Golden generic lightlike submanifolds of a golden semi-Riemannian manifold

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Abstract. In this paper, we introduce the concept of golden generic lightlike submanifolds of a golden semi-Riemannian manifold. We also present one non-trivial example of this class of lightlike submanifolds. Then, we derive necessary and sufficient conditions for the induced connection of golden generic lightlike submanifolds to be a metric connection. Further, we establish a characterization result for the integrability of the distributions associated with golden generic lightlike submanifolds. Finally, we analyze totally umbilical and minimal generic lightlike submanifolds of a golden semi-Riemannian manifold.

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1 Introduction

The study of differentiable manifolds with structure has attracted the attention of many mathematicians due to their own elegance as well as their applications in physics. The concept of an almost complex structure and almost product structure has been studied by many geometers. Yano [18] as a generalization of an almost complex structure and an almost contact structure introduced the notion of f-structure which is a tensor field of type (1,1) satisfying the equation $f^3 + f = 0$. On a similar note, Crasmareanu and Hretcanu [3, 4] studied the concept of golden structures and their geometric properties on Riemannian manifold with the structure polynomial $f(x) = x^2 - x - Id = 0$. After this, various new type of submanifolds such as invariant, anti-invariant submanifolds, slant submanifolds and generic submanifolds of golden Riemannian manifolds have been studied by others (see, [2, 17]).
On the other hand, during the generalization of submanifold theory from Riemannian manifolds to semi-Riemannian manifolds, the induced metric on the submanifolds becomes degenerate and gives rise to the class of lightlike submanifolds in the semi-Riemannian category. As a result of degenerate metric, in a lightlike submanifold

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the tangent vector bundle intersects with the normal vector bundle. This unique geometric feature makes the study of lightlike submanifolds significant as well as more complicated in comparison to Riemannian submanifolds. The basic theory of lightlike submanifolds of semi-Riemannian manifolds was developed by Kupeli and Duggal-Bejancu (see, [6, 12]). Later the theory of lightlike submanifolds has been enriched by Duggal-Sahin and many others (see, [5, 8, 11] etc.). Recently, Poyraz and Yasar [14] initiated the study of golden structure in lightlike hypersurfaces of semi-Riemannian manifolds and proved several results on lightlike hypersurfaces of golden semi-Riemannian manifolds. In [15], Poyraz and Yasar investigated lightlike submanifolds of golden semi-Riemannian manifolds. In the sequel, Poyraz [16] studied the concept of golden GCR-lightlike submanifolds of a golden semi-Riemannian manifold. Since many significant applications of lightlike submanifolds have been found in the study of black holes, asymptotically flat spacetimes, Killing horizons and electromagnetic and radiation fields (see, [6, 9]) and very limited information available on general theory of lightlike submanifolds motivated us to study golden generic lightlike submanifolds of a golden semi-Riemannian manifold. After defining golden generic lightlike submanifold, we present one non-trivial example of this class of lightlike submanifolds. Then, we derive necessary and sufficient conditions for the induced connection of golden generic lightlike submanifolds to be a metric connection. Further, we establish a characterization result for the integrability of the distributions associated with golden generic lightlike submanifolds. We also show that for a totally umbilical golden generic lightlike submanifold, the distribution D is always integrable. Finally, we analyze minimal generic lightlike submanifolds of a golden semi-Riemannian manifold.

2 Preliminaries

2.1 Geometry of lightlike submanifolds

In this section, we recall some essential formulae and notations for lightlike submanifolds given by Duggal and Bejancu [6].

Assume (M_n, g) be an immersed submanifold of a semi-Riemannian manifold $(\tilde{M}_{m+n}, \tilde{g})$, where the metric $\tilde{g}$ with index q (constant), provided $m, n \geq 1$ and $1 \leq q \leq m+n-1$. If the metric $\tilde{g}$ is degenerate on TM, then T_pM and $T_pM^{\perp}$ both become degenerate and there exists a radical subspace $Rad(T_pM)$ such that $Rad(T_pM) = T_pM \cap T_pM^{\perp}$. If $Rad(TM) : p \in M \to Rad(T_pM)$ be a smooth distribution with rank $r(> 0)$ on M such that $1 \leq r \leq n$, then M is known as an r-lightlike submanifold of $\tilde{M}$. While $Rad(TM)$ of TM is defined as follows

$$Rad(TM) = \cup_{p\in M}\{\xi \in T_pM | g(u, \xi) = 0,\ \forall\ u \in T_pM, \xi \neq 0\}.$$

Thus, the tangent bundle TM and normal bundle $TM^{\perp}$ are given by

$$TM = Rad(TM) \perp S(TM) \text{ and } TM^{\perp} = Rad(TM) \perp S(TM^{\perp}). \tag{2.1}$$

Moreover there exists a local null frame $\{N_i\}$ of null sections with values in orthogonal complement of $S(TM^{\perp})$ in $S(TM^{\perp})^{\perp}$ satisfying

$$\tilde{g}(N_i, N_j) = 0, \quad \tilde{g}(N_i, \xi_j) = \delta_{ij}, \text{ for } \quad i, j \in \{1, 2, .., r\}, \tag{2.2}$$

where $\{\xi_j\}$ is a local basis of $\Gamma(Rad(TM))$. It implies that $tr(TM)$ and $ltr(TM)$, respectively, be the vector bundles in $T\tilde{M}\mid_M$ and $S(TM^{\perp})^{\perp}$ with the property

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}) \tag{2.3}$$

and

$$T\tilde{M}|_M = TM \oplus tr(TM) = S(TM) \perp (Rad(TM) \oplus ltr(TM)) \perp S(TM^{\perp}). \tag{2.4}$$

Let $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$, then in view of decomposition (2.4), the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + h^l(Y_1, Y_2) + h^s(Y_1, Y_2), \tag{2.5}$$

$$\tilde{\nabla}_{Y_1} W = -A_W Y_1 + \nabla^s_{Y_1} W + D^l(Y_1, W), \tag{2.6}$$

$$\tilde{\nabla}_{Y_1} N = -A_N Y_1 + \nabla^l_{Y_1} N + D^s(Y_1, N), \tag{2.7}$$

for $W \in \Gamma(S(TM^{\perp}))$, $Y_1, Y_2 \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$.
Furthermore employing Eqs. (2.5)-(2.7), we get

$$g(A_W Y_1, Y_2) = \tilde{g}(h^s(Y_1, Y_2), W) + \tilde{g}(Y_2, D^l(Y_1, W)), \tag{2.8}$$
$$\tilde{g}(D^s(Y_1, N), W) = \tilde{g}(A_W Y_1, N).$$

Moreover from Eq. (2.1), for $\xi \in \Gamma(Rad(TM))$, $Y_1, Y_2 \in \Gamma(TM)$ and for projection morphism S of TM on $S(TM)$, some new geometric objects of $S(TM)$ on M are

$$\nabla_{Y_1} SY_2 = \nabla^*_{Y_1} SY_2 + h^*(Y_1, Y_2),\ \nabla_{Y_1}\xi = -A^*_{\xi} Y_1 + \nabla^{*t}_{Y_1}\xi, \tag{2.9}$$

where $\{\nabla^*_{Y_1} SY_2, A^*_{\xi} Y_1\} \in \Gamma(S(TM))$, $\{h^*(Y_1, Y_2), \nabla^{*t}_{Y_1}\xi\} \in \Gamma(Rad(TM))$. Using Eqs. (2.5), (2.7) and (2.9), we obtain

$$\tilde{g}(h^*(Y_1, SY_2), N) = g(A_N Y_1, SY_2) \ \ \textit{and} \ \ \tilde{g}(h^l(Y_1, SY_2), \xi) = g(A^*_{\xi} Y_1, SY_2), \tag{2.10}$$

for $\xi \in \Gamma(Rad(TM))$, $Y_1, Y_2 \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$ and it is well-known that ∇^* is the metric connection on $S(TM)$.

2.2 Golden semi-Riemannian manifolds

Let $\tilde{M}$ be a C^{∞} – manifold. If a tensor field $\tilde{P}$ of the type (1, 1) satisfies the following equation

$$\tilde{P}^2 = \tilde{P} + I, \tag{2.11}$$

then $\tilde{P}$ is known as a golden structure on $\tilde{M}$, where I is the identity transformation [10]. Let $(\tilde{M}, \tilde{g})$ be a semi-Riemannian manifold and $\tilde{P}$ be a golden structure on $\tilde{M}$. If $\tilde{P}$ satisfies

$$\tilde{g}(\tilde{P}Y_1, Y_2) = \tilde{g}(Y_1, \tilde{P}Y_2), \tag{2.12}$$

then $(\tilde{M}, \tilde{g}, \tilde{P})$ is known as a golden semi-Riemannian manifold [13]. If $\tilde{P}$ is a golden structure, then Eq. (2.12) is equivalent to

$$\tilde{g}(\tilde{P}Y_1, \tilde{P}Y_2) = \tilde{g}(\tilde{P}Y_1, Y_2) + \tilde{g}(Y_1, Y_2), \tag{2.13}$$

for any $Y_1, Y_2 \in \Gamma(T\tilde{M})$. Also, we have

$$(\tilde{\nabla}_{Y_1}\tilde{P})Y_2 = 0. \tag{2.14}$$

3 Golden generic lightlike submanifolds

At first, we define golden generic lightlike submanifolds of golden semi-Riemannian manifolds following [5] as

Definition 3.1. Let M be a real r-lightlike submanifold of a semi-Riemannian manifold $\tilde{M}$. Then, M is said to be a golden generic lightlike submanifold if the screen distribution $S(TM)$ of M is expressed as

$$\begin{aligned} S(TM) &= \tilde{P}(S(TM)^{\perp}) \perp D_0 \\ &= \tilde{P}(Rad(TM)) \oplus \tilde{P}(ltr(TM)) \perp \tilde{P}(S(TM^{\perp})) \perp D_0, \end{aligned} \tag{3.1}$$

where D_0 is a non-degenerate almost complex distribution on M with respect to $\tilde{P}$, i.e., $\tilde{P}(D_0) = D_0$ and D' is a r-lightlike distribution on S(TM) such that $\tilde{P}(D') \subset tr(TM)$, where $D' = \tilde{P}(ltr(TM)) \perp \tilde{P}(S(TM^{\perp}))$.

Therefore using Eq. (3.1), the general decompositions of Eqs. (2.1) and (2.4) becomes

$$TM = D \oplus D', \quad T\tilde{M}|_M = D \oplus D' \oplus tr(TM),$$

where $D = Rad(TM) \perp \tilde{P}(Rad(TM)) \perp D_0$ is a $2r$-lightlike almost complex distribution on M.
Consider Q, P_1 and P_2 be the projections from TM to D, $\tilde{P}ltr(TM)$ and $\tilde{P}S(TM^{\perp})$, respectively. Then for $Y \in \Gamma(TM)$, we have

$$Y = QY + P_1Y + P_2Y, \tag{3.2}$$

applying $\tilde{P}$ to Eq. (3.2), we obtain

$$\tilde{P}Y = fY + \omega P_1Y + \omega P_2Y \tag{3.3}$$

and we can write Eq. (3.3) as

$$\tilde{P}Y = fY + \omega Y, \tag{3.4}$$

where fY and ωY, respectively, denote the tangential and transversal components of $\tilde{P}Y$. Similarly,

$$\tilde{P}V = EV, \tag{3.5}$$

for $V \in \Gamma(tr(TM))$, where EV is the section of TM.

Lemma 3.1. *Consider a golden generic lightlike submanifold M of a golden semi-Riemannian manifold $\tilde{M}$. Then for $Y_1, Y_2 \in \Gamma(TM)$, one has*

$$(\nabla_{Y_1} f)Y_2 = A_{\omega Y_2}Y_1 + Eh(Y_1, Y_2) \tag{3.6}$$

and

$$(\nabla^t_{Y_1} \omega)Y_2 = -h(Y_1, fY_2), \tag{3.7}$$

where

$$(\nabla_{Y_1} f)Y_2 = \nabla_{Y_1} fY_2 - f\nabla_{Y_1} Y_2, \quad (\nabla^t_{Y_1} \omega)Y_2 = \nabla^t_{Y_1} \omega Y_2 - \omega \nabla_{Y_1} Y_2.$$

Proof. For $Y_1, Y_2 \in \Gamma(TM)$, using Eq. (2.14), we have $\tilde{\nabla}_{Y_1}\tilde{P}Y_2 = \tilde{P}\tilde{\nabla}_{Y_1}Y_2$. Then, employing Eqs. (2.5)-(2.6), (3.4)-(3.5) and further comparing the tangential and transversal components, the assertions follow. □

Theorem 3.2. *Let M be a golden generic lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then*

$$f^2Y_1 = fY_1 + Y_1 - E\omega Y_1, \tag{3.8}$$

$$\omega f Y_1 = \omega Y_1, \tag{3.9}$$

$$fEV = EV, \quad V = \omega EV, \tag{3.10}$$

$$g(fY_1, Y_2) - g(Y_1, fY_2) = \tilde{g}(Y_1, \omega Y_2) - \tilde{g}(\omega Y_1, Y_2), \tag{3.11}$$

$$\begin{aligned} g(fY_1, fY_2) =& g(fY_1, Y_2) + g(Y_1, Y_2) + \tilde{g}(\omega Y_1, Y_2) - \tilde{g}(fY_1, \omega Y_2) \\ & - \tilde{g}(\omega Y_1, fY_2) - \tilde{g}(\omega Y_1, \omega Y_2), \end{aligned} \tag{3.12}$$

for $Y_1, Y_2 \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$.

Proof. For $Y_1, Y_2 \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, applying $\tilde{P}$ on both sides of Eq. (3.4) and using Eq. (2.11), we have

$$\begin{aligned} \tilde{P}^2Y_1 =& \tilde{P}fY_1 + \tilde{P}\omega Y_1, \\ fY_1 + \omega Y_1 + Y_1 =& f^2Y_1 + \omega fY_1 + E\omega Y_1. \end{aligned} \tag{3.13}$$

Then, equating the tangential components of above equation, we get Eq. (3.8). Further comparing the transversal components of Eq. (3.13), we obtain Eq. (3.9).
On a similar note, applying $\tilde{P}$ on both sides of Eq. (3.5) and using Eq. (2.11), we derive

$$\begin{aligned} \tilde{P}^2V =& \tilde{P}EY_1, \\ EV + V =& fEV + \omega EV \end{aligned} \tag{3.14}$$

and comparing the tangential and transversal parts of Eq. (3.14), we get Eq. (3.10). Further, employing Eqs. (2.12) and (3.4), we have

$$\tilde{g}(fY_1, Y_2) + \tilde{g}(\omega Y_1, Y_2) = \tilde{g}(Y_1, fY_2) + \tilde{g}(Y_1, \omega Y_2).$$

Thus, Eq. (3.11) is obtained. Moreover, using Eqs. (2.13) and (3.4), we derive

$$\begin{aligned} \tilde{g}(fY_1, \tilde{P}Y_2) + \tilde{g}(\omega Y_1, Y_2) + \tilde{g}(Y_1, Y_2) =& \tilde{g}(fY_1, fY_2) + \tilde{g}(fY_1, \omega Y_2) \\ & + \tilde{g}(\omega Y_1, fY_2) + \tilde{g}(\omega Y_1, \omega Y_2) \end{aligned}$$

and we obtain Eq. (3.12). Hence, the proof is completed. □

Theorem 3.3. *For a golden generic lightlike submanifold M of a golden semi-Riemannian manifold $\tilde{M}$, the Nijenhius tensor field with respect to the golden structure $\tilde{P}$ vanishes.*

Proof. For $Y, Z \in \Gamma(TM)$, employing Eqs. (2.11) and (2.14), the Nijenhius tensor field becomes

$$\begin{aligned} N(Y,Z) =& [\tilde{P}Y, \tilde{P}Z] + \tilde{P}^2[Y,Z] - \tilde{P}[\tilde{P}Y, Z] - \tilde{P}[Y, \tilde{P}Z] \\ =& \tilde{\nabla}_{\tilde{P}Y}\tilde{P}Z - \tilde{\nabla}_{\tilde{P}Z}\tilde{P}Y + \tilde{P}\{\tilde{\nabla}_Y Z - \tilde{\nabla}_Z Y\} + \tilde{\nabla}_Y Z - \tilde{\nabla}_Z Y \\ & - \tilde{P}\{\tilde{\nabla}_{\tilde{P}Y} Z - \tilde{\nabla}_Z \tilde{P}Y\} - \tilde{P}\{\tilde{\nabla}_Y \tilde{P}Z - \tilde{\nabla}_{\tilde{P}Z} Y\}. \\ =& \tilde{P}\tilde{\nabla}_Y Z - \tilde{P}\tilde{\nabla}_Z Y + \tilde{\nabla}_Y Z - \tilde{\nabla}_Z Y + \tilde{P}\tilde{\nabla}_Z \tilde{P}Y - \tilde{P}\tilde{\nabla}_Y \tilde{P}Z \\ =& \tilde{P}\tilde{\nabla}_Y Z - \tilde{P}\tilde{\nabla}_Z Y + \tilde{\nabla}_Y Z - \tilde{\nabla}_Z Y + \tilde{P}^2\tilde{\nabla}_Z Y - \tilde{P}^2\tilde{\nabla}_Y Z \\ =& 0. \end{aligned}$$

Hence, the result holds. □

Theorem 3.4. *Let M be a golden generic lightlike submanifold of a golden semi-Riemannian manifold $\tilde{M}$. Then f is a golden structure on D.*

Proof. By the definition of golden generic lightlike submanifolds, we have $\omega Y = 0$ for any $Y \in \Gamma(D)$. Further from Eq. (3.8), we have $f^2Y = fY + Y$. Thus, f becomes a golden structure on D. □

Next, we present a non-trivial example of golden generic lightlike submanifolds in a golden semi-Riemannian manifold $\tilde{M}$.

Example 3.2. Let M be a 6-dimensional submanifold of $(R^{12}_4, \tilde{g})$ with signature $(+,+,+,-,+,-,+,+,+,-,-,+)$ given by

$$x^1 = u^2, \quad x^2 = u^1, \quad x^3 = u^2\cos\theta, \quad x^4 = u^1\cos\theta, \quad x^5 = u^2\sin\theta,$$

$$x^6 = u^1\sin\theta, \quad x^7 = u^3 + 2u^4, \quad x^8 = u^3 + u^4, \quad x^9 = x^{10} = x^{11} = u^5,$$

$$x^{12} = 0, \quad \textit{where} \quad \theta \in R - \{\frac{n\pi}{2},\ n \in Z\}.$$

Then TM is spanned by Z_1, Z_2, Z_3, Z_4, Z_5, where

$$Z_1 = \partial x_2 + \cos\theta\partial x_4 + \sin\theta\partial x_6, \quad Z_2 = \partial x_1 + \cos\theta\partial x_3 + \sin\theta\partial x_5,$$

$$Z_3 = \partial x_7 + \partial x_8, \quad Z_4 = 2\partial x_7 + \partial x_8, \quad Z_5 = \partial x_9 + \partial x_{10} + \partial x_{11}.$$

Thus M is a 1-lightlike submanifold with $Rad(TM) = Span\{Z_1\}$. As $\tilde{P}Z_1 = Z_2$ implies that $\tilde{P}Rad(TM) \subset S(TM)$. As $\tilde{P}Z_3 = Z_4$ gives that $D_0 = Span\{Z_3, Z_4\}$. Further, by direct calculations, $S(TM^{\perp}) = Span\{W = 2\partial x_9 + \partial x_{10} + \partial x_{11} + \partial x_{12}\}$. Thus, we have $\tilde{P}Z_5 = W$. On the other hand, the lightlike transversal bundle $ltr(TM)$ is spanned by

$$N = \frac{1}{2}(\partial x_2 - \cos\theta\partial x_4 - \sin\theta\partial x_6).$$

Hence $D' = Span\{\tilde{P}W, \tilde{P}N\}$. Thus M is a proper golden generic lightlike submanifold of $(R^{12}_4, \tilde{g})$.

Theorem 3.5. *Consider a golden generic lightlike submanifold M of a golden semi-Riemannian manifold $\tilde{M}$. Then the induced connection is a metric connection if and only if*

$$f(\nabla^*_Y \tilde{P}\xi + h^*(Y, \tilde{P}\xi)) - \nabla^*_Y \tilde{P}\xi - h^*(Y, \tilde{P}\xi) \in \Gamma(Rad(TM))$$

and $Eh(Y, \tilde{P}\xi) = 0$, for $\xi \in \Gamma(Rad(TM))$ and $Y \in \Gamma(TM)$.

Proof. For the golden structure $\tilde{P}$ of a golden semi-Riemannian manifold $\tilde{M}$, one has $\tilde{\nabla}_Y \xi = \tilde{P}\tilde{\nabla}_Y \tilde{P}\xi - \tilde{\nabla}_Y \tilde{P}\xi$, where $\xi \in \Gamma(Rad(TM))$ and $Y \in \Gamma(TM)$. Further, employing Eqs. (2.5), (2.9), (3.4)-(3.5) and comparing the tangential components, we derive

$$\begin{aligned}\nabla_Y \xi =& f\nabla_Y \tilde{P}\xi + Eh(Y, \tilde{P}\xi) - \nabla_Y \tilde{P}\xi\\ =& f(\nabla^*_Y \tilde{P}\xi + h^*(Y, \tilde{P}\xi)) + Eh(Y, \tilde{P}\xi) - \nabla^*_Y \tilde{P}\xi - h^*(Y, \tilde{P}\xi).\end{aligned}$$

Then, $\nabla_Y \xi \in \Gamma(Rad(TM))$, if and only if

$$f(\nabla^*_Y \tilde{P}\xi + h^*(Y, \tilde{P}\xi)) - \nabla^*_Y \tilde{P}\xi - h^*(Y, \tilde{P}\xi) \in \Gamma(Rad(TM))$$

and $Eh(Y, \tilde{P}\xi) = 0$, which proves the result. □

Next we discuss the conditions for the integrability of distributions associated with golden generic lightlike submanifolds.

Theorem 3.6. *Let M be a golden generic lightlike submanifold of a golden semi-Riemannian manifold $\tilde{M}$. Then,*

(i) the distribution D is integrable, if and only if, $h(\tilde{P}Y_1, Y_2) = h(Y_1, \tilde{P}Y_2)$ for $Y_1, Y_2 \in \Gamma(D)$.

(ii) the distribution D' is integrable, if and only if, $A_{\omega Z}Z' = A_{\omega Z'}Z$ for $Z, Z' \in \Gamma(D')$.

Proof. For $Y_1, Y_2 \in \Gamma(D)$, using the Gauss formula, we have

$$\nabla_{Y_1} Y_2 = \tilde{\nabla}_{Y_1} Y_2 - h(Y_1, Y_2). \tag{3.15}$$

Then, applying $\tilde{P}$ on both sides of Eq. (3.15) and using Eqs. (3.4)-(3.5), we obtain

$$f\nabla_{Y_1} Y_2 + \omega\nabla_{Y_1} Y_2 = \nabla_{Y_1} \tilde{P}Y_2 + h(Y_1, \tilde{P}Y_2) - Eh(Y_1, Y_2). \tag{3.16}$$

Further, on comparing the transversal components of above equation, we derive

$$\omega\nabla_{Y_1} Y_2 = h(Y_1, \tilde{P}Y_2). \tag{3.17}$$

Then, interchanging the role of Y_1 and Y_2 in Eq. (3.17), we get

$$\omega\nabla_{Y_2} Y_1 = h(\tilde{P}Y_1, Y_2). \tag{3.18}$$

Further, from Eqs. (3.17)-(3.18), we obtain

$$\omega[Y_1, Y_2] = h(Y_2, \tilde{P}Y_1) - h(Y_1, \tilde{P}Y_2). \tag{3.19}$$

On the other hand, for $Z, Z' \in \Gamma(D')$, again using Gauss formula, we consider

$$\nabla_Z Z' = \tilde{\nabla}_Z Z' - h(Z, Z'). \tag{3.20}$$

Then, applying $\tilde{P}$ on both sides of Eq. (3.20) and using Eqs. (3.4)-(3.5), we obtain

$$f\nabla_Z Z' + \omega\nabla_Z Z' = -\,A_{\omega Z'}Z + \nabla^t_Z \omega Z' - Eh(Z, Z'). \tag{3.21}$$

Further, on comparing the tangential parts of above equation, we derive

$$f\nabla_Z Z' = -A_{\omega Z'}Z - Eh(Z, Z'). \tag{3.22}$$

Further, interchanging the role of Z and Z' in Eq. (3.22), we derive

$$f\nabla_{Z'} Z = -A_{\omega Z}Z' - Eh(Z, Z'). \tag{3.23}$$

Then, from Eqs. (3.22)-(3.23), we get

$$f[Z, Z'] = A_{\omega Z}Z' - A_{\omega Z'}Z. \tag{3.24}$$

Hence, the result follows from Eqs. (3.19) and (3.24). □

Definition 3.3. A golden generic lightlike submanifold M is said to be mixed geodesic if second fundamental form h satisfies

$$h(Y, Z) = 0, \quad for\ Y \in \Gamma(D)\ and\ Z \in \Gamma(D'). \tag{3.25}$$

Lemma 3.7. *Consider a golden generic lightlike submanifold M of a golden semi-Riemannian manifold $\tilde{M}$. Then, M is mixed geodesic, if and only if,*

$$\omega A_{\omega Z}Y = -\nabla^t_Y \omega Z,$$

for $Y \in \Gamma(D)$ and $Z \in \Gamma(D')$.

Proof. For the almost golden structure $\tilde{P}$ of a golden semi-Riemannian manifold $\tilde{M}$, one has $\tilde{\nabla}_Y Z = \tilde{P}\tilde{\nabla}_Y \tilde{P}Z - \tilde{\nabla}_Y \tilde{P}Z$ for $Y, Z \in \Gamma(TM)$. Then for $Z \in \Gamma(D')$ and $Y \in \Gamma(D)$, employing Eqs. (2.5)-(2.6), (3.4)-(3.5) and equating the transversal components of resulting equation, we derive

$$h(Y, Z) = -\,\omega A_{\omega Z}Y - \nabla^t_Y \omega Z. \tag{3.26}$$

Hence, the proof follows from Eq. (3.26). □

Definition 3.4. [7] A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ is called totally umbilical, if there exists smooth transversal curvature vector field $H \in \Gamma(tr(TM))$ on M satisfying

$$h(Y_1, Y_2) = Hg(Y_1, Y_2), \tag{3.27}$$

for $Y_1, Y_2 \in \Gamma(TM)$. Using Eqs. (2.5)-(2.7), M is totally umbilical, if and only if, there exist smooth vector fields $H^s \in \Gamma(S(TM^\perp))$ and $H^l \in \Gamma(ltr(TM))$ satisfying

$$h^s(Y_1, Y_2) = H^s g(Y_1, Y_2),\ h^l(Y_1, Y_2) = H^l g(Y_1, Y_2)\ and\ D^l(Y_1, W) = 0, \tag{3.28}$$

for $W \in \Gamma(S(TM^\perp))$ and $Y_1, Y_2 \in \Gamma(TM)$.

Theorem 3.8. *Let M be a totally umbilical golden generic lightlike submanifold of a golden semi-Riemannian manifold $\tilde{M}$. Then the distribution D is always integrable.*

Proof. For $Y_1, Y_2 \in \Gamma(D)$, from Definition 3.4, we have

$$h^s(\tilde{P}Y_1, Y_2) = H^s g(\tilde{P}Y_1, Y_2). \tag{3.29}$$

Then, employing Eq. (2.13), we derive

$$g(\tilde{P}Y_1, Y_2) = g(\tilde{P}Y_1, \tilde{P}Y_2) - g(Y_1, Y_2). \tag{3.30}$$

Further, using Definition 3.4 and Eqs. (3.29)-(3.30), we get

$$h^s(\tilde{P}Y_1, Y_2) = h^s(\tilde{P}Y_1, \tilde{P}Y_2) - h^s(Y_1, Y_2). \tag{3.31}$$

On the other hand, from Eqs. (2.12) and (3.30), we obtain

$$g(Y_1, \tilde{P}Y_2) = g(\tilde{P}Y_1, \tilde{P}Y_2) - g(Y_1, Y_2)$$

and using Definition 3.4, we get

$$h^s(Y_1, \tilde{P}Y_2) = h^s(\tilde{P}Y_1, \tilde{P}Y_2) - h^s(Y_1, Y_2). \tag{3.32}$$

On using Eqs. (3.31)-(3.32) with Theorem 3.6, the result holds. □

4 Minimal golden generic lightlike submanifolds

Bejan and Duggal introduced the generalized definition of a minimal lightlike submanifold of a semi-Riemannian manifold in [1] as follows

Definition 4.1. A lightlike submanifold $(M, g, S(TM))$ of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ is said to be minimal if

(i) $h^s(\xi_1, \xi_2) = 0$ for $\xi_1, \xi_2 \in \Gamma(Rad(TM))$.

(ii) trace $h|_{S(TM)} = 0$.

It is notable that Definition 4.1 depends on $tr(TM)$ rather than the choice of $S(TM)$ and $S(TM^{\perp})$.
Next, we prove some characterizations on minimal golden generic lightlike submanifolds.

Theorem 4.1. *Consider a totally umbilical golden generic lightlike submanifold M of a golden semi-Riemannian manifold $\tilde{M}$. Then, M is minimal, if and only if, M is totally geodesic.*

Proof. Let us suppose that M is minimal. Then for $Y, Z \in \Gamma(Rad(TM))$, we have $h^s(Y, Z) = 0$. Further, as M is totally umbilical, which gives that $h^l(Y, Z) = H^l g(Y, Z) = 0$ for $Y, Z \in \Gamma(Rad(TM))$. Next, choose an orthonormal basis $\{e_1, e_2, ..., e_{m-r}\}$ of $S(TM)$ and using Eq. (3.28), we have

$$\begin{aligned} trace\ h(e_i, e_i) =& \sum_{i=1}^{n-r} \epsilon_i \{h^l(e_i, e_i) + h^s(e_i, e_i)\} \\ =&(n-r)\{H^l + H^s\}. \end{aligned}$$

Further, in view of Definition 4.1, we get $trace\ h|_{S(TM)} = 0$ and then, employing Eq. (2.3), we obtain $H^l = 0$ and $H^s = 0$, which implies that M is totally geodesic. The proof of the converse part is trivial. □

Theorem 4.2. *Consider M be a totally umbilical golden generic lightlike submanifold of a golden semi-Riemannian manifold $\tilde{M}$. Then M is minimal, if and only if,*

$$traceA^*_{\xi_k}|_{D_0 \perp S(TM^\perp)} = traceA_{W_p}|_{D_0 \perp S(TM^\perp)} = 0,$$

for $W_p \in \Gamma(S(TM^\perp))$, where $k \in 1, 2, ..., r$ and $p \in 1, 2, ..., n - r$.

Proof. Since M is totally umbilical, then using Eq. (3.28), it is clear that $h^s(Y_1, Y_2) = 0$ for $Y_1, Y_2 \in \Gamma(Rad(TM))$. Next, using the definition of a golden generic lightlike submanifold, we have

$$\begin{aligned} trace\ h|_{S(TM)} =& \sum_{i=1}^{2p} h(Z_i, Z_i) + \sum_{j=1}^{r} h(\tilde{P}\xi_j, \tilde{P}\xi_j) + \sum_{j=1}^{r} h(\tilde{P}N_j, \tilde{P}N_j) \\ &+ \sum_{l=1}^{n-r} h(\tilde{P}W_l, \tilde{P}W_l), \end{aligned}$$

where $2p = dim(D_0)$, $r = dim(Rad(TM))$ and $n - r = dim(S(TM^\perp))$. Again, using Eq. (3.27), we obtain $h(\tilde{P}\xi_j, \tilde{P}\xi_j) = h(\tilde{P}N_j, \tilde{P}N_j) = 0$. Thus, the above equation reduces to

$$\begin{aligned} trace\ h|_{S(TM)} =& \sum_{i=1}^{2p} h(Z_i, Z_i) + \sum_{l=1}^{n-r} h(\tilde{P}W_l, \tilde{P}W_l) \\ =& \sum_{i=1}^{2p} \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(h^l(Z_i, Z_i), \xi_k)N_k + \sum_{i=1}^{2p} \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(h^s(Z_i, Z_i), W_p)W_p \\ &+ \sum_{l=1}^{n-r} \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(h^l(\tilde{P}W_l, \tilde{P}W_l), \xi_k)N_k \\ &+ \sum_{l=1}^{n-r} \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(h^s(\tilde{P}W_l, \tilde{P}W_l), W_p)W_p, \end{aligned} \tag{4.1}$$

where $\{W_1, W_2, ..., W_{n-r}\}$ is an orthonormal basis of $S(TM^\perp)$. Using Eqs. (2.8) and (2.10) in Eq. (4.1), we obtain

$$\begin{aligned} trace\ h|_{S(TM)} =& \sum_{i=1}^{2p} \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(A^*_{\xi_k} Z_i, Z_i)N_k + \sum_{i=1}^{2p} \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(A_{W_p} Z_i, Z_i)W_p \\ &+ \sum_{l=1}^{n-r} \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(A^*_{\xi_k} \tilde{P}W_l, \tilde{P}W_l)N_k + \sum_{l=1}^{n-r} \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(A_{W_p} \tilde{P}W_l, \tilde{P}W_l)W_p. \end{aligned}$$

Thus, from the above equation, trace $h|_{S(TM)} = 0$, if and only if, trace $A_{W_p} = 0$ and trace $A^*_{\xi_k} = 0$ on $D_0 \perp \tilde{P}S(TM^\perp)$, which proves the theorem. □

Example 4.2. Let M be a 6-dimensional submanifold in $(R_2^{14}, \tilde{g})$ with

$$x^1 = u^2 + \tfrac{1}{2}u^3, \quad x^2 = u^1, \quad x^3 = u^2 - \tfrac{1}{2}u^3, \quad x^4 = u^1, \quad x^5 = u^5 \cos\alpha,$$
$$x^6 = u^4 \cos\alpha, \quad x^7 = -u^5 \sin\alpha, \quad x^8 = -u^4 \sin\alpha, \quad x^9 = \sin u^6 \sinh u^7, \quad x^{10} = 0,$$
$$x^{11} = \sin u^6 \cosh u^7, \quad x^{12} = 0, \quad x^{13} = \sqrt{2}\cos u^6 \cosh u^7, \quad x^{14} = 0,$$

where $\alpha, u^6, u^7 \in R - \{\frac{n\pi}{2},\ n \in Z\}$ and g is of signature $(+,+,-,-,+,+,+,+,+,+,+,+,+,+)$. Then, TM is spanned by $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7$, such that

$$Z_1 = \partial x_2 + \partial x_4, \quad Z_2 = \partial x_1 + \partial x_3,$$

$$Z_3 = \frac{1}{2}\{\partial x_1 - \partial x_3\}, \quad Z_4 = \cos\alpha \partial x_6 - \sin\alpha \partial x_8, \quad Z_5 = \cos\alpha \partial x_5 - \sin\alpha \partial x_7,$$

$$Z_6 = \cos u^6 \sinh u^7 \partial x_9 + \cos u^6 \cosh u^7 \partial x_{11} - \sqrt{2}\sin u^6 \cosh u^7 \partial x_{13},$$

$$Z_7 = \sin u^6 \cosh u^7 \partial x_9 + \sin u^6 \sinh u^7 \partial x_{11} + \sqrt{2}\cos u^6 \sinh u^7 \partial x_{13}.$$

Clearly, M is a 1-lightlike submanifold with $Rad(TM) = Span\{Z_1\}$. As $\tilde{P}Z_1 = Z_2 \in \Gamma(S(TM))$. Moreover, $\tilde{P}Z_4 = Z_5$, thus $D_0 = Span\{Z_4, Z_5\}$. Further, $ltr(TM)$ is spanned by

$$N = \frac{1}{2}(\partial x_2 - \partial x_4),$$

such that $\tilde{P}N = \frac{1}{2}(\partial x_1 - \partial x_3) = Z_3$. By direct calculations, $S(TM^{\perp}) = Span\{W_1, W_2, W_3, W_4\}$, which are given by

$$W_1 = \cos u^6 \sinh u^7 \partial x_{10} + \cos u^6 \cosh u^7 \partial x_{12} - \sqrt{2}\sin u^6 \cosh u^7 \partial x_{14},$$

$$W_2 = \sin u^6 \cosh u^7 \partial x_{10} + \sin u^6 \sinh u^7 \partial x_{12} + \sqrt{2}\cos u^6 \sinh u^7 \partial x_{14},$$

$$W_3 = -\sqrt{2}\sinh u^7 \cosh u^7 \partial x_{10} + \sqrt{2}(\sin^2 u^6 + \sinh^2 u^7)\partial x_{12} + \sin u^6 \cos u^6 \partial x_{14},$$

$$W_4 = -\sqrt{2}\sinh u^7 \cosh u^7 \partial x_9 + \sqrt{2}(\sin^2 u^6 + \sinh^2 u^7)\partial x_{11} + \sin u^6 \cos u^6 \partial x_{14}.$$

Since $\tilde{P}W_1 = Z_6$ and $\tilde{P}W_2 = Z_7$, then $D' = Span\{Z_3, Z_6, Z_7\}$. Therefore, M is a proper golden generic lightlike submanifold of R_2^{14}. On the other hand, by direct calculations, we obtain

$$\tilde{\nabla}_{Z_i} Z_j = 0,\ 1 \le i \le 5, 1 \le j \le 7$$

and

$$h^s(Z_6, Z_6) = -\frac{\sqrt{2}\sin u^6 \cosh u^7}{(\sin^2 u^6 + 2\sinh^2 u^7)(1 + \sin^2 u^6 + 2\sinh^2 u^7)} W_4,$$

$$h^s(Z_7, Z_7) = -\frac{\sqrt{2}\sin u^6 \cosh u^7}{(\sin^2 u^6 + 2\sinh^2 u^7)(1 + \sin^2 u^6 + 2\sinh^2 u^7)} W_4.$$

Thus, we get

$$trace\ h_{g|S(TM)} = h^s(Z_6, Z_6) + h^s(Z_7, Z_7) = 0.$$

Therefore, M is a minimal proper golden generic lightlike submanifold of R_2^{14}.

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