A spacetime admitting semiconformal curvature tensor

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Abstract. In the present paper, our main objective is to study spacetimes which admit a semiconformal curvature tensor. First, we prove that the energy-momentum tensor with vanishing semiconformal curvature tensor, satisfying Einstein's field equations (with cosmological constant), is covariantly constant. Next, we prove that if in a perfect fluid spacetime with divergence-free semiconformal curvature tensor satisfying Einstein field equations without cosmological constant, has constant pressure and density. Finally, we prove that if the perfect fluid spacetime has vanishing semiconformal curvature tensor satisfying Einstein field equations without cosmological constant, then the spacetime has constant energy density and isotropic pressure, and the perfect fluid always behaves as having a cosmological constant.

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1 Introduction

In Riemannian manifolds, the structure of the Lorentzian manifold is the most important subclass of semi-Riemannian manifolds and plays a significant role in Cosmology and in General Relativity. The spacetime of General Relativity can be viewed as an associated four-dimensional semi-Riemannian manifold with the Lorentzian metric of signature $(-,+,+,+)$.

The distribution of matter contents and its energy-momentum in the spacetime of General Relativity manifests through the components of energy the momentum tensor, which is divergence-free, due to the Einstein field equations ([15]), and this will be done if it is covariantly constant ([7]). M.C. Chaki and Sarbari Ray ([7]) also showed that a spacetime admitting covariantly constant energy-momentum tensor is Ricci symmetric, i.e.,, $\nabla\mathcal{R} = 0$, where $\mathcal{R}$ is the Ricci tensor of the spacetime. The study of different tensors for the spacetime of General Relativity attracted many

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researchers. Recently, Z. Ahsan and M. Ali studied the symmetries along different tensors for Petrov type D ([2]), Petrov type N ([3]) gravitational fields and vanishing of a curvature tensor in perfect fluid settings ([1]). The spacetimes with semisymmetric energy-momentum tensor were discussed by U.C. De and L. Velimirovič ([9]), spacetimes with pseudo-quasi-conformal curvature tensor, by Y. J. Suh et al. in ([19]), pseudo $\mathcal{Z}$ symmetric spacetimes, by C. A. Mantica and Y. J. Suh ([14]) and the literature is still expanding.

The motivational aspect of the study of spacetime models in cosmology is to acquire data about stages of the Universe evolution that are categorized into three stages, namely, the initial phase soon after the Big Bang when the viscosity and heat flux effects were very articulated, the intermediate phase when the viscosity impact was no longer important but the heat flux was not negligible, and the final phase (extending to the current state) when the impact of both viscosity and heat flux become negligible. It is notable that, in the standard cosmological models, the matter content of the universe is assumed to behave like a perfect fluid.

Recently in 2017, J. Kim ([12], [13]) introduced a curvature-like tensor such that its (1, 3) components remains invariant under conharmonic transformations ([18]); this curvature-like tensor $\mathcal{P}$ of type (1, 3) on a Riemannian manifold, is called semiconformal curvature tensor, and is defined by

$$\mathcal{P}^h_{ijk} = -(n-2)b\mathcal{W}^h_{ijk} + [a + (n-2)b]\mathcal{H}^h_{ijk}, \tag{1.1}$$

where $\mathcal{W}$ is the Weyl conformal curvature tensor, $\mathcal{H}$ is the conharmonic curvature tensor and a, b are constants which are not simultaneously zero. The conformal and the conharmonic curvature tensor of the type (1, 3) are defined respectively, by

$$\mathcal{W}^h_{ijk} = \mathcal{R}^h_{ijk} + \frac{1}{n-2}(\delta^h_j\mathcal{R}_{ik} - \delta^h_k\mathcal{R}_{ij} + g_{ik}\mathcal{R}^h_j - g_{ij}\mathcal{R}^h_k) + \frac{r}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}), \tag{1.2}$$

and

$$\mathcal{H}^h_{ijk} = \mathcal{R}^h_{ijk} + \frac{1}{n-2}(\delta^h_j\mathcal{R}_{ik} - \delta^h_k\mathcal{R}_{ij} + g_{ik}\mathcal{R}^h_j - g_{ij}\mathcal{R}^h_k), \tag{1.3}$$

If $b = 0$, then from equation (1.1) it follows that the semiconformal curvature tensor reduces to the conharmonic curvature tensor, provided that $a \neq 0$, and if $a+(n-2)b = 0$, then the semiconformal curvature tensor is equivalent to the conformal curvature tensor, provided that $b \neq 0$. In view of equation (1.1), we can easily verify that the curvature tensor $\mathcal{P}_{hijk}$ of the type (0, 4) satisfies the following properties

$$\mathcal{P}_{hijk} = -\mathcal{P}_{ihjk} = -\mathcal{P}_{hikj} = \mathcal{P}_{jkih} \tag{1.4}$$

and

$$\mathcal{P}_{hijk} + \mathcal{P}_{jhik} + \mathcal{P}_{ijhk} = 0. \tag{1.5}$$

U.C. De and Y.J. Suh ([10]) obtained certain notable results on weakly semiconformally symmetric manifolds. Further, the semiconformal symmetry of spacetime was studied by N. A. Pundeer et al. ([17]). They introduced a new symmetry known as the semiconformal curvature collineation, which is helpful in solving the Einstein field equations. However, the semiconformal curvature tensor is not used directly in solving these equations.

The plan of this paper is as follows. In Section 2, we show that the covariant derivative of energy-momentum tensor vanishes in semiconformally flat spacetimes. In Section 3, we consider the perfect fluid and semiconformally flat spacetimes and we obtain the expression of the Ricci operator and derive some results regarding the behavior of the fluid.

2 Semiconformal curvature tensor with perfect fluid spacetime

The semiconformal curvature tensor for $n = 4$ is defined as

$$\mathcal{P}^h_{ijk} = a[\mathcal{R}^h_{ijk} + \frac{1}{2}(\delta^h_j \mathcal{R}_{ik} - \delta^h_k \mathcal{R}_{ij} + g_{ik}\mathcal{R}^h_j - g_{ij}\mathcal{R}^h_k)] - \frac{br}{3}(\delta^h_k g_{ij} - \delta^h_j g_{ik}). \tag{2.1}$$

If $\mathcal{P}^h_{ijk} = 0$, then (2.1) leads to

$$a\mathcal{R}^h_{ijk} = -\frac{a}{2}(\delta^h_j \mathcal{R}_{ik} - \delta^h_k \mathcal{R}_{ij} + g_{ik}\mathcal{R}^h_j - g_{ij}\mathcal{R}^h_k) - \frac{br}{3}(\delta^h_k g_{ij} - \delta^h_j g_{ik}). \tag{2.2}$$

Now, taking the transvection over h and j, we get

$$\mathcal{R}_{ik} = -\left(\frac{a + 2b}{a}\right)\frac{r}{4} g_{ik}. \tag{2.3}$$

Einstein's field equation with cosmological constant is given by

$$\mathcal{R}_{ij} - \frac{r}{2} g_{ij} + \lambda g_{ij} = \mathcal{K}T_{ij}. \tag{2.4}$$

Using the relation (2.3) in (2.4), we get

$$T_{ij} = \frac{1}{\mathcal{K}}\left[\lambda - \left(\frac{3a + 2b}{a}\right)\frac{r}{4}\right] g_{ij}. \tag{2.5}$$

Now, taking the covariant derivative of (2.5), we obtain

$$T_{ij;e} = \left[\frac{\lambda}{\mathcal{K}} - \left(\frac{3a + 2b}{a}\right)\frac{r}{\mathcal{K}4} g_{ij}\right]_{;e} \tag{2.6}$$

Since a semiconformally flat spacetime is an Einstein space, the scalar curvature r is constant, that is, thhe equation will take the following form, as the metric tensor is covariantly constant:

$$T_{ij;e} = 0. \tag{2.7}$$

Thus, we have the following theorem.

Theorem 2.1. *The energy-momentum tensor of a semiconformally flat spacetime satisfying the Einstein field equations with cosmological term, is covariantly constant.*

The energy-momentum tensor for a perfect fluid spacetime is given by

$$T_{ij} = (\mu + p)u_i u_j + p g_{ij}, \tag{2.8}$$

where μ is the energy density, p is the isotropic pressure and u_i is the fluid-four velocity. After transvecting (2.8) and using $u_i u^i = -1$, we get

$$T = -\mu + 3p. \tag{2.9}$$

The Einstein field equations without cosmological constant are

$$\mathcal{R}_{ij} - \frac{1}{2} g_{ij} r = \mathcal{K} T_{ij}, \tag{2.10}$$

which, after transvection, takes the form

$$r = -\mathcal{K} T. \tag{2.11}$$

M. Ali et al. ([5]), defined the divergence of the semiconformal curvature tensor as

$$\mathcal{P}^h_{ijk;h} = a\left[(\mathcal{R}_{ij} - \frac{1}{2} r g_{ij})_{;k} - (\mathcal{R}_{ik} - \frac{1}{2} r g_{ik})_{;j}\right] - \frac{b}{3}(r_{;k} g_{ij} - r_{;j} g_{ik}). \tag{2.12}$$

Using (2.10) and (2.11), we note that (2.12) leads to

$$\mathcal{P}^h_{ijk;h} = a[T_{ij;k} - T_{ik;j}] + \frac{b\mathcal{K}}{3}[T_{;k} g_{ij} - T_{;j} g_{ik}] \tag{2.13}$$

Now, for spacetimes having a divergence-free semiconformal curvature tensor, (2.13), reduces to

$$a[T_{ij;k} - T_{ik;j}] + \frac{b\mathcal{K}}{3}[T_{;k} g_{ij} - T_{;j} g_{ik}] = 0. \tag{2.14}$$

By substituting expression of T_{ij} and T from (2.8) and (2.9) respectively, we get

$$\begin{aligned} & a[(\mu+p)_{;k} u_i u_j + (\mu+p) u_{i;k} u_j + (\mu+p) u_i u_{j;k} + p_{;k} g_{ij} - (\mu+p)_{;j} u_i u_k \\ - \quad & (\mu+p) u_{i;j} u_k - (\mu+p) u_i u_{k;j} - p_{;j} g_{ik}] - \frac{b\mathcal{K}}{3}[(\mu - 3p)_{;k} g_{ij} - (\mu - 3p)_{;j} g_{jk}] = 0 \end{aligned}$$

By contracting the above equation by u^k, we yield

$$\begin{aligned} & a[(\mu+p)^{\cdot} u_i u_j + (\mu+p) u_i^{\cdot} u_j + (\mu+p) u_i u_j^{\cdot} + p^{\cdot} g_{ij} + (\mu+p)_{;j} u_i \\ + \quad & (\mu+p) u_{i;j} - p_{;j} u_i] - \frac{b\mathcal{K}}{3}[(\mu - 3p)^{\cdot} g_{ij} - (\mu - 3p)_{;j} u_i] = 0, \end{aligned} \tag{2.15}$$

were we denote by a overhead dot the covariant derivative along the fluid flow vector. Also, from the conservation of energy-momentum tensor ($T^{ij}_{;j} = 0$), we get

$$\begin{aligned} & (\mu+p) u_i^{\cdot} = -p_{;i} + p^{\cdot} u_i \qquad \text{(force equation)} \\ & \text{and} \\ & \mu^{\cdot} = -(\mu+p)\theta \qquad \text{(energy equation)}, \end{aligned} \tag{2.16}$$

where $\theta = u^i_{;i}$ is the expansion scalar. The covariant derivative of the velocity vector can be split into kinematical quantities (see [11]):

$$u_{i;j} = \frac{1}{3}\theta(g_{ij} + u_i u_j) - u^{\cdot}_i u_j + \sigma_{ij} + \omega_{ij}, \tag{2.17}$$

Making use of the (2.15) and (2.16), we get

$$\begin{aligned} & a[\mu^{\cdot} u_i u_j + 3p^{\cdot} u_i u_j - p_{;i} u_j + p^{\cdot} g_{ij} + \mu_{;j} u_i + (\mu + p) u_{i;j} - p_{;j} u_i] \\ - \quad & \frac{b\mathcal{K}}{3}[\mu^{\cdot} g_{ij} - 3p^{\cdot} g_{ij} - \mu_{;j} u_i + 3p_{;j} u_i] = 0 \end{aligned}$$

Now, contracting the above equation by u^i, we get

$$-\left(\frac{3a + b\mathcal{K}}{3}\right)\mu^{\cdot} u_j - (3a - b\mathcal{K})p^{\cdot} u_j - \left(\frac{3a + b\mathcal{K}}{3}\right)\mu_{;j} + (a + b\mathcal{K})p_{;j} = 0. \tag{2.18}$$

We can state now the following result

Theorem 2.2. *For a perfect fluid spacetime with divergence-free semiconformal curvature tensor, both pressure and density are constant.*

3 Cosmological models with vanishing semiconformal curvature tensor

In this section, we consider a perfect fluid spacetime admitting a vanishing semiconformal curvature tensor, which satisfies the Einstein field equations (without cosmological constant).

In view of equations (2.8) and (2.10), we get

$$\mathcal{R}_{ij} - \frac{r}{2} g_{ij} = \mathcal{K}[p g_{ij} + (\mu + p) u_i u_j]. \tag{3.1}$$

Transvecting (3.1), we obtain

$$r = \mathcal{K}(\mu - 3p). \tag{3.2}$$

In view of (2.3) and (3.2), the Ricci tensor of the semiconformally flat spacetime can be written as

$$\mathcal{R}_{ij} = -\left(\frac{a + 2b}{a}\right)\frac{[\mathcal{K}(\mu - 3p)]}{4} g_{ij},$$

or, in global form,

$$\mathcal{R}(X, Y) = -\left(\frac{a + 2b}{a}\right)\frac{[\mathcal{K}(\mu - 3p)]}{4} g(X, Y). \tag{3.3}$$

Let $\mathcal{Q}$ be the Ricci operator given by

$$g(\mathcal{Q}X, Y) = \mathcal{R}(X, Y) \qquad \text{and} \qquad \mathcal{R}(\mathcal{Q}X, Y) = \mathcal{R}^2(X, Y). \tag{3.4}$$

Now, with the help of (3.3) and (3.4), we obtain

$$\mathcal{R}(\mathcal{Q}X, Y) = \left(\frac{a+2b}{a}\right)^2 \left[\frac{\mathcal{K}(\mu - 3p)}{4}\right]^2 g(X, Y). \tag{3.5}$$

By contraction over X and Y, we get

$$||\mathcal{Q}X||^2 = \left(\frac{a+2b}{a}\right)^2 \left[\frac{\mathcal{K}(\mu - 3p)}{2}\right]^2. \tag{3.6}$$

Thus, we obtain the following

Theorem 3.1. *If in a semiconformally flat perfect fluid spacetime the Einstein field equations without cosmological constant hold good, then the square of the length of the Ricci operator is* $\left(\frac{a+2b}{a}\right)^2 \left[\frac{\mathcal{K}(\mu-3p)}{2}\right]^2$.

The energy-momentum tensor for the dust case is

$$T_{ij} = \mu u_i u_j. \tag{3.7}$$

In view of equations (2.10) and (3.7), we get

$$\mathcal{R}_{ij} - \frac{r}{2} g_{ij} = \mathcal{K}\mu u_i u_j. \tag{3.8}$$

Now, multiplying equation (3.8) by g_{ij}, we infer

$$r = \mathcal{K}\mu. \tag{3.9}$$

Making use of (2.3) and (3.9), we have

$$\mathcal{R}_{ij} = -\left(\frac{a+2b}{a}\right) \frac{\mathcal{K}\mu}{4} g_{ij},$$

or,

$$\mathcal{R}(X, Y) = -\left(\frac{a+2b}{a}\right) \frac{\mathcal{K}\mu}{4} g(X, Y). \tag{3.10}$$

From (3.4) and (3.10), we yield

$$\mathcal{R}(\mathcal{Q}X, Y) = \left(\frac{a+2b}{a}\right)^2 \left[\frac{\mathcal{K}\mu}{4}\right]^2 g(X, Y) \tag{3.11}$$

Now, putting $X = Y$, we get

$$||\mathcal{Q}X||^2 = \left(\frac{a+2b}{a}\right)^2 \left[\frac{\mathcal{K}\mu}{2}\right]^2. \tag{3.12}$$

Thus, we may state the following

Theorem 3.2. *In a dust cosmological model for the semiconformally flat spacetime in which the Einstein equations without consmological constant hold, the square of the length of the Ricci operator is given by* $\left(\frac{a+2b}{a}\right)^2\left[\frac{\mathcal{K}\mu}{2}\right]^2$.

By virtue of (2.3), equation (3.1) may take the form:

$$\left[-\left\{\frac{3a+2b}{4a}\right\}-\mathcal{K}p\right]g_{ij}=\mathcal{K}(\mu+p)u_iu_j. \tag{3.13}$$

After contraction by g^{ij}, equation (3.13) leads to

$$r=\frac{\mathcal{K}a}{3a+2b}(\mu-3p). \tag{3.14}$$

Again, by multiplying with g^{ik} and putting $k=j=l$ in (3.13), we get

$$r=\frac{4\mathcal{K}a\mu}{3a+2b}. \tag{3.15}$$

Equations (3.14) and (3.15) imply $\mu+p=0$. Further, (2.8) leads to

$$T_{ij}=pg_{ij}. \tag{3.16}$$

M. Ali and N. A. Pundeer ([4]) proved that the scalar curvature r of a semiconformally flat spacetime is constant, and therefore from (3.15), we readily find that $\mu=$ constant, and thus from $\mu+p=0$ we obtain $p=$ constant. Now, $\mu+p=0$ implies that the fluid behaves as a cosmological constant ([16]). This is also known as *the phantom barrier* ([8]). In Cosmology, such a choice $\mu=-p$ leads to a rapid expansion of the spacetime, which is now termed as *inflation* ([6]).

Thus, we may state the following

Theorem 3.3. *If a perfect fluid spacetime with vanishing semiconformal curvature tensor obeys the Einstein field equations without cosmological constant, then the energy density and the isotropic pressure are constant. Also, the spacetime represents an inflation and the fluid behaves as a cosmological constant.*

References

[1] Z. Ahsan and M. Ali, *Curvature tensor for the spacetime of General Relativity*, Int. Journal Geometric Methods in Modern Physics, 14 (5) (2017), 1750078.

[2] Z. Ahsan and M. Ali, *Symmetries of type D pure radiation fields* Int. Jour. of Theo. Phys., 51 (2012), 2044-2055.

[3] Z. Ahsan and M. Ali, *Symmetries of type N pure radiation fields*, Int. Jour. Theor. Phys., 54 (5) (2015), 1397-1407.

[4] M. Ali and N.A. Pundeer, *Semiconformal curvature tensor and Spacetime of General Relativity*, Differential Geometry - Dynamical Systems, 21 (2019), 14-22.

[5] M. Ali, N. A. Pundeer and A. Ali, *Semiconformal curvature tensor and perfect fluid spacetimes in General Relativity*, Journal of Taibah University for Science, 14 (1) (2020), 205-210.

[6] K. Arslan, R. Deszcz, R. Ezentas, M. Hotlos and C. Murathan, *On generalized Robertson-Walker spacetimes satisfying some curvature condition*, Turk. J. of Math., 38 (2014), 353-373.

[7] M.C. Chaki and S. Ray, *Spacetimes with covariantly constant energy-momentum tensor*, Int. J. Theor. Physics, 35 (1996), 1027-1032.

[8] S. Chakraborty, N. Mazumder, and R. Biswas, *Cosmological evolution across phantom crossing and the nature of the horizon*, Astrophysics Space Science, 334 (2011), 183-186.

[9] U. C. De and L. Velimirovič, *Spacetimes with semisymmetric energy-momentum tensor*, International Journal of Theoretical Physics, 54 (6) (2015), 1779-1783.

[10] U. C. De and Y.J. Suh, *On weakly semiconformally symmetric manifolds*, Acta Mathematica Hungerica, 157 (2) (2019), 503-521.

[11] Ellis GFR., *Relativistic Cosmology,* Gen. Rel. Grav., 47 (1971), 104-82.

[12] J. Kim, *A type of conformal curvature tensor*, Far East J. Math. Sci., 99 (1) (2016), 61-74.

[13] J. Kim, *On pseudo semiconformally symmetric manifolds*, Bull. Korean Math. Soc., 54(1) (2017), 177-186.

[14] C.A. Mantica and Y.J. Suh, *Pseudo-Z symmetric space-times*, Journal of Mathematical Physics, 55 (4) (2014), 12.

[15] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[16] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, *Exact Solutions of Einstein's Field Equations*, 2nd ed. Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2003.

[17] N. A. Pundeer, M. Ali, N. Ahmad and Z. Ahsan, *Symmetry - a new symmetry of the spacetime manifold of the General Relativity*, 10 January (2019), (accepted in JMCS Scopus on December 12, 2019).

[18] Y. Ishii, *On conharmonic transformations*, Tensor (N.S.), 7 (1957), 73-80.

[19] Y. J. Suh, V. Chavan and N. A. Pundeer, *Pseudo-quasi-conformal curvature tensor and spacetimes of General Relativity.*, Filomat, 35 (2) (2021), 657-666.

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