty}\Lambda^p M$. The kernels L_1 and L_2 consist of closed p-forms and Killing p-forms, respectively, and the last ones, for (pseudo-)Riemannian manifolds, are called Killing-Yano tensors (see [26, p. 559]). For a Riemannian manifold (M, g), the decomposition (2.1) is pointwise orthogonal, i.e., $g(L_1\omega,\, L_2\omega) = 0$ for any $\omega \in C^{\infty}\Lambda^p M$.

Note that $d : C^{\infty}\Lambda^p M \to C^{\infty}\Lambda^{p+1} M$ has a formally adjoint operator $d^* : C^{\infty}\Lambda^{p+1} M \to C^{\infty}\Lambda^p M$ with respect to Riemannian metric on M, called codifferential (see [3, c. 54]). Thus, for L_2 there exists a formally adjoint operator $L_2^* = p(p{+}1)^{-1} d^*$. Using these operators, we build the second order differential operator

$$L_2^* L_2 = p(p+1)^{-1}(\bar{\Delta} - (p+1)^{-1} d^* d). \tag{2.2}$$

The main symbol $\sigma(L_2^* L_2)(\xi, \omega_x)$ of the operator (2.2) has the form

$$\sigma(L_2^* L_2)\,(\xi, \omega_x) = -\frac{p}{p+1}\Big(\frac{p}{p+1}\,\|\xi\|^2 \omega_x + \frac{1}{p+1}\,\xi \wedge (\iota_\xi \omega_x)\Big) \tag{2.3}$$

according to the following formulas (see [3, p. 461]):

$$\sigma(\nabla)(\xi, \omega_x) = \xi \otimes \omega_x, \quad \sigma(\nabla^*)(\xi, \omega_x) = -\iota_\xi \theta_x,$$
$$\sigma(d)\,(\xi, \omega_x) = \xi \wedge \omega_x, \quad \sigma(d^*)(\xi, \omega_x) = -\iota_\xi \omega_x$$

for all $\xi \in R^*_x M \setminus \{0\}$, $\omega_x \in \Lambda^r(T^*_x M)$ and $\theta_x \in T^*_x M \otimes \Lambda^r(T^*_x M)$ at each point $x \in M$. From (2.3) we obtain the following inequality:

$$-g(\sigma(L_2^* L_2)(\xi,\, \omega_x),\, \omega_x) = \frac{p}{(p+1)^2}\big(p\, g(\xi,\, \xi)\, \omega_x + g(\iota_\xi\, \omega_x,\, \iota_\xi\, \omega_x)\big) > 0$$

for any nonzero ξ and ω_x. Thus, (2.2) is an elliptic operator (see [3, p. 462]). On a compact manifold M, the kernel of $L_2^* L_2$ consists of Killing-Yano p-tensors (see [23]), because of the inequality $\int_M g(L_2^* L_2 \omega, \omega)\, d\mathrm{V}_g = \int_M g(L_2\omega, L_2\omega)\, d\mathrm{V}_g \geq 0$, where $d\mathrm{V}_g$ is the volume form of g; moreover, according to [3, p. 464], as a consequence of ellipticity of $L_2^* L_2 : C^\infty \Lambda^p M \to C^\infty \Lambda^p M$ we get the decomposition $C^\infty \Lambda^{p+1} M = \ker L_2^* \oplus \mathrm{Im}\, L_2$ with respect to the L^2-global scalar product on (M, g), defined by $\langle \omega,\, \omega' \rangle = \frac{1}{p!}\int_M g(\omega, \omega')\, d\mathrm{V}_g$, where $\omega, \omega' \in C^\infty \Lambda^p M$. As the result, we get

Proposition 2.1. *For any $\omega \in C^\infty \Lambda^p M$ and its $\mathrm{SL}(n, \mathbb{R})$-gradients $L_1\omega = (p+1)^{-1} d\,\omega$ and $L_2\omega = \nabla\,\omega - (p+1)^{-1} d\,\omega$ on $\Lambda^p M$ the orthogonal decomposition* (2.1) *holds. If (M, g) is compact, then the orthogonal decomposition $C^\infty \Lambda^{p+1} M = \ker L_2^* \oplus \mathrm{Im}\, L_2$ holds. Moreover, $L_2^* L_2$ in* (2.2) *is a nonnegative definite elliptic operator, whose kernel is a finite-dimensional vector space over $\mathbb{R}$ consisting of Killing-Yano p-tensors.*

Bourguignon [5] studied first order *natural differential operators* on the spaces of C^∞-sections of bundle of $\Lambda^p M$ on (M, g) with the structural group $\mathrm{O}(n, \mathbb{R})$ and the Levi-Civita connection ∇ (see the theory in [13]). By definition, if the symbols of these operators are projectors on pointwise $\mathrm{O}(n, \mathbb{R})$-irreducible subbundles of $T^*M \otimes \Lambda^p M$, they are called *fundamental*. Fundamental differential operators of Bourguignon are Stein-Weiss operators. Bourguignon proved that $T^*M \otimes \Lambda^p M$ is decomposed into three pointwise $\mathrm{O}(n, \mathbb{R})$-irreducible subbundles. Based on this fact, Bourguignon defined fundamental operators d and d^* and indicated the existence of a third fundamental operator. He also noted that apart from the case $p = 1$, the third fundamental operator does not have a simple geometric interpretation. As a consequence, this allows for each $\omega \in C^\infty \Lambda^p M$ to obtain an expansion of $\nabla\,\omega \in C^\infty(T^*M \otimes \Lambda^p M)$ in the sum of three pointwise $\mathrm{O}(n, \mathbb{R})$-irreducible components

$$\nabla\omega = G_1\omega + G_2\omega + G_3\omega. \tag{2.4}$$

Then, all three Stein-Weiss operators were found explicitly in [22]:

$$G_1 = (p+1)^{-1} d, \quad G_2 = (n-p+1)^{-1} g \wedge d^*, \quad G_3 = \nabla - G_1 - G_2, \tag{2.5}$$

and it was proved in [27] that the kernel of G_3 consists of conformal Killing p-forms.

Further, in [23], the operator G_3^* formally conjugated to G_3 on (M, g) was found, the following second order differential operator was constructed and studied:

$$G_3^* G_3 = \frac{p}{p+1}\Big(\bar\Delta - \frac{1}{p+1}\, d^* d - \frac{1}{n-p+1}\, d\, d^*\Big).$$

For $n = 2p$ we get $G_3^* G_3 = \frac{p}{p+1}\big(\bar\Delta - \frac{1}{p+1}\,\Delta_H\big)$ for the *Hodge-de Rham Laplacian* $\Delta_H = d^* d + d\, d^*$ (e.g., [16, p. 260]). The *Hodge-de Rham Laplacian* Δ_H admits the Weitzenböck decomposition (e.g., [3, p. 57]) $\Delta_H = \bar\Delta + \Re$, where $\Re$ depends linearly in a known way on the curvature tensor and the Ricci tensor Ric of ∇. Moreover, for $n = 2\,p$ we get the equality $G_3^* G_3 = (\frac{p}{p+1})^2(\bar\Delta - \frac{1}{p}\,\Re)$, where $\Delta_L = \bar\Delta - p^{-1}\Re$ is the Lichnerovich Laplacian (see [9]). Thus, the following is valid.

Proposition 2.2. *Let for each differential p-form $\omega \in C^\infty \Lambda^p M$ the expansion of its covariant derivative $\nabla\omega \in C^\infty(T^*M \otimes \Lambda^p M)$ in the sum (2.4) of pointwise $\mathrm{O}(n,\mathbb{R})$-irreducible components with Stein-Weiss operators (2.5) hold. Then for $n = 2p$ the operator $p^{-2}(p+1)^2 G_3^* G_3$ is the Lichnerovich Laplacian.*

The *Bochner-Weitzenböck formula* (e.g., [16, p. 106]), can be rewritten as

$$\frac{1}{2}\Delta \|\,\omega\,\|^2 = -g(\Delta_H\,\omega,\,\omega) - g(\Re(\omega),\,\omega) + \|\,G_1\omega\,\|^2 + \|\,G_2\omega\,\|^2 + \|\,G_3\omega\,\|^2.$$

The operator $G_3^*G_3$ is elliptic for $2 \le p \le n-1$ (see [18, 10], where it lacks the normalizing factor $p(p+1)^{-1}$ calculated in [23]): on a compact (M,g) the kernel of $G_3^*G_3$ is formed by conformal Killing p-forms.

3 The Stein-Weiss operator on symmetric tensors

Let $C^\infty \mathrm{S}^p M$ be the space of C^∞-sections of the bundle $\mathrm{S}^p M$ of symmetric p-tensors on M. Consider T_xM at any point $x \in M$ as an n-dimensional vector space V with the structure group $\mathrm{GL}(n,\mathbb{R})$. Let $\mathrm{S}^p V$ denote the p-th symmetric power of the space V^* dual to V. The fiber of $T^*M \otimes \mathrm{S}^p M$ is the tensor space $V^* \otimes \mathrm{S}^p V$, which will be regarded as the representation space of $\mathrm{GL}(n,\mathbb{R})$. Define an endomorphism $S^{p+1} : V^* \otimes \mathrm{S}^p V \to \mathrm{S}^{p+1} V \subset V^* \otimes \mathrm{S}^p V$, called the *Young symmetrizer*, see [1], by

$$\begin{aligned}(S^{p+1}(\phi))_{i_0 i_1 \dots i_{p-1} i_p} &:= \phi_{(i_0 i_1 \dots\, i_{p-1}\, i_p)} \\ &= \frac{1}{p+1}\left(\phi_{i_0 i_1 \dots i_{p-1} i_p} + \phi_{i_1 \dots i_{p-1} i_p i_0} + \dots + \phi_{i_p i_0 i_1 \dots i_p i_{p-1}}\right)\end{aligned}$$

for components $\phi_{i_0\, i_1 \dots i_{p-1}\, i_p} = \phi(\,e_{i_0},\, e_{i_1}, \dots, e_{i_p})$ of any $\phi \in V^* \otimes \mathrm{S}^p V$ in any basis $e_1, \dots, e_n$ of V. The endomorphism S^{p+1} is $\mathrm{GL}(n,\mathbb{R})$-invariant and $S^{p+1}(S^{p+1}(\varphi)) = S^{p+1}(\varphi)$, i.e., S^{p+1} is an idempotent in $V^* \otimes \mathrm{S}^p V$. Thus, the $\mathrm{GL}(n,\mathbb{R})$-invariant decomposition of $V^* \otimes \mathrm{S}^p V$ into a direct sum $V^* \otimes \mathrm{S}^p V = \mathrm{Im}\, S^{p+1} \oplus \ker\, S^{p+1}$ of two subspaces $V^* \otimes \mathrm{S}^p V$ holds, where $\mathrm{Im}\, S^{p+1} = \mathrm{S}^{p+1} V$, and $\ker S^{p+1} := \mathrm{Im}(\mathrm{id} - S^{p+1})$ consists of tensors of the form $\phi - S^{p+1}(\phi)$.

Lemma 3.1. *Let $\mathrm{GL}(n,\mathbb{R})$ act on fibers of tensor bundles on M. Then the following pointwise $\mathrm{GL}(n,\mathbb{R})$-irreducible decomposition holds:*

$$T^*M \otimes \mathrm{S}^p M = \mathrm{S}^{p+1} M \oplus \ker\, S^{p+1}. \tag{3.1}$$

Proof. The first component of the expansion $\mathrm{S}^{p+1} V$ for $V = T_xM$ and any point $x \in M$ is irreducible $\mathrm{GL}(n,\mathbb{R})$ – a module. To find $\mathrm{GL}(n,\mathbb{R})$-irreducible subspaces in $\mathrm{S}^{p+1} V$, we need a list of all correctly filled $(n, p+1)$-Young schemes, which in this case contains only one simple scheme $\boxed{1}\boxed{2}\boxed{\;\dots\;}\boxed{p}\boxed{p+1}$.

Thus, there are no other $\mathrm{GL}(n,\mathbb{R})$-irreducible subspaces in $\mathrm{S}^{p+1} V$ other than $\mathrm{S}^{p+1} V$. To determine what weights with respect to the maximal tori (diagonal matrices) have elements of $\ker\, S^{p+1}$, we decompose $V^* \otimes \mathrm{S}^p V$ into weighted spaces, where the weight vectors are tensors of the form

$$\bar\varphi^{\,(l, j_1, \dots, j_l)}_{\,k, i_1, \dots, i_l} = \begin{cases} 1, & \text{if } l = k \text{ and } i_1, \dots, i_l \text{ is a permutation of } j_1, \dots, j_l, \\ 0, & \text{otherwise.} \end{cases}$$

The above tensor has weight $\mathrm{diag}(t_1,\ldots,t_n) \mapsto t_k t_{i_1}\ldots t_{i_p}$. Then the maximum weight with respect to the order of domination $\lambda \ge \mu \Leftrightarrow \forall m : \sum_{i=1}^m \lambda_i \ge \sum_{i=1}^m \mu_i$ has tensor $\phi^{(2,1,\ldots,1)} \ne 0$, since $\phi^{(1,1,\ldots,1)} = 0$. The weight of this nonzero vector is $(p,1,0,\ldots,0)$. It follows that $\ker S^{p+1} \cong V((p,1,0,\ldots,0))$. Since the module $\mathrm{S}^{p+1}V$ is $\mathrm{GL}(n,\mathbb{R})$-irreducible, the decomposition (3.1) is also $\mathrm{GL}(n,\mathbb{R})$-irreducible. Based on the above, we conclude that there are only two Stein-Weiss differential operators defined on the space of sections $C^\infty\mathrm{S}^pM$ of S^pM. We define the first-order linear differential operator $\delta^* : C^\infty\mathrm{S}^pM \to C^\infty\mathrm{S}^{p+1}M$ by means of the equality $\delta^*\varphi = (p+1)\,S^{p+1}(\nabla\varphi)$. It has the following form in local coordinates $x^1,\ldots,x^n$:

$$(\delta^*\varphi)_{k\,i_1\ldots i_{p-1}i_p} := \nabla_k\,\varphi_{\,i_1\ldots i_{p-1}i_p} + \ldots + \nabla_{i_p}\,\varphi_{\,i_1\ldots i_{p-1}\,k}$$

where $\nabla_k = \nabla_{\partial/\partial x^k}$, and $\varphi \in C^\infty\mathrm{S}^pM$. The value on $\xi \in C^\infty T_x^*M$ of the symbol $\sigma(\delta^*)$ of the operator δ^* is a homomorphism

$$\sigma(\delta^*)(\xi,x) : \varphi_x \in \mathrm{S}^p(T_xM) \to (p+1)\,\xi\odot\varphi_x \in \mathrm{S}^{p+q}(T_xM),$$

according to the law of symmetric multiplication $\varphi_x \odot \varphi'_x = S^{p+q}(\varphi_x \otimes \varphi'_x)$ for the pointwise defined symmetric multiplication $S^{p+q} : \mathrm{S}^p(T_xM)\otimes\mathrm{S}^q(T_xM) \to \mathrm{S}^{p+q}(T_xM)$ and any tensors $\varphi \in C^\infty\mathrm{S}^pM$ and $\varphi' \in C^\infty\mathrm{S}^qM$. Therefore, $P_1 = (p+1)^{-1}\,\delta^*$ is the first Stein-Weiss operator defined as symmetrization of the covariant derivative. □

Consider further an operator of the form $P_2 = \nabla - (p+1)^{-1}\delta^*$. The value of its symbol $\sigma(P_2)$ on any 1-form $\xi \in C^\infty T^*M$ is the homomorphism

$$\sigma(P_2)(\xi,x) : \varphi_x \in S^p(T_xM) \to (\xi\otimes\varphi_x - (p+1)\,\xi\odot\varphi_x) \in \ker S^{p+1}(T_xM)$$

defined at any point $x \in M$. Thus, the second operator will be P_2.

Since for any $\varphi \in C^\infty\mathrm{S}^pM$ there is a pointwise $\mathrm{GL}(n,\mathbb{R})$-irreducible decomposition

$$\nabla\varphi = P_1\varphi + P_2\varphi, \tag{3.2}$$

then due to Stein-Weiss approach in [20], the above P_1 and P_2 are Stein-Weiss operators on the space of symmetric p-tensors, because $P_1\varphi$ and $P_2\varphi$ are pointwise $\mathrm{GL}(n,\mathbb{R})$-irreducible components of the decomposition of $\nabla\varphi$. Thus, we get

Proposition 3.2. *Let M be a smooth n-dimensional $(n \ge 2)$ manifold with a linear connection ∇ without torsion. Then there are two Stein-Weiss differential operators $P_1 = \frac{1}{p+1}\delta^*$ and $P_2 = \nabla - \frac{1}{p+1}\delta^*$ on the space of sections $C^\infty\mathrm{S}^pM$.*

The kernel of P_1 consists of *symmetric Killing p-tensors*, that is, tensor fields $\varphi \in C^\infty\mathrm{S}^pM$ such that $S^{p+1}(\nabla\varphi) = 0$. The kernel of P_2 consists of *Codazzi p-tensors* $\varphi \in C^\infty\mathrm{S}^pM$, for which $\nabla\varphi \in C^\infty\mathrm{S}^{p+1}M$. According to [3, p. 35], the operator $\delta^* : C^\infty\mathrm{S}^pM \to C^\infty\mathrm{S}^{p+1}M$ has the formally adjoint operator $\delta : C^\infty\mathrm{S}^{p+1}M \to C^\infty\mathrm{S}^pM$, called *divergence* and defined by the equality $\delta\,\varphi = -\mathrm{trace}_g\nabla\,\varphi$. Here, the trace_g is given by the formula $(\mathrm{trace}_g\,\varphi)\,(a_3,\ldots,a_p) = \sum_{i=1}^n \varphi\,(e_i,\,e_i,\,a_3,\ldots,a_p)$ for any vectors $a_3,\ldots,a_p$ and orthonormal basis $\{\,e_1,\ldots,e_n\}$ of T_xM at any point $x \in M$. Therefore, the formally adjoint to P_1 operator has the form $P_1^* = (p+1)^{-1}\delta$. Let us construct a second-order differential operator $P_1^*P_1 = (p+1)^{-2}\delta\,\delta^*$. The operator $P_1^*P_1 : C^\infty\mathrm{S}^pM \to C^\infty\mathrm{S}^pM$ is elliptic, since its principal symbol satisfies

$$\begin{aligned}-g(\sigma(P_1^*P_1)(\xi,x)\varphi_x,\varphi_x) &= g(\xi,\xi)g(\varphi_x,\varphi_x) - (p+1)\,g(\xi\odot\varphi_x(\xi,\cdot),\varphi_x)\\ &= g(\xi,\xi)\cdot g(\varphi_x,\varphi_x) + p\cdot g(i_\xi\varphi_x, i_\xi\varphi_x) > 0 \end{aligned} \tag{3.3}$$

for any $\xi \in T_x^*M \setminus \{0\}$ and nonzero φ_x at any point $x \in M$. Thus, on a compact (M, g) the kernel of $P_1^* P_1$ is a finite-dimensional vector space over $\mathbb{R}$. A local estimate for the dimension of this space was found in [2]:

$$\dim_{\mathbb{R}} \ker P_1^* P_1 \le C_p^{n+p}\, C_p^{n+p-1} - C_{p+1}^{n+p}\, C_{p-1}^{n+p-1},$$

where the equality is attained on the Euclidean sphere. Since $\int_M g(P_1^* P_1 \varphi, \varphi)\, d\,\mathrm{V}_g = \int_M g(P_1\varphi, P_1\varphi)\, d\,\mathrm{V}_g \ge 0$, the kernel of $P_1^* P_1$ consists of symmetric Killing tensors $\varphi \in C^\infty \mathrm{S}^p M$. By [3, p. 464], the following orthogonal decomposition is valid:

$$C^\infty \mathrm{S}^{p+1} M = \ker\, P_1^* \oplus \mathrm{Im}\, P_1 \tag{3.4}$$

for the L^2-global scalar product on a compact (M, g). Summing up, we formulate

Proposition 3.3. *For any tensor field $\varphi \in C^\infty \mathrm{S}^p M$ there is a pointwise orthogonal decomposition* (3.2)*, where $P_1 = \frac{1}{p+1}\delta^*$ and $P_2 = \nabla - \frac{1}{p+1}\,\delta^*$. On a compact manifold (M, g), the second-order differential operator $P_1^* P_1 = (p+1)^{-2} \delta\, \delta^*$ is a nonnegative elliptic operator, whose kernel is a finite-dimensional vector space of symmetric Killing p-tensors. Moreover, the orthogonal decomposition* (3.4) *is valid.*

If (M, g) is a compact Riemannian manifold of nonpositive sectional curvature, then ker $P_1^* P_1$ consists of parallel symmetric p-tensors, that is, tensors φ obeying the condition $\nabla \varphi = 0$ (see [7]). If, in addition, M is connected and there is a point at which all sectional curvatures are negative, then ker $P_1^* P_1$ consists of symmetric p-tensors of the form $C \cdot g^k$ for some real constant C (see also [7]).

4 The Stein-Weiss operators on traceless symmetric tensors

Bourguignon studied first order *natural differential operators* on the spaces of C^∞-sections of the bundle $\mathrm{S}_0^2 M$ of symmetric traceless 2-tensors on (M, g), e.g., [6]. The symbols of such operators are projectors onto pointwise $\mathrm{O}(n, \mathbb{R})$-irreducible subbundles of $T^*M \otimes \mathrm{S}_0^2 M$. The following decomposition is valid:

$$\begin{aligned} T^*M \otimes \mathrm{S}_0^2 M \quad = \quad & \mathrm{Pr}_{\,\mathrm{S}_0^3 M}(T^*M \otimes \mathrm{S}_0^2 M) \oplus\ \mathrm{Pr}_{T^*M}(T^*M \otimes \mathrm{S}_0^2 M) \oplus \\ & \oplus\ \mathrm{Pr}_{\ker S^3 \bigcap \ker\ \mathrm{trace}_g}(T^*M \otimes \mathrm{S}_0^2 M). \end{aligned}$$

As a consequence, we have the pointwise $\mathrm{O}(n, \mathbb{R})$-irreducible decomposition

$$\nabla \varphi = D_1\, \varphi + D_2\, \varphi + D_3\, \varphi \tag{4.1}$$

for any traceless symmetric 2-form, or, the field of 2-tensors $\varphi \in C^\infty \mathrm{S}_0^2 M$. Based on this fact, Bourguignon defined all three operators D_1, D_2 and D_3 and proved that the kernel of the operator D_1 consists of the divergence-free 2-tensors $\varphi \in C^\infty \mathrm{S}_0^2 M$. He argued that the kernels of D_2 and D_3 do not have a simple geometric interpretation. In [25], these arguments were applied to a pseudo-Riemannian manifold (M, g), all three Stein-Weiss operators were redefined on C^∞-sections of $\mathrm{S}_0^2 M$, and a geometric interpretation of traceless symmetric 2-tensors lying in the kernel of each of them was given. It was proved that the kernel of D_1 consists of (traceless) *symmetric conformal Killing 2-tensors* (see [26, p. 559]), and the kernel of D_2 consists of traceless *conformal*

Codazzi 2-tensors defined in [24]. The main difference of these tensors from well-known *Codazzi 2-tensors* (e.g., [3, pp. 434; 436–440]) is their conformal invariance.

Consider a bundle $\mathrm{S}_0^p M$ $(p \geq 2)$ of traceless symmetric p-tensors on M. For each $\varphi \in \mathrm{S}_0^p M$, the equality $\mathrm{trace}_g\, \varphi = 0$ is valid.

Lemma 4.1. *Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. Then the following pointwise $\mathrm{O}(n, \mathbb{R})$-irreducible decomposition is valid:*

$$
\begin{aligned}
T^*M \otimes \mathrm{S}_0^p M \quad &= \mathrm{Pr}_{\,\mathrm{S}_0^{p+1} M}(T^*M \otimes \mathrm{S}_0^p M) \oplus \mathrm{Pr}_{\,\mathrm{S}_0^{p-1} V}(T^*M \otimes \mathrm{S}_0^p M) \oplus \\
&\oplus\ \mathrm{Pr}_{\,\ker S^{p+1} \bigcap\, \ker \mathrm{trace}_g}(T^*M \otimes \mathrm{S}_0^p M).
\end{aligned}
$$

Proof. The fiber of $T_x^*M \otimes \mathrm{S}_0^p(T_x^*M)$ at any point $x \in M$ is an n-dimensional $(n > 1)$ cotangent vector space T_x^*M. We will consider this tensor space as the space of representations $V^* \otimes \mathrm{S}_0^p V$ of $\mathrm{O}(n, \mathbb{R})$. There are three orthogonal subspaces $\ker S^{p+1} \bigcap \ker \mathrm{trace}_g$, $\mathrm{S}_0^{p+1} V$ and $\mathrm{S}_0^{p-1} V$ of $V^* \otimes \mathrm{S}_0^p V$ such that (see [1])

$$
\begin{aligned}
V^* \otimes \mathrm{S}_0^p V \quad &= \quad \mathrm{Pr}_{\,\mathrm{S}_0^{p+1} V}(V^* \otimes \mathrm{S}_0^p V) \oplus \mathrm{Pr}_{\,\mathrm{S}_0^{p-1} V}(V^* \otimes \mathrm{S}_0^p V) \oplus \\
&\oplus \mathrm{Pr}_{\,\ker\ S^{p+1} \bigcap\ \ker\ \mathrm{trace}_g}(V^* \otimes \mathrm{S}_0^p V).
\end{aligned}
$$

The irreducibility of the components of the decomposition of $V^* \otimes \mathrm{S}_0^p V$ under the action of $\mathrm{O}(n, \mathbb{R})$ follows from Theorem by G. Weyl on quadratic $\mathrm{O}(n, \mathbb{R})$-invariant forms (see [6, pp. 313–314]). There are three such independent invariant quadratic forms, which are specified using components $\phi_{i_0\, i_1 \ldots i_p} = \phi(e_{i_0},\, e_{i_1}, \ldots,\, e_{i_p})$ of $\phi \in V^* \otimes \mathrm{S}_0^p V$ in the orthonormal basis $e_1, \ldots,\, e_n$ of V, and have the form

$$
\begin{aligned}
\Psi_1(\phi) \quad &= \quad \sum\nolimits_{i_0, i_1, \ldots, i_p = 1}^{n} (\phi_{i_0 i_1 \ldots i_p})^2, \quad \Psi_2(\phi) = \sum\nolimits_{i, i_2, \ldots, i_p = 1}^{n} (\phi_{i i\, i_2 \ldots i_p})^2, \\
\Psi_3(\phi) \quad &= \quad \sum\nolimits_{i_0, i_1, i_2, \ldots, i_p = 1}^{n} \phi_{i_0 i_1 i_2 \ldots i_p} \phi_{i_1 i_0 i_2 \ldots i_p}.
\end{aligned}
$$

They represent all possible traces of the $\phi \otimes \phi$-form. Since there are three such forms, the decomposition $V^* \otimes \mathrm{S}_0^p V$, which also has three tensor components, is $\mathrm{O}(n, \mathbb{R})$-irreducible according to result of H. Weil (see [6, pp. 313–314]). $\square$

Let $\mathrm{Diff}(\mathrm{S}_0^p M,\ T^*M \otimes \mathrm{S}_0^p M)$ denote the $C^\infty M$-module of first-order linear differential operators $D : C^\infty \mathrm{S}_0^p M \to C^\infty(\,T^*M \otimes \mathrm{S}_0^p M)$ on the space of smooth sections $C^\infty \mathrm{S}_0^p M$ of the bundle $\mathrm{S}_0^p M$. Due to the pointwise orthogonal decomposition of the bundle $T^*M \otimes \mathrm{S}_0^p M$ from [1], we get the pointwise $\mathrm{O}(n, \mathbb{R})$-irreducible decomposition (4.1) of the covariant derivative of any tensor field $\varphi \in C^\infty \mathrm{S}_0^p M$. Then certain D_1, D_2 and D_3 are Stein-Weiss operators on $C^\infty \mathrm{S}_0^p M$. The Stein-Weiss operator D_1, whose symbol is the projector onto the pointwise irreducible component $\mathrm{S}_0^p M$, is

$$
D_1\, \varphi = \frac{1}{p+1}\Big(\delta^* \varphi + \frac{p(p+1)}{n + 2(p-1)}\, g\, \odot \delta\, \varphi\Big) \tag{4.2}
$$

for any $\varphi \in C^\infty \mathrm{S}_0^p M$ and an algebraic operator $g\,\odot : \mathrm{S}^{p-1} M \to \mathrm{S}^{p+1} M$ defined pointwise by $g\,\odot := (2p-1) S^{p+1}(g\,\otimes)$ (see [1]). In local coordinates $x^1, \ldots,\, x^n$ on (M, g), the expression (4.2) appears as

$$
(D_1 \varphi)_{i_0 i_1 i_2 \ldots i_p} = \frac{1}{p+1}\Big(\delta^* \varphi_{\ i_0 i_1 i_2 \ldots i_p} + \frac{p(p+1)}{n + 2(p-1)}\, g_{(i_0 i_1}\, \delta\, \varphi_{\ i_2 \ldots i_p)}\Big). \tag{4.3}
$$

Using the identity $g_{(i_0 i_1}\,\delta\,\varphi_{\,i_2\ldots i_p)} = g_{(i_0(i_1}\,\delta\,\varphi_{\,i_2\ldots i_p))}$ for the pointwise symmetrization operator $S^{p+1}\,(g \otimes \delta\,\varphi)_{i_0 i_1 \ldots i_p} = g_{(i_0 i_1}\,\delta\,\varphi_{\,i_2\ldots i_p)}$, we rewrite (4.3) in the form

$$
\begin{aligned}
(D_1\varphi)_{i_0 i_1 i_2\ldots i_p} = {} & \frac{1}{p+1}\Big(\delta^*\varphi_{i_0 i_1 i_2 i_3\ldots i_{p-1} i_p} + \frac{1}{n+2(p-1)}\Big(g_{i_0 i_1}\delta\varphi_{i_2 i_3\ldots i_{p-1} i_p}\\
& + g_{i_0 i_2}\delta\varphi_{i_3\ldots i_{p-1} i_p i_1} + \ldots + g_{i_0 i_{p-1}}\delta\varphi_{i_p i_1 i_2\ldots i_{p-2}} + g_{i_0 i_p}\delta\varphi_{i_1 i_2 i_3\ldots i_{p-1}}\\
& + g_{i_1 i_2}\delta\varphi_{i_3 i_4\ldots i_{p-1} i_p i_0} + g_{i_1 i_3}\delta\varphi_{i_4\ldots i_{p-1} i_p i_0 i_2} + \ldots + g_{i_1 i_p}\delta\varphi_{i_0 i_2 i_3\ldots i_{p-1}}\\
& + g_{i_1 i_0}\delta\varphi_{i_2 i_3 i_4\ldots i_{p-1} i_p} + g_{i_2 i_3}\delta\varphi_{i_4 i_5\ldots i_p i_0 i_1} + g_{i_2 i_4}\delta\varphi_{i_5\ldots i_p i_0 i_1 i_3} + \ldots\\
& + g_{i_2 i_0}\delta\varphi_{i_1 i_3 i_4\ldots i_p} + g_{i_2 i_1}\delta\varphi_{i_3 i_4\ldots i_p i_0} + \cdots + g_{i_p i_0}\delta\varphi_{i_1 i_2 i_3\ldots i_{p-1}}\\
& + g_{i_p i_1}\delta\varphi_{i_2 i_3\ldots i_{p-1} i_0} + \ldots + g_{i_p i_{p-2}}\delta\varphi_{i_{p-1} i_0 i_1 i_3\ldots i_{p-3}} + g_{i_p i_{p-1}}\delta\varphi_{i_0 i_1 i_2 i_3\ldots i_{p-2}}\Big)\Big).
\end{aligned}
\tag{4.4}
$$

Based on (4.4), we get $D_1\varphi \in C^\infty \mathrm{S}_0^{p+1}M$. We call $\varphi \in C^\infty \mathrm{S}_0^p M$ a *symmetric conformal Killing p-tensor*, if $D_1\varphi = 0$, which coincides with the notion of a conformal Killing p-tensor, e.g., [7, 8]. For $p = 1$ condition $D_1\varphi = 0$ takes the form of well-known equations of a *conformal Killing vector* (see [26, pp. 559]). Formally conjugate to (4.2) operator $D_1^* : C^\infty \mathrm{S}_0^{p+1}M \to C^\infty \mathrm{S}_0^p M$ is given for any $\bar\varphi \in C^\infty \mathrm{S}_0^{p+1}M$ by

$$
D_1^*\bar\varphi = \frac{1}{p+1}\Big(\delta\,\bar\varphi + \frac{p(p+1)}{n+2(p-1)}\,(g\odot\delta)^*\bar\varphi\Big) = \frac{1}{p+1}\,\delta\,\bar\varphi, \tag{4.5}
$$

because $(g\,\otimes)^* = \operatorname{trace}_g$. Therefore, $(g\odot\delta)^*\bar\varphi = (2p-1)\delta^*(\operatorname{trace}_g\bar\varphi) = 0$ for any traceless tensor $\bar\varphi \in C^\infty \mathrm{S}_0^{p+1}M$. Based on the operators D_1 and D_1^*, we define a second-order differential operator of the form $D_1^* D_1 : C^\infty \mathrm{S}_0^p M \to C^\infty \mathrm{S}_0^p M$, which according to (4.4) and (4.5) is given by the following equality:

$$
D_1^* D_1\varphi = \frac{1}{(p+1)^2}\Big(\delta\,\delta^*\varphi + \frac{1}{n+2(p-1)}\,\big(-2\,\delta^*\delta\,\varphi + p(p-1)\,g\odot\delta\,\delta\,\varphi\big)\Big). \tag{4.6}
$$

For the Sampson Laplacian operator $\Delta_S = \delta\,\delta^* - \delta^*\delta$, (4.6) can be rewritten as

$$
D_1^* D_1\varphi = \frac{1}{(p+1)^2(n+2(p-1))}\big(2\Delta_S\,\varphi + (n+2(p-2))\delta\,\delta^*\varphi + p(p-1)\,g\odot\delta\,\delta\,\varphi\big). \tag{4.7}
$$

Let us prove the ellipticity of the operator $D_1^* D_1$. First, note that at each point $x \in M$ for any $\varphi \in C^\infty \mathrm{S}_0^p M$ and $\xi \in T_x^* M \setminus \{0\}$ the equality $g(\sigma(g\odot\delta\,\delta)(\xi, x)\varphi_x, \varphi_x) = 0$ holds, which is a consequence of the tracelessness of the tensor field φ. Second, for any nonzero $\varphi \in C^\infty \mathrm{S}_0^p M$ the inequality $-g(\sigma(\Delta_S)(\xi,\varphi_x),\varphi_x) = g(\xi,\xi)g(\varphi_x\,\varphi_x) > 0$ holds (see [15]). By (3.3), $-g(\sigma(\delta\,\delta^*)(\xi,\varphi_x),\varphi_x) > 0$ holds. Thus, the inequality $-g(\sigma(\delta\,\delta^*)(\xi,\varphi_x),\varphi_x) > 0$ takes place; hence, $D_1^* D_1$ is elliptic. Then its kernel on a compact (M, g) is finite-dimensional. Moreover, $\int_M g(D_1^* D_1\,\varphi,\,\varphi)\,d\,\mathrm{V}_g = \int_M g(D_1\varphi,\,D_1\varphi)\,d\,\mathrm{V}_g \geq 0$, thus, this vector space consists of symmetric conformal Killing p-tensors $\varphi \in C^\infty \mathrm{S}_0^p M$. The following orthogonal decomposition takes place:

$$
C^\infty \mathrm{S}^{p+1}M = \ker\,D_1^* \oplus \operatorname{Im} D_1 \tag{4.8}
$$

for the L^2-global scalar product on the compact (M, g). Summing up, we formulate

Proposition 4.2. *The pointwise* $\mathrm{O}(n,\mathbb{R})$*-irreducible decomposition* (4.1) *of the covariant derivative of any tensor field* $\varphi \in C^\infty \mathrm{S}_0^p M$ *holds. On a compact* (M, g), *a second-order differential operator* $D_1^* D_1$ *for the Stein-Weiss operator*

$$
D_1\,\varphi = (p+1)^{-1}\,\big(\delta^*\varphi + (n+2(p-1))^{-1}(g\odot\delta\,\varphi)\big),
$$

and its formally conjugate D_1^*, is a nonnegative elliptic operator, whose kernel is a finite-dimensional vector space over $\mathbb{R}$ and consists of symmetric conformal Killing p-tensors. Moreover, the orthogonal decomposition (4.8) *is valid.*

The second Stein-Weiss differential operator D_2, whose symbol is the projector onto the second pointwise irreducible component of the decomposition $TM^* \otimes \mathrm{S}_0^p M$ is

$$(D_2\,\varphi)_{i_0\,i_1\,i_2\ldots i_{p-2}\,i_{p-1}\,i_p} = -p(n+p-1)^{-1}\,g_{i_0\,(\,i_1}\delta\,\varphi_{i_2\ldots i_p)}$$

(see [1]), and its kernel consists of traceless divergence-free p-tensors.

The third Stein-Weiss differential operator D_3, whose symbol is the projector onto the third pointwise irreducible component of the decomposition $TM^* \otimes \mathrm{S}_0^p M$, is

$$\begin{aligned}(D_3\,\varphi)_{i_0\,i_1\,i_2\ldots i_{p-2}\,i_{p-1}\,i_p} &= \nabla_{i_0}\varphi_{i_1 i_2\ldots i_p} + \frac{p}{n+p-1} g_{i_0(\,i_1}\,\delta\,\varphi_{i_2\ldots i_p)}\\ &\quad -\frac{1}{p+1}\Big(\delta^*\varphi_{i_0 i_1 i_2\ldots i_p} + \frac{p(p+1)}{n+2(p-1)}\,g_{(i_0 i_1}\,\delta\,\varphi_{\,i_2\ldots i_p)}\Big)\end{aligned}$$

for any $\varphi \in C^\infty \mathrm{S}_0^p M$ (see [1]). For any $\varphi \in \ker D_3$, the following equations hold:

$$\nabla_{i_0}\varphi_{i_1 i_2\ldots i_p} - \nabla_{i_1}\varphi_{i_0 i_2\ldots i_p} = \frac{p}{n+p-1}\,\big(g_{i_0\,(\,i_1}\,\delta\,\varphi_{i_2\ldots i_p)} - g_{i_1\,(\,i_0}\,\delta\,\varphi_{i_2\ldots i_p)}\big)\,. \tag{4.9}$$

5 Global Riemannian geometry of conformal Killing tensors

The kernel of D_1 consists of p-tensors $\varphi \in C^\infty \mathrm{S}_0^p M$ for $p \geq 2$ that satisfy

$$\delta^*\varphi = -\frac{p(p+1)}{n-2(p-1)}\,g\odot\delta\,\varphi. \tag{5.1}$$

Each such p-tensor is a symmetric *conformal Killing p-tensor* (e.g., [7, 8]). Note that the requirement of tracelessness is included here in the definition of the conformal Killing p-tensor ($p \geq 2$) as well as in [26, p. 559] for the case $p = 2$. The condition $\varphi \in \ker\, D_1 \bigcap \ker\,\delta$ defines a symmetric Killing p-tensor $\varphi \in C^\infty \mathrm{S}_0^p M$, because (5.1) implies that $\delta^*\varphi = 0$. Taking into account (4.7), we find

$$g(\Delta_S\,\varphi,\,\varphi) = -2^{-1}(n+2(p-2))\,g(\delta\,\delta^*\varphi,\,\varphi) \tag{5.2}$$

for Sampson Laplacian $\Delta_S = \delta\,\delta^* - \delta^*\delta$ and conformal Killing tensors $\varphi \in C^\infty \mathrm{S}_0^p M$. From (5.2) we conclude that the symmetric divergence-free (traceless) conformal Killing tensor, or, equivalently, the symmetric traceless p-Killing tensor belongs to the kernel of Δ_S. For a compact manifold (M, g), it follows from (5.2) that

$$\int_M g(\Delta_S\,\varphi,\,\varphi)\,d\,\mathrm{V}_g = -2^{-1}(n+2(p-2))\int_M g(\delta\,\varphi,\,\delta\,\varphi)\,d\,\mathrm{V}_g.$$

Thus, any traceless conformal Killing p-tensor belonging to the kernel of the Sampson Laplacian is divergence-free, thus it is a Killing p-tensor. We get the following

Proposition 5.1. *On a compact Riemannian manifold, a symmetric (traceless) conformal Killing p-tensor belongs to the kernel of the Sampson Laplacian if and only if it is a traceless p-Killing tensor.*

For any Killing p-tensor ($p \geq 2$), direct calculations lead to the following formula: $2\,\delta\,\varphi = \delta^*(\mathrm{trace}_g\,\varphi)$. Thus, *on a compact Riemannian manifold of negative Ricci curvature, every symmetric Killing tensor of rank 3 is traceless.* The Sampson Laplacian $\Delta_S : C^\infty \mathrm{S}^p M \to \mathrm{C}^\infty \mathrm{S}^p M$ admits the Weitzenböck decomposition (see [15])

$$\Delta_S\,\varphi = \bar{\Delta}\,\varphi - \Re(\varphi). \tag{5.3}$$

The formula (5.3) indicates that Δ_S is a particular form of *Lichnerovich's Laplacian* (see [3, p. 79] and [9]). Here, $\Re$ is linearly expressed in terms of the Riemannian curvature tensor and the Ricci tensor of the Levi-Civita connection and satisfies $g(\Re(\varphi),\,\varphi') = g(\Re(\varphi'),\,\varphi)$ for any $\varphi,\,\varphi' \in C^\infty \mathrm{S}^p M$ (see [15]). Thus, $\Phi_p(\varphi_x,\,\varphi_x) = g(\Re(\varphi_x),\,\varphi_x)$ is a quadratic form for any $\varphi_x \in S^p(T_x^*M)$ and $x \in M$. Since Δ_S is an elliptic operator, by [3, p. 632], the $L^2(M)$-orthogonal decomposition $C^\infty \mathrm{S}^p M = \ker \Delta_S \oplus \mathrm{Im}\,\Delta_S$ is valid. The symmetric tensor $\varphi \in C^\infty \mathrm{S}^p M$ such that $\varphi \in \ker \Delta_S$ is called *Δ_S-harmonic section* (see [16, p. 104]), and the space of such tensors on a compact Riemannian manifold (M, g) is finite-dimensional. The following is valid.

Proposition 5.2. *On a compact Riemannian manifold (M, g) the space of Δ_S-harmonic sections is finite-dimensional.*

Using Proposition 5.2 and (5.3), we can formulate the following

Corollary 5.3. *On a Riemannian manifold (M, g), any divergence-free or, e.g., traceless Killing p-tensor is a Δ_S-harmonic section.*

From (5.3) we deduce the Bochner-Weitzenböck formula (e.g., [15] and [16, p. 106])

$$\frac{1}{2}\,\Delta\,\|\,\varphi\|^2 = -g(\Delta_S\,\varphi,\,\varphi) - g(\Re(\varphi), \varphi) + \|\,\nabla\,\varphi\,\|^2\,,$$

where for $\nabla\varphi$ the pointwise $\mathrm{O}(n, \mathbb{R})$-irreducible decomposition (4.1) holds. Thus,

$$\frac{1}{2}\,\Delta\,\|\,\varphi\,\|^2 = -g(\Delta_S\,\varphi,\,\varphi) - g(\Re(\varphi),\,\varphi) + \|\,D_1\varphi\,\|^2 + \|\,D_2\varphi\,\|^2 + \|\,D_3\varphi\,\|^2\,. \tag{5.4}$$

For a symmetric conformal Killing p-tensor, the formula (5.4) takes the form

$$\frac{1}{2}\,\Delta\,\|\varphi\|^2 = 2^{-1}(n + 2(p-2))\,g(\delta\,\delta^*\varphi,\,\varphi) - g(\Re(\varphi),\,\varphi) + \|\,D_2\varphi\|^2 + \|\,D_3\varphi\|^2. \tag{5.5}$$

Suppose that M is compact, then integrating (5.5) we obtain

$$\int_M g(\Re(\varphi), \varphi)\,d\,\mathrm{V}_g = 2^{-1}(n{+}2p{-}4)\int_M \|\delta^*\varphi\|^2 d\,\mathrm{V}_g + \int_M (\|D_2\varphi\|^2 {+} \|D_3\varphi\|^2)\,d\,\mathrm{V}_g \geq 0,$$

because $\int_M g\big(\delta\,\delta^*\varphi,\,\varphi\big)\,d\,\mathrm{V}_g = \int_M \|\,\delta^*\varphi\,\|^2 d\,\mathrm{V}_g \geq 0$. On (M, g) of nonpositive curvature $\Phi_p(\varphi, \varphi) = g(\Re(\varphi),\,\varphi) \leq 0$ holds for any $\varphi \in \mathrm{S}_0^p M$ (see [8, 7]). If there is a point at which the sectional curvature is negative, then $\Phi_p(\varphi,\,\varphi) = g(\Re(\varphi),\,\varphi) < 0$ for any symmetric p-form $\varphi \in \mathrm{S}_0^p M$. Based on the above equality, we get the following

Proposition 5.4. *On a compact Riemannian manifold (M, g) of nonpositive sectional curvature* sec*, each symmetric conformal Killing tensor $\varphi \in C^\infty \mathrm{S}_0^p M$ is parallel, i.e., $\nabla\,\varphi = 0$. Moreover, if there is a point at which* $\sec < 0$*, then on (M, g) there are no nonzero symmetric conformal Killing p-tensors $\varphi \in C^\infty \mathrm{S}_0^p M$.*

One can show $\frac{1}{2}\Delta\|\varphi\|^2 = \|\varphi\|\,\Delta\|\varphi\| + \|\,d\,\|\varphi\|\,\|^2$, where $\|\nabla\varphi\|^2 \geq \|\,d\,\|\,\varphi\,\|\,\|^2$ by *Kato's inequality* (e.g., [16, p. 105]). Thus, the above equality takes the following form:

$$\|\varphi\|\,\Delta\|\varphi\| = \frac{1}{2}\Delta\|\varphi\|^2 - \|\,d\,\|\varphi\|\,\|^2 \geq \frac{1}{2}\Delta\|\varphi\|^2 - \|\nabla\varphi\|^2,$$

where $\Delta\|\varphi\|^2$ due to (5.4) satisfies the inequality

$$\frac{1}{2}\Delta\|\varphi\|^2 \geq -g(\Delta_S\,\varphi,\,\varphi) - g(\Re(\varphi),\,\varphi).$$

Summing up, we get the following inequality:

$$\|\varphi\|\,\Delta\|\varphi\| \geq -g(\Delta_S\,\varphi,\varphi) - \Phi_p(\varphi,\varphi). \tag{5.6}$$

Let further $\varphi \in C^\infty \mathrm{S}^p_0 M$ be a Killing p-tensor, for which, as was proved above, $\Delta_S\,\varphi = 0$, then the inequality (5.6) can be rewritten as

$$\|\varphi\|\,\Delta\|\varphi\| \geq -\Phi_p(\varphi,\varphi). \tag{5.7}$$

For (M,g) of nonpositive curvature, from (5.7) we find $\Delta\|\varphi\| \geq 0$, thus, $\|\varphi\|$ is a nonnegative subharmonic function for any Killing p-tensor $\varphi \in \mathrm{S}^p_0 M$. There is a well-known theorem (see [14, p. 288]): On a complete simply connected Riemannian manifold (M,g) of nonpositive curvature, any nonnegative subharmonic function $f \in C^2(M)$ satisfying $\int_M f^q\,d\,\mathrm{V}_g < \infty$ for some $q \in (0,\infty)$, is constant. Setting $f = \|\varphi\|$, we find $\|\varphi\| = C$ for some real constant C, thus, $\nabla\varphi = 0$. On the other hand, in this case

$$\int_M \|\varphi\|^q\,d\,\mathrm{V}_g = C^q \int_M d\,\mathrm{V}_g = C^q\,\mathrm{Vol}(M,g).$$

Since we assume $\|\varphi\| \in L^q(M)$ for some $0 < q < \infty$, then for $C \neq 0$ the volume of (M,g) must be finite. If the volume of (M,g) is infinite, then necessarily $\varphi \equiv 0$. The following has been proven.

Theorem 5.5. *If a simply connected complete (M,g) has nonpositive sectional curvature, then the symmetric Killing p-tensor $(p \geq 2)$ $\varphi \in \mathrm{S}^p_0 M$ such that*

$$\int_M \|\varphi\|^q\,d\,\mathrm{V}_g < \infty \tag{5.8}$$

for some $q \in (0,\infty)$ is parallel; and if (M,g) has infinite volume, then $\varphi \equiv 0$.

A Riemannian manifold (M,g) with $\delta^*\mathrm{Ric} = 0$ was popular [3, pp. 450-451]. In this case, $\Delta_S\,\mathrm{Ric} = 0$, thus, by Theorem 5.5, $\mathrm{Ric} = 0$ (for a compact M, see [3, p. 451]).

Let $M = G/H$ be a Riemannian symmetric space of noncompact type with a G-invariant metric g. Then (M,g) is a complete Riemannian manifold of nonpositive sectional curvature and negative definite Ricci tensor, thus, it is irreducible (see [12, pp. 226, 236]). Therefore, it is true the following

Corollary 5.6. *On a Riemannian symmetric space (M,g) of noncompact type, each symmetric Killing p-tensor $(p \geq 2)$ $\varphi \in \mathrm{S}^p_0 M$ such that (5.8) holds for some $q \in (0,\infty)$, is parallel. Moreover, if $p = 2$, then $\varphi \equiv 0$.*

6 Global Riemannian geometry of rank $p \geq 2$ Codazzi tensors

For a Codazzi p-tensor ($p > 3$) $\varphi \in C^\infty S^p M$, from $\nabla\varphi \in C^\infty S^{p+1} M$ we conclude that $\nabla(\text{trace}_g \varphi) \in C^\infty S^{p-2} M$. From the condition (also defining the Codazzi p-tensor)

$$P_2\varphi = \nabla \varphi - \frac{1}{p+1}\,\delta^*\varphi = 0, \tag{6.1}$$

it follows that $\delta\,\varphi = -\nabla\,(\text{trace}_g\,\varphi)$ for any $p \geq 2$. Therefore, the following is true.

Proposition 6.1. *For any Codazzi p-tensor $\varphi \in S^p M$, where $p > 3$, on the Riemannian manifold (M, g) the symmetric form $\text{trace}_g\,\varphi$ is a Codazzi $(p-2)$-tensor. For $p \geq 2$, each traceless Codazzi p-tensor φ has zero divergence.*

Based on (6.1) for the divergence-free Codazzi tensor $\varphi \in S^p M$, we obtain

$$\bar{\Delta}\,\varphi = \frac{1}{p+1}\,P_1^* P_1\,\varphi = \frac{1}{p+1}\,\Delta_S\,\varphi.$$

Thus, it follows from the Weitzenböck expansion (5.3) that

$$\bar{\Delta}\,\varphi = -\frac{1}{p+1}\,\Re(\varphi). \tag{6.2}$$

Therefore, we can formulate the following

Proposition 6.2. *Any divergence-free Codazzi p-tensor φ on a Riemannian manifold (M, g) belongs to the kernel of the Lichnerovich Laplacian $\Delta_L = \bar{\Delta} + \frac{1}{p+1}\,\Re$.*

From (6.2) we get the Bochner-Weitzenböck formula

$$\frac{1}{2}\,\Delta\,\|\,\varphi\,\|^2 = \frac{1}{p+1}\,\Phi_p(\,\varphi,\,\varphi) + \|\,\nabla\,P_1\,\|^2. \tag{6.3}$$

Using (6.3), we obtain the inequality

$$\|\,\varphi\,\|\,\Delta\,\|\,\varphi\,\| \geq \frac{1}{p+1}\,\Phi_p(\varphi,\,\varphi). \tag{6.4}$$

On (M, g) of nonnegative sectional curvature, we have the inequality $\Phi_p(\varphi,\,\varphi) \geq 0$ for any $\varphi \in S^p M$ (see [4]). If this assumption is true, then from (6.4) we get $\Delta\,\|\,\varphi\,\| \geq 0$. As a result, $\|\,\varphi\,\|$ becomes a nonnegative subharmonic function for any divergence-free Codazzi p-tensor $\varphi \in S^p M$. Due to S.T. Yau (see [16, p. 262] and [28]), on a complete (M, g) of infinite volume the only nonnegative subharmonic function f satisfying $f \in L^q(M)$ for some $1 < q < \infty$, is $f \equiv 0$. Since a complete noncompact Riemannian manifold of nonnegative sectional curvature has infinite volume (see [14]), we get $\varphi \equiv 0$. The following theorem is proved.

Theorem 6.3. *On a complete noncompact Riemannian manifold (M, g) of nonnegative sectional curvature there is no nonzero divergence-free Codazzi tensor $\varphi \in S^p M$ $(p \geq 2)$ such that (5.8) holds for some $q > 1$.*

Remark 6.1. There are no complete noncompact conformally flat (M, g) of nonnegative sectional curvature and constant scalar curvature such that Ric satisfies (5.8) for some $q > 1$, since, in this case, Ric is a Codazzi divergence-free tensor, [3, p. 432].

Let $M = G/H$ be a Riemannian symmetric space of compact type with a G-invariant metric g. Then (M, g) is compact with nonnegative sectional curvature and positive definite Ricci tensor, thus, it is irreducible (see [12, p. 256]).

The following theorem generalizes the result from [10].

Corollary 6.4. *On a Riemannian symmetric space (M, g) of compact type, any divergence-free Codazzi p-tensor $\varphi \in \mathrm{S}^p M$ for $p \geq 2$ has a constant length. In particular, if $p = 2$, then $\varphi = C\, g$ for some real constant C.*

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